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LOWER SEMICONTINUOUS ENVELOPES IN $W^{1,1} \times L^p$

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Abstract. The lower semicontinuity of functionals of the type $\int_\Omega f(x,u,v,\nabla u)\,dx$ with respect to the $(W^{1,1} \times L^p)$-weak* topology is studied. Moreover, in absence of lower semicontinuity, an integral representation in $W^{1,1} \times L^p$ for the lower semicontinuous envelope is also provided.

1. Introduction. In this paper we consider energies depending on two vector fields with different behaviours: $u \in W^{1,1}(\Omega;\mathbb{R}^n)$, $v \in L^p(\Omega;\mathbb{R}^m)$, Ω being a bounded open set of $\mathbb{R}^N$. Let $1 < p \leq +\infty$, for every $(u,v) \in W^{1,1}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$ define the functional

$$J(u,v) := \int_\Omega f\big(x, u(x), v(x), \nabla u(x)\big)\,dx \tag{1}$$

where $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0,+\infty)$ is a continuous function with linear growth in the last variable and p-growth in the third variable (cf. $(\mathrm{H1}_p)$ and $(\mathrm{H1}_\infty)$ below).

The energies (1), which generalize those considered by [14], [15] and [8], have been introduced to deal with equilibria for systems depending on elastic strain and chemical composition. In this context a multiphase alloy is represented by the set Ω, the deformation gradient is given by ∇u, and v (when $m = 1$) denotes the chemical composition of the system. We also recall that our result may find applications also in the framework of Elasticity, when dealing with Cosserat's theory, see [19]. In [14], the density $f \equiv f(v,\nabla u)$ is a convex-quasiconvex function, while in our model we also take into account heterogeneities and the deformation, without imposing any convexity restriction.

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We are interested in studying the lower semicontinuity and relaxation of (1) with respect to the L^1-strong$\times L^p$-weak convergence. Clearly, bounded sequences $\{u_n\} \subset W^{1,1}(\Omega;\mathbb{R}^n)$ may converge in L^1, up to a subsequence, to a BV function. In this paper we restrict our analysis to limits u which are in $W^{1,1}(\Omega;\mathbb{R}^n)$. Thus, our results can be considered as a step towards the study of relaxation in $\mathrm{BV}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$ of functionals (1). (For the definition and properties of BV spaces we refer to [3].)

We will consider separately the cases $1 < p < \infty$ and $p = \infty$. To this end we introduce for $1 < p < +\infty$ the functional

$$\overline{J}_p(u,v) := \inf\bigl\{\liminf J(u_n,v_n) : u_n \in W^{1,1}(\Omega;\mathbb{R}^n),\ v_n \in L^p(\Omega;\mathbb{R}^m),\\ u_n \to u \text{ in } L^1, v_n \rightharpoonup v \text{ in } L^p\bigr\}, \quad (2)$$

for any pair $(u,v) \in W^{1,1}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$, and for $p = \infty$ the functional

$$\overline{J}_\infty(u,v) := \inf\bigl\{\liminf J(u_n,v_n) : u_n \in W^{1,1}(\Omega;\mathbb{R}^n),\ v_n \in L^\infty(\Omega;\mathbb{R}^m),\\ u_n \to u \text{ in } L^1,\ v_n \overset{*}{\rightharpoonup} v \text{ in } L^\infty\bigr\}, \quad (3)$$

for any pair $(u,v) \in W^{1,1}(\Omega;\mathbb{R}^n) \times L^\infty(\Omega;\mathbb{R}^m)$.

For any $p \in (1,+\infty]$ we will achieve the following integral representation (see Theorems 12 and 14):

$$\overline{J}_p(u,v) = \int_\Omega CQf\bigl(x,u(x),v(x),\nabla u(x)\bigr)\,dx,$$

where CQf represents the convex-quasiconvexification of f defined in (6).

2. Notation and general facts.

In this section we introduce the sets of assumptions we will make to obtain our results. We prove some properties related to convex-quasiconvex functions and we recall several facts that will be useful through the paper.

2.1. Assumptions.

Let $1 < p < +\infty$, to obtain a characterization of the relaxed functional $\overline{J}_p$ in (2), we will make several assumptions on the continuous function

$$f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0,+\infty).$$

They are inspired by the set of assumptions in [17] for the case with no dependence on v.

($\mathrm{H1}_p$) There exists a constant C such that

$$\frac{1}{C}(|v|^p + |\xi|) - C \le f(x,u,v,\xi) \le C(1 + |v|^p + |\xi|),$$

for every $(x,u,v,\xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N}$.

($\mathrm{H2}_p$) For every compact set K of $\Omega \times \mathbb{R}^n$ there exists a continuous function $\omega_K : \mathbb{R} \to [0,+\infty)$ with $\omega_K(0) = 0$ such that

$$|f(x,u,v,\xi) - f(x',u',v,\xi)| \le \omega_K(|x-x'| + |u-u'|)(1 + |v|^p + |\xi|)$$

for every (x,u,v,ξ) and (x',u',v,ξ) in $K \times \mathbb{R}^m \times \mathbb{R}^{n\times N}$.

Moreover, given $x_0 \in \Omega$, and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| \le \delta$ then

$$\forall (u,v,\xi) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N} \quad f(x,u,v,\xi) - f(x_0,u,v,\xi) \ge -\varepsilon(1 + |v|^p + |\xi|).$$

In order to characterize the functional $\overline{J}_\infty$ defined in (3) we will replace assumptions ($\mathrm{H1}_p$) and ($\mathrm{H2}_p$) by the following ones.

(H1$_\infty$) Given $M > 0$, there exist a bounded continuous function $G_M : \Omega \times \mathbb{R}^n \to [0, +\infty)$ and $C_M > 0$ such that if $|v| \le M$ then for all $(x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times N}$

$$\frac{1}{C_M} G_M(x,u)|\xi| - C_M \le f(x,u,v,\xi) \le C_M G_M(x,u)(1+|\xi|).$$

(H2$_\infty$) For every $M > 0$, and for every compact set K of $\Omega \times \mathbb{R}^n$ there exists a continuous function $\omega_{M,K} : \mathbb{R} \to [0, +\infty)$ with $\omega_{M,K}(0) = 0$ such that if $|v| \le M$ then

$$|f(x,u,v,\xi) - f(x',u',v,\xi)| \le \omega_{M,K}(|x-x'| + |u-u'|)(1+|\xi|)$$

for every $(x,u,\xi), (x',u',\xi) \in K \times \mathbb{R}^{n\times N}$.

Moreover, given $M > 0$, $x_0 \in \Omega$, and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|v| \le M$ and $|x - x_0| \le \delta$ then

$$\forall (u,\xi) \in \mathbb{R}^n \times \mathbb{R}^{n\times N} \quad f(x,u,v,\xi) - f(x_0,u,v,\xi) \ge -\varepsilon G_M(x,u)(1+|\xi|),$$

where the function G_M is as in (H1$_\infty$).

2.2. Convex-quasiconvex functions. We start recalling the notion of convex-quasiconvex function, presented in [14] (see also [19, Definition 4.1], [15] and [13]). This notion plays, in the context of lower semicontinuity problems where the density depends on two fields $v, \nabla u$, the role of the well known notion of quasiconvexity introduced by Morrey for the lower semicontinuity of functionals where the dependence is just on ∇u.

Definition 1. A Borel measurable function $h : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to \mathbb{R}$ is said to be *convex-quasiconvex* if there exists a bounded open set D of $\mathbb{R}^N$ such that

$$h(v,\xi) \le \frac{1}{|D|} \int_D h\big(v + \eta(x), \xi + \nabla\varphi(x)\big)\, dx, \tag{4}$$

for every $(v,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$, $\varphi \in W_0^{1,\infty}(D;\mathbb{R}^n)$ and $\eta \in L^\infty(D;\mathbb{R}^m)$ with $\int_D \eta(x)\,dx = 0$.

Remark 2.

(i) It can be easily seen that, if h is convex-quasiconvex, then condition (4) is true for any bounded open set $D \subset \mathbb{R}^N$.

(ii) We recall that a convex-quasiconvex function is separately convex.

(iii) Through this paper we will work with functions f defined in $\Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N}$ and when saying that f is convex-quasiconvex we mean the above definition with respect to the last two variables of f.

The following result adapts to the context of $W^{1,1} \times L^p$, i.e. growth conditions expressed by (H1$_p$), a well known result due to Marcellini in [20] (see also Proposition 2.32 in [10] or Lemma 5.42 in [3]). Indeed, the following proposition follows as a particular case of [9, Proposition 2.11].

Proposition 3. *Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N} \to \mathbb{R}$ be a separately convex function in each entry of the variables (v,ξ), satisfying the growth condition*

$$|f(x,u,v,\xi)| \le c(1 + |\xi| + |v|^p) \quad \forall (x,u,v,\xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N}$$

for some $p > 1$. Then, if we denote by p' the conjugate exponent of p, there exists a constant $\gamma > 0$ such that

$$|f(x,u,v,\xi) - f(x,u,v',\xi')| \le \gamma\Big[|\xi - \xi'| + \big(1 + |v|^{p-1} + |v'|^{p-1} + |\xi|^{1/p'} + |\xi'|^{1/p'}\big)|v - v'|\Big]$$

for every $\xi, \xi' \in \mathbb{R}^{n\times N}$, $v, v' \in \mathbb{R}^m$ and $(x,u) \in \Omega \times \mathbb{R}^n$.

A similar result holds for $W^{1,1} \times L^\infty$ (i.e. growth conditions expressed by $(\mathrm{H1}_\infty)$).

Proposition 4. *Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N} \to \mathbb{R}$ be a separately convex function in each entry of the variables (v, ξ), satisfying assumption* $(\mathrm{H1}_\infty)$*. Then, given $M > 0$, there exists a constant $\beta(M, n, m, N)$ such that*

$$|f(x,u,v,\xi) - f(x,u,v',\xi')| \le \beta(1+|\xi|+|\xi'|)|v-v'| + \beta|\xi-\xi'|, \tag{5}$$

for every $v, v' \in \mathbb{R}^m$, such that $|v| \le M$ and $|v'| \le M$, for every $\xi, \xi' \in \mathbb{R}^{n\times N}$ and for every $(x,u) \in \Omega \times \mathbb{R}^n$.

We introduce the notion of convex-quasiconvexification with respect to the last variables for a function $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0, +\infty)$. This notion is crucial in order to deal with the subsequent relaxation processes.

If $h : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to \mathbb{R}$ is any given Borel measurable function bounded from below, the *convex-quasiconvex envelope* of h can be defined as the largest convex-quasiconvex function below h:

$$CQh(v,\xi) := \sup\{g(v,\xi) : g \le h,\ g \text{ convex-quasiconvex}\}. \tag{6}$$

Moreover, by Theorem 4.16 in [19]

$$CQh(v,\xi) = \inf\Big\{\frac{1}{|D|}\int_D h\big(v+\eta(x), \xi+\nabla\varphi(x)\big)\,dx : \eta \in L^\infty(D;\mathbb{R}^m),\ \int_D \eta(x)\,dx = 0,\ \varphi \in W_0^{1,\infty}(D;\mathbb{R}^n)\Big\}. \tag{7}$$

Consequently, given a Carathéodory function $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N} \to \mathbb{R}$, by $CQf(x,u,v,\xi)$ we denote the convex-quasiconvexification of $f(x,u,v,\xi)$ with respect to the last two variables.

As for convex-quasiconvexity, condition (7) can be stated for any bounded open set $D \subset \mathbb{R}^N$ and it can be also showed that if f satisfies a growth condition of the type $(\mathrm{H1}_p)$ then in (4) and (7) the spaces L^∞ and $W_0^{1,\infty}$ can be replaced by L^p and $W_0^{1,1}$, respectively.

The following results will be exploited in the sequel. We omit the proofs since they are very similar to [21, Proposition 2.2], in turn inspired by [10].

Proposition 5. *Let $1 < p < +\infty$. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0, +\infty)$ be a continuous function satisfying* $(\mathrm{H1}_p)$ *and* $(\mathrm{H2}_p)$*. Let CQf be the convex-quasiconvexification of f in (7). Then CQf satisfies* $(\mathrm{H1}_p)$, $(\mathrm{H2}_p)$ *and is a continuous function.*

Analogously we have

Proposition 6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, let $\alpha : [0, +\infty) \to [0, +\infty)$ be a convex and increasing function, such that $\alpha(0) = 0$ and let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0, +\infty)$ be a continuous function satisfying the following conditions*:

- *For a.e. $(x,u) \in \Omega \times \mathbb{R}^n$ and for every $(v,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$*

$$\frac{1}{C}\left(\alpha(|v|) + |\xi|\right) - C \le f(x,u,v,\xi) \le C\big(1 + \alpha(|v|) + |\xi|\big). \tag{8}$$

- *For every compact set $K \subset \Omega \times \mathbb{R}^n$ there exists a continuous function $\omega'_K : \mathbb{R} \to [0, +\infty)$ such that $\omega'_K(0) = 0$ and*
$$|f(x,u,v,\xi) - f(x',u',v,\xi)| \le \omega'_K\big(|x-x'| + |u-u'|\big)\big(1 + \alpha(|v|) + |\xi|\big), \qquad (9)$$
for all $(x,u), (x',u') \in K$ and $(v,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$.
- *For every $x_0 \in \Omega$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*
$$|x - x_0| \le \delta \;\Rightarrow\; f(x,u,v,\xi) - f(x_0,u,v,\xi) \ge -\varepsilon\big(1 + \alpha(|v|) + |\xi|\big), \qquad (10)$$
for all $(u,\xi) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N}$.

Let CQf be the convex-quasiconvexification of f (see (7)). Then CQf satisfies conditions analogous to (8), (9) and (10). Moreover, CQf is a continuous function.

Remark 7. We observe that from one hand (8), (9), (10) generalize ($\mathrm{H1}_p$) and ($\mathrm{H2}_p$), and from the other hand they can be regarded as a stronger version of ($\mathrm{H1}_\infty$) and ($\mathrm{H2}_\infty$).

In order to provide an integral representation for $\overline{J}_p$ in (2) and $\overline{J}_\infty$ in (3) on $W^{1,1} \times L^p$ $(1 < p < +\infty)$ and $W^{1,1} \times L^\infty$ respectively, we prove some preliminary results.

For every $p \in (1, +\infty]$ we introduce the functional $J_{CQf} : L^1(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m) \to \mathbb{R} \cup \{+\infty\}$ defined as

$$J_{CQf}(u,v) := \begin{cases} \int_\Omega CQf\big(x,u(x),v(x),\nabla u(x)\big)\,dx & \text{if } (u,v) \in W^{1,1}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

and its relaxed one, also defined in $L^1(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$,

$$\overline{J_{CQf}}(u,v) := \inf\Big\{\liminf_n J_{CQf}(u_n,v_n) : (u_n,v_n) \in W^{1,1}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m), \\ u_n \to u \text{ in } L^1,\; v_n \overset{*}{\rightharpoonup} v \text{ in } L^p\Big\}.$$

Lemma 8. *Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0,+\infty)$ be a continuous function. Let $p \in (1,+\infty]$ and consider the functionals J and J_{CQf} and their corresponding relaxed functionals $\overline{J}_p$ and $\overline{J_{CQf}}$. If f satisfies conditions ($\mathrm{H1}_p$)–($\mathrm{H2}_p$) (if $p \in (1,+\infty)$), and both f and CQf satisfy ($\mathrm{H1}_\infty$)–($\mathrm{H2}_\infty$) (if $p = +\infty$), then*
$$\overline{J}_p(u,v) = \overline{J_{CQf}}(u,v) \qquad (11)$$
for every $(u,v) \in W^{1,1}(\Omega,\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$.

Remark 9. First we observe that the same proof shows that equality (11) holds also in $\mathrm{BV}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$, $p \in (1,+\infty]$, up to the extension to $L^1(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$ of J in (1) as

$$J(u) = \begin{cases} \displaystyle\int_\Omega f\big(x,u(x),v(x),\nabla u(x)\big)\,dx & \text{if } (u,v) \in W^{1,1}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

and the subsequent extension of $\overline{J}_p$ and $\overline{J}_\infty$ to $L^1(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$ via formulae (2) and (3).

We recall that a function $u \in L^1(\Omega;\mathbb{R}^N)$ is said to be *of bounded variation*, and this is denoted by $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$, if the distributional derivative of u is representable by a Radon measure in Ω.

For more details concerning the theory of functions of bounded variation we refer the reader to [3].

We emphasize also that in the above lemma, by virtue of Proposition 5, if $p \in (1, +\infty)$, it is enough to assume growth and continuity hypotheses just on f (and not on CQf). If $p = +\infty$, by virtue of Proposition 6, we can also only make assumptions on f, replacing conditions (H1$_\infty$) and (H2$_\infty$) by (8)–(10).

Proof. The argument is close to the proof of [21, Lemma 3.1]. First we observe that, since $CQf \leq f$, we have $\overline{J_{CQf}} \leq \overline{J}_p$. Next we prove the opposite inequality in the nontrivial case that $\overline{J_{CQf}}(u,v) < +\infty$. For fixed $\delta > 0$, we can consider $(u_n, v_n) \in W^{1,1}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$ with $u_n \to u$ strongly in $L^1(\Omega;\mathbb{R}^n)$, $v_n \overset{*}{\rightharpoonup} v$ in $L^p(\Omega;\mathbb{R}^m)$ and such that

$$\overline{J_{CQf}}(u,v) \geq \lim_n \int_\Omega CQf\big(x, u_n(x), v_n(x), \nabla u_n(x)\big)\, dx - \delta.$$

By the results from [8] and [9], for each n there exists a sequence $\{(u_{n,k}, v_{n,k})\}$ converging to (u_n, v_n) weakly in $W^{1,1}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$ such that

$$\int_\Omega CQf\big(x, u_n(x), v_n(x), \nabla u_n(x)\big)\, dx = \lim_k \int_\Omega f\big(x, u_{n,k}(x), v_{n,k}(x), \nabla u_{n,k}(x)\big)\, dx.$$

Consequently

$$\overline{J_{CQf}}(u,v) \geq \lim_n \lim_k \int_\Omega f\big(x, u_{n,k}(x), v_{n,k}(x), \nabla u_{n,k}(x)\big)\, dx - \delta, \tag{12}$$

$$\lim_n \lim_k \|u_{n,k} - u\|_{L^1} = 0,$$

and

$$v_{n,k} \overset{*}{\rightharpoonup} v \text{ in } L^p \text{ as } k \to +\infty \text{ and } n \to +\infty.$$

Via a diagonal argument (remind that weak L^p and weak* L^∞-topologies are metrizable on bounded sets), there exists a sequence $\{(u_{n,k_n}, v_{n,k_n})\}$ satisfying $u_{n,k_n} \to u$ in $L^1(\Omega;\mathbb{R}^n)$, $v_{n,k_n} \overset{*}{\rightharpoonup} v$ in $L^p(\Omega;\mathbb{R}^m)$ and realizing the double limit in the right hand side of (12). Thus

$$\overline{J_{CQf}}(u,v) \geq \lim_n \int_\Omega f\big(x, u_{n,k_n}(x), v_{n,k_n}(x), \nabla u_{n,k_n}(x)\big)\, dx - \delta \geq \overline{J}_p(u,v) - \delta.$$

If we let δ go to 0 the conclusion follows. ∎

2.3. Some results on measure theory. Let Ω be a generic open subset of $\mathbb{R}^N$, we denote by $\mathcal{M}(\Omega)$ the space of all signed Radon measures in Ω with bounded total variation. By the Riesz Representation Theorem, $\mathcal{M}(\Omega)$ can be identified to the dual of the separable space $\mathcal{C}_0(\Omega)$ of continuous functions on Ω vanishing on the boundary $\partial\Omega$. The N-dimensional Lebesgue measure in $\mathbb{R}^N$ is designated as $\mathcal{L}^N$ while $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure. If $\mu \in \mathcal{M}(\Omega)$ and $\lambda \in \mathcal{M}(\Omega)$ is a nonnegative Radon measure, we denote by $\frac{d\mu}{d\lambda}$ the Radon–Nikodým derivative of μ with respect to λ. By a generalization of the Besicovitch Differentiation Theorem (see [2, Proposition 2.2]), it can be proved that there exists a Borel set $E \subset \Omega$ such that $\lambda(E) = 0$ and

$$\frac{d\mu}{d\lambda}(x) = \lim_{\rho \to 0^+} \frac{\mu(x + \rho C)}{\lambda(x + \rho C)}$$

for all $x \in \operatorname{Supp} \mu \setminus E$ and any open convex set C containing the origin. (Recall that the set E is independent of C.)

We also recall the following generalization of Lebesgue–Besicovitch Differentiation Theorem, as stated in [18, Theorem 2.8].

THEOREM 10. *If μ is a nonnegative Radon measure and if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mu)$ then*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\mu(x+\varepsilon C)} \int_{x+\varepsilon C} |f(y)-f(x)|\, d\mu(y) = 0,$$

for μ-a.e. $x \in \mathbb{R}^d$ and for every bounded, convex, open set C containing the origin.

In particular, if $v \in L^\infty(\Omega;\mathbb{R}^m)$, then, for $\mathcal{L}^N$-a.e. $x \in \Omega$

$$\lim_{\varepsilon\to 0} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |v(y)-v(x)|\, dy = 0. \tag{13}$$

In the sequel we exploit the Calderón–Zygmund theorem for $u \in \mathrm{BV}$, cf. [3, Theorem 3.83, page 176]

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |u(y)-u(x)-\nabla u(x)(y-x)|\, dy = 0 \quad \mathcal{L}^N\text{-a.e. } x\in\Omega. \tag{14}$$

3. Lower semicontinuity in $W^{1,1} \times L^p$, $1 < p < +\infty$. This section is devoted to provide a lower bound for the integral representation of $\overline{J}_p$ in (2) under assumptions (H1$_p$) and (H2$_p$), as stated in Theorem 14. Clearly this is equivalent to proving the lower semicontinuity with respect to the L^1-strong $\times$ L^p-weak topology of

$$\int_\Omega CQf\big(x,u(x),v(x),\nabla u(x)\big)\, dx,$$

when $(u,v) \in W^{1,1}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$.

Indeed, we prove the following result

THEOREM 11. *Let Ω be a bounded open set of $\mathbb{R}^N$, and let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0,+\infty)$ be a continuous function. Assuming that f satisfies hypotheses* (H1$_p$) *and* (H2$_p$), *and it is convex-quasiconvex, we deduce that $\int_\Omega f\big(x,u(x),v(x),\nabla u(x)\big)\,dx$ is lower semicontinuous in $W^{1,1}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$ with respect to the (L^1-strong $\times$ L^p-weak) convergence.*

Proof. The proof is mostly a combination of the theorems in [18] and [14], which used already some ideas from [17]. For convenience of the reader we present here some details, however we may refer to some separate results in the papers mentioned above.

Let

$$G(u,v) = \int_\Omega f\big(x,u(x),v(x),\nabla u(x)\big)\, dx.$$

It is enough to prove that for every $(u,v) \in W^{1,1}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$, $G(u,v) \le \liminf J(u_n,v_n)$ for any $u_n \to u$ in L^1 with $u_n \in W^{1,1}(\Omega;\mathbb{R}^n)$ and $v_n \rightharpoonup v$ in L^p.

Using the same arguments as in [1, Proof of Theorem II.4] (see also [17, Proposition 2.4]) and the density of smooth functions in L^p, we can reduce to the case where $u_n \in C_0^\infty(\mathbb{R}^N;\mathbb{R}^n)$ and $v_n \in C_0^\infty(\mathbb{R}^N;\mathbb{R}^m)$.

Moreover, we can also suppose

$$\liminf_{n\to\infty} J(u_n,v_n) = \lim_{n\to\infty} J(u_n,v_n) < +\infty.$$

Then $J(u_n, v_n)$ is bounded and so, up to a subsequence, $\mu_n := f(x, u_n, v_n, \nabla u_n)dx \overset{*}{\rightharpoonup} \mu$ in the sense of measures for some positive measure μ.

By the Radon–Nikodým theorem, $\mu = g\mathcal{L}^N + \mu_s$ for some $g \in L^1(\Omega)$, with μ_s singular with respect to $\mathcal{L}^N$. It will be enough to prove the inequality

$$g(x) \geq f\big(x, u(x), v(x), \nabla u(x)\big), \quad \mathcal{L}^N\text{-a.e. } x \in \Omega. \tag{15}$$

Indeed, once proved (15), since $\mu_n \overset{*}{\rightharpoonup} \mu$, by the lower semicontinuity of μ, and since μ_s is nonnegative,

$$\begin{aligned}\lim_{n\to+\infty} J(u_n, v_n) &= \lim_{n\to+\infty} \int_\Omega f\big(x, u_n(x), v_n(x), \nabla u_n(x)\big)\, dx \\ &\geq \int_\Omega d\mu(x) = \int_\Omega g(x)\, dx + \int_\Omega d\mu_s(x) \geq \int_\Omega f\big(x, u(x), v(x), \nabla u(x)\big)\, dx.\end{aligned}$$

In order to prove (15), we follow the proofs of Theorem 2.1 in [17] and condition (2.3) in [14]. We start by freezing the terms x and u. This will be achieved through Steps 1–5.

By the Besicovitch derivation theorem

$$g(x) = \lim_{\varepsilon\to 0} \frac{\mu(B_\varepsilon(x))}{|B_\varepsilon(x)|} \in \mathbb{R} \quad \mathcal{L}^N\text{-a.e. } x \in \Omega. \tag{16}$$

Let x_0 be any element of Ω satisfying (16), (14) and (13) (notice that such an x_0 can be taken in Ω up to a set of Lebesgue measure zero) and prove that $g(x_0) \geq f\big(x_0, u(x_0), v(x_0), \nabla u(x_0)\big)$. First remark that, as noticed before, since $v_n \rightharpoonup v$ in L^p, we have $\|v_n\|_{L^p}, \|v\|_{L^p} \leq C$.

Step 1. Localization. This part can be reproduced in the same way as in [17], pages 1085–1086. We present some details for the reader's convenience. We start providing a first estimate for g. Observe that we can choose a sequence $\varepsilon \to 0^+$ such that $\mu\big(\partial B_\varepsilon(x_0)\big) = 0$. Let $B := B_1(0)$. Applying Proposition 1.203 *iii*) in [16], we have

$$\begin{aligned}g(x_0) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \frac{\mu(B_\varepsilon(x_0))}{|B|} \\ &= \limsup_{\varepsilon\to 0} \lim_{n\to+\infty} \frac{1}{\varepsilon^N\, |B|} \int_{B_\varepsilon(x_0)} f\big(y, u_n(y), v_n(y), \nabla u_n(y)\big)\, dy \\ &= \limsup_{\varepsilon\to 0} \lim_{n\to+\infty} \frac{1}{|B|} \int_B f\big(x_0 + \varepsilon x, u_n(x_0 + \varepsilon x), v_n(x_0 + \varepsilon x), \nabla u_n(x_0 + \varepsilon x)\big)\, dx \\ &\geq \limsup_{\varepsilon\to 0} \lim_{n\to+\infty} \frac{1}{|B|} \int_B f\big(x_0 + \varepsilon x, u(x_0) + \varepsilon w_{n,\varepsilon}(x), v_n(x_0 + \varepsilon x), \nabla w_{n,\varepsilon}(x)\big)\, dx\end{aligned}$$

where $w_{n,\varepsilon}(x) = (u_n(x_0 + \varepsilon x) - u(x_0))/\varepsilon$.

Step 2. Blow-up. Next we will "identify the limits" of $w_{n,\varepsilon}$ and $v_n(x_0 + \varepsilon\,\cdot)$ in a sense to be made precise below. Define $w_0 : B \to \mathbb{R}^n$ by $w_0(x) = \nabla u(x_0)x$. Then

$$\begin{aligned}\lim_{\varepsilon\to 0} \lim_{n\to+\infty} \|w_{n,\varepsilon} - w_0\|_{L^1(B)} &= \lim_{\varepsilon\to 0} \lim_{n\to+\infty} \int_B \left| \frac{u_n(x_0 + \varepsilon x) - u(x_0)}{\varepsilon} - \nabla u(x_0)x \right| dx \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N+1}} \int_{B_\varepsilon(x_0)} |u(y) - u(x_0) - \nabla u(x_0)(y - x_0)|\, dy = 0\end{aligned}$$

where we have used (14) in the last identity.

Let q be the Hölder conjugate exponent of p. Since L^q is separable, consider $\{\varphi_l\}$ a countable dense set of functions in $L^q(B)$. Then

$$\lim_{\varepsilon\to 0}\lim_{n\to+\infty}\left|\int_B \big(v_n(x_0+\varepsilon x)-v(x_0)\big)\varphi_l(x)\,dx\right| = \lim_{\varepsilon\to 0}\left|\int_B \big(v(x_0+\varepsilon x)-v(x_0)\big)\varphi_l(x)\,dx\right| = 0$$

where we have used in the last identity the fact that x_0 is a Lebesgue point for v.

Step 3. Diagonalization. Arguing as in [18] and [14] we can use a diagonalization argument to find $\varepsilon_n \in \mathbb{R}^+$, $w_n \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^n)$ and $v_n \in L^p(B;\mathbb{R}^m)\cap C_0^\infty(\mathbb{R}^N;\mathbb{R}^m)$, such that $\varepsilon_n \to 0$, $w_n \to w_0$ in $L^1(B;\mathbb{R}^n)$, $v_n \rightharpoonup v(x_0)$ in $L^p(B;\mathbb{R}^m)$ as $n\to+\infty$ and

$$g(x_0) \geq \lim_{n\to+\infty}\frac{1}{|B|}\int_B f\big(x_0+\varepsilon_n x, u(x_0)+\varepsilon_n w_n(x), v_n(x), \nabla w_n(x)\big)\,dx.$$

Step 4. Truncation. We show that the sequences $\{w_n\}$ and $\{v_n\}$ constructed in the preceding steps can be replaced by sequences $\{\widetilde{w}_n\} \subset W^{1,\infty}_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^n)$ and $\{\widetilde{v}_n\} \subset L^p(B;\mathbb{R}^m)\cap C_0^\infty(\mathbb{R}^N;\mathbb{R}^m)$ such that $\|\widetilde{w}_n\|_{W^{1,1}(B;\mathbb{R}^n)} \leq C$, $\widetilde{w}_n \to w_0$ in $L^\infty(B;\mathbb{R}^n)$, $\|\widetilde{v}_n\|_{L^p(B;\mathbb{R}^n)} \leq C$, $\widetilde{v}_n \rightharpoonup v(x_0)$ in $L^p(B;\mathbb{R}^m)$ and

$$g(x_0) \geq \lim_{n\to\infty}\frac{1}{|B|}\int_B f\big(x_0+\varepsilon_n x, u(x_0)+\varepsilon_n\widetilde{w}_n(x), \tilde{v}_n(x), \nabla\widetilde{w}_n(x)\big)\,dx.$$

Let $0<s<t<1$ and $\lambda>1$ and define $\varphi_{s,t}$ a cut-off function such that $0\leq\varphi_{s,t}\leq 1$, $\varphi_{s,t}(\tau)=1$ if $\tau\leq s$, $\varphi_{s,t}(\tau)=0$ if $\tau\geq t$ and $\|\varphi'_{s,t}\|_\infty \leq \frac{C}{t-s}$.

Set

$$\begin{aligned}
\hat{w}_n(x;\lambda) &:= |w_n(x)-w_0(x)| + \frac{|v_n(x)|}{\lambda},\\
w^{n,\lambda}_{s,t}(x) &:= w_0(x)+\varphi_{s,t}(\hat{w}_n(x;\lambda))(w_n(x)-w_0(x)),\\
v^{n,\lambda}_{s,t}(x) &:= v(x_0)+\varphi_{s,t}(\hat{w}_n(x;\lambda))(v_n(x)-v(x_0)).
\end{aligned}$$

Clearly,

$$\big\|w^{n,\lambda}_{s,t}-w_0\big\|_\infty \leq t \quad\text{and}\quad v^{n,\lambda}_{s,t} \rightharpoonup v(x_0) \text{ in } L^p \text{ as } n\to+\infty. \tag{17}$$

Define

$$h_n(x,s,b,A) := f\big(x_0+\varepsilon_n x, u(x_0)+\varepsilon_n s, b, A\big). \tag{18}$$

By the growth conditions there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$,

$$c(|b|^p+|A|)-C \leq h_n(x,s,b,A) \leq C(|b|^p+|A|+1) \tag{19}$$

for some constants $c, C>0$. Consequently there exists a constant $C>0$, possibly depending on x_0, such that

$$-C \leq h_n(x, w_0(x), v(x_0), \nabla w_0(x)) \leq C. \tag{20}$$

Also

$$\begin{aligned}
&\int_B h_n\big(x, w^{n,\lambda}_{s,t}(x), v^{n,\lambda}_{s,t}(x), \nabla w^{n,\lambda}_{s,t}(x)\big)\,dx\\
&= \int_{B\cap\{\hat{w}_n(x;\lambda)\leq s\}} h_n\big(x, w_n, v_n, \nabla w_n\big)\,dx\\
&\quad + \int_{B\cap\{s<\hat{w}_n(x;\lambda)\leq t\}} h_n\big(x, w^{n,\lambda}_{s,t}, v^{n,\lambda}_{s,t}, \nabla w^{n,\lambda}_{s,t}\big)\,dx\\
&\quad + \int_{B\cap\{\hat{w}_n(x;\lambda)>t\}} h_n\big(x, w_0(x), v(x_0), \nabla w_0(x)\big)\,dx := I_1+I_2+I_3.
\end{aligned} \tag{21}$$

By the growth conditions and the definition of h_n we have

$$I_3 \leq C\Big|\Big\{x \in B : \hat{w}_n(x;\lambda) > t\Big\}\Big|. \tag{22}$$

On the other hand, if $s < \hat{w}_n(x;\lambda) < t$ then

$$\nabla w^{n,\lambda}_{s,t}(x) = \nabla u(x_0) + \varphi_{s,t}(\hat{w}_n(x;\lambda))\big(\nabla w_n(x) - \nabla w_0(x)\big) + \big(w_n(x) - w_0(x)\big) \otimes \varphi'_{s,t}\big(\hat{w}_n(x;\lambda)\big)\nabla\big(\hat{w}_n(x;\lambda)\big).$$

By (19) we have

$$I_2 \leq C \int_{B\cap\big\{s<\hat{w}_n(x;\lambda)\leq t\big\}} 1 + |\nabla w_n(x) - \nabla w_0(x)| + |v_n(x) - v(x_0)|^p\, dx + \frac{C}{t-s}\int_{B\cap\{s<\hat{w}_n(x;\lambda)\leq t\}} |w_n(x) - w_0(x)|\, |\nabla(\hat{w}_n(x;\lambda))|\, dx. \tag{23}$$

We remark that for almost every t and λ

$$\lim_{s\to t^-} \int_{\{\hat{w}_n(x;\lambda)<t\}} \big(1 + |\nabla w_n(x) - \nabla u(x_0)| + |v_n(x) - v(x_0)|^p\big)\, dx = 0 \tag{24}$$

and by the coarea formula

$$\begin{aligned} &\lim_{s\to t^-} \frac{1}{t-s}\int_{B\cap\{s<\hat{w}_n(x;\lambda)\leq t\}} |w_n(x) - w_0(x)|\, \big|\nabla\big(\hat{w}_n(x;\lambda)\big)\big|\, dx \\ &\leq \lim_{s\to t^-} \frac{1}{t-s}\int_{B\cap\{s<\hat{w}_n(x;\lambda)\leq t\}} \big(\hat{w}_n(x;\lambda)\big)\, \big|\nabla\big(\hat{w}_n(x;\lambda)\big)\big|\, dx \\ &\leq t\mathcal{H}^{N-1}\big(\{x \in B : \hat{w}_n(x;\lambda) = t\}\big). \end{aligned} \tag{25}$$

Due to the fact that $\{v_n\}$ is a $C^\infty_0(\mathbb{R}^N;\mathbb{R}^m)$ sequence, for every $C > 0$, for every n there exists $\lambda_n \in [1, +\infty)$ such that $\lambda_n \leq \lambda_{n+1}$, $\lambda_n \to +\infty$ as $n \to +\infty$ and

$$\int_B \frac{\big|\nabla|v_n|\big|}{\lambda_n}\, dx \leq C. \tag{26}$$

On the other hand, by (19) and (20)

$$\begin{aligned} &\int_{B\cap\{\hat{w}_n(x;\lambda_n)\leq 1\}} |\nabla(|w_n(x) - w_0(x)|)|\, dx \\ &\leq \int_{B\cap\{\hat{w}_n(x;\lambda_n)\leq 1\}} (|\nabla w_n(x)| + C)\, dx \\ &\leq C\int_B \big(h_n\big(x, u_n(x), v_n(x), \nabla w_n(x)\big) + 1\big)\, dx \leq C \end{aligned}$$

since $\{w_n\}$ and $\{v_n\}$ are convergent.

Thus, by (26)

$$\int_{B\cap\{\hat{w}_n(x;\lambda_n)\leq 1\}} \big|\nabla(\hat{w}_n(x;\lambda_n))\big|\, dx \leq C.$$

Recall that by the previous step $\{v_n\}$ is weakly converging in L^p, thus $\int_B |v_n|^p\, dx \leq C$ and by Hölder inequality also $\int_B |v_n|\, dx \leq C$. Consequently, since $\lambda_n \to +\infty$ it follows that $\int_B \frac{|v_n|}{\lambda_n}\, dx \to 0$ as $n \to +\infty$. Hence, by Lemma 2.6 in [17] there exists

$$t_n \in \left[\left(\|w_n - w_0\|_{L^1} + \frac{\|v_n\|_{L^1}}{\lambda_n}\right)^{1/2}, \left(\|w_n - w_0\|_{L^1} + \frac{\|v_n\|_{L^1}}{\lambda_n}\right)^{1/3}\right]$$

such that (24) and (25) hold (with $t = t_n$), and

$$t_n \mathcal{H}^{N-1}\big(\{x \in B : \hat{w}_n(x;\lambda_n) = t_n\}\big) \le \frac{C}{\ln\big(\|w_n - w_0\|_{L^1} + \frac{\|v_n\|_{L^1}}{\lambda_n}\big)^{-1/6}}. \tag{27}$$

Observe that the right hand side of (27) tends to 0 as $n \to +\infty$ since $\|w_n - w_0\|_{L^1} \to 0$, and $\frac{\|v_n\|_{L^1}}{\lambda_n} \to 0$ as $n \to +\infty$.

According to (24) and (25) we may choose $0 < s_n < t_n$ such that

$$\int_{\{s_n < \hat{w}_n(x;\lambda_n) \le t_n\}} \big(1 + |\nabla w_n(x) - \nabla u(x_0)| + |v_n(x) - v(x_0)|^p\big)\, dx = O(1/n), \tag{28}$$

$$\begin{aligned} &\frac{1}{t_n - s_n} \int_{B \cap \{s_n < \hat{w}_n(x;\lambda_n) \le t_n\}} \big(\hat{w}_n(x;\lambda_n)\big)\, |\nabla(\hat{w}_n(x;\lambda_n))|\, dx \\ &\qquad \le t_n \mathcal{H}^{N-1}\big(\{x \in B : \hat{w}_n(x;\lambda_n) = t_n\}\big) + O\Big(\frac{1}{n}\Big). \end{aligned} \tag{29}$$

Set

$$\widetilde{w}_n(x) := w^{n,\lambda_n}_{s_n,t_n}(x), \qquad \widetilde{v}_n(x) := v^{n,\lambda_n}_{s_n,t_n}(x)$$

thus by (17)

$$\|\widetilde{w}_n - w_0\|_\infty \le t_n \to 0, \qquad \widetilde{v}_n \rightharpoonup v(x_0) \text{ in } L^p \text{ as } n \to \infty.$$

We get the following estimates using Step 3 in the first inequality, (18) in the second inequality, while for the third inequality we exploit the equality defining I_1, I_2 and I_3 in (21), and the estimates obtained for I_3 in (22) and for I_2 in (23), together with (27), (28) and (29):

$$\begin{aligned} g(x_0) &\ge \lim_{n\to\infty} \frac{1}{|B|} \int_B f\big(x_0 + \varepsilon_n x, u(x_0) + \varepsilon_n w_n(x), v_n(x), \nabla w_n(x)\big)\, dx \\ &\ge \liminf_{n\to\infty} \frac{1}{|B|} \int_{B \cap \{\hat{w}_n(x;\lambda_n) \le s\}} h_n\big(x, w_n(x), v_n(x), \nabla w_n(x)\big)\, dx \\ &\ge \liminf_{n\to\infty} \frac{1}{|B|} \int_B h_n\big(x, \widetilde{w}_n(x), \widetilde{v}_n(x), \nabla \widetilde{w}_n(x)\big)\, dx - O\Big(\frac{1}{n}\Big) \\ &\quad - \frac{C}{\ln\big(\|w_n - w_0\|_{L^1(B)} + \frac{\|v_n\|_{L^1(B')}}{\lambda_n}\big)^{-1/6}} - C\,|\{x \in B : \hat{w}_n(x;\lambda_n) > t_n\}| \\ &= \liminf_{n\to\infty} \frac{1}{|B|} \int_B h_n\big(x, \widetilde{w}_n(x), \widetilde{v}_n(x), \nabla \widetilde{w}_n(x)\big)\, dx, \end{aligned}$$

since

$$t_n \ge \Big(\|w_n - w_0\|_{L^1(B)} + \frac{\|v_n\|_{L^1(B)}}{\lambda_n}\Big)^{1/2}$$

and thus

$$\begin{aligned} |\{x \in B : \hat{w}_n(x;\lambda_n) > t_n\}| &\le \frac{1}{t_n}\Big(\|w_n - w_0\|_{L^1(B)} + \frac{\|v_n\|_{L^1(B)}}{\lambda_n}\Big) \\ &\le \Big(\|w_n - w_0\|_{L^1(B)} + \frac{\|v_n\|_{L^1(B)}}{\lambda_n}\Big)^{1/2} \to 0. \end{aligned}$$

The bound of $\{\|\nabla \widetilde{w}_n\|_{L^1}\}$ follows from (19).

Step 5. Fixing x_0 and $u(x_0)$. We now fix in f the value of x and u. Indeed, using hypothesis (H2$_p$) and the fact that $\nabla \tilde{w}_n$ and $|\tilde{v}_n|^p$ have bounded L^1 norm, one gets

$$\begin{aligned} g(x_0) &\geq \limsup_{n\to+\infty} \frac{1}{|B|}\int_B f\big(x_0+\varepsilon_n x, u(x_0)+\varepsilon_n \tilde{w}_n(x), \tilde{v}_n(x), \nabla \tilde{w}_n(x)\big)\,dx \\ &\geq \limsup_{n\to+\infty} \frac{1}{|B|}\int_B f\big(x_0, u(x_0), \tilde{v}_n(x), \nabla \tilde{w}_n(x)\big)\,dx. \end{aligned}$$

Step 6. Slicing. At this point we are in an analogous context to [14] and the desired inequality follows in the same way. It relies on the slicing method in order to modify the sequences $\{\tilde{v}_n\}$ and $\{\tilde{w}_n\}$ obtained in the previous steps, and exploit the convex-quasiconvexity of f. Namely it is possible to find new sequences, denoted by $\{\bar{v}_j\} \subset L^p(B;\mathbb{R}^m)\cap C_0^\infty(\mathbb{R}^N;\mathbb{R}^m)$ and $\{\bar{w}_j\}$ such that

$$\frac{1}{|B|}\int_B \bar{v}_j(z)\,dz = v(x_0) \quad\text{and}\quad \bar{w}_j \in w_0 + W_0^{1,\infty}(B;\mathbb{R}^n).$$

Let, for each $k\in\mathbb{N}$, $L_k = \{z\in B : \operatorname{dist}(z,\partial B) < 1/k\}$ that we call a layer. Consider the layer L_2 and recall that, by construction of $\tilde{v}_j$ and $\tilde{w}_j$ made in Step 4, there exists $c\in\mathbb{R}^+$ such that $\sup_{j\in\mathbb{N}}\|\tilde{v}_j\|_{L^p(B)} + \sup_{j\in\mathbb{N}}\|\nabla\tilde{w}_j\|_{L^1(B)} \leq c$. Then if we divide L_2 in two sublayers, say S_2^1 and S_2^2, we have

$$\forall j\in\mathbb{N} \quad \int_{S_2^1}\big(|\tilde{v}_j(z)|^p + |\nabla\tilde{w}_j(z)|\big)\,dz \leq \frac{c}{2} \quad\text{or}\quad \int_{S_2^2}\big(|\tilde{v}_j(z)|^p + |\nabla\tilde{w}_j(z)|\big)\,dz \leq \frac{c}{2}\,.$$

Thus for some subsequences of $\tilde{v}_j$ and $\tilde{w}_j$, say $\tilde{v}_{j_2}$ and $\tilde{w}_{j_2}$, and one of the sublayers S_2^1 or S_2^2, say S_2, we have

$$\forall j_2\in\mathbb{N} \quad \int_{S_2}\big(|\tilde{v}_{j_2}(z)|^p + |\nabla\tilde{w}_{j_2}(z)|\big)\,dz \leq \frac{c}{2}\,.$$

Note that for some $0\leq\alpha_2<\beta_2\leq 1/2$ we can write

$$S_2 = \{z\in B : \alpha_2 < \operatorname{dist}(z,\partial B) < \beta_2\}.$$

Define then a cutoff function $\eta_2 : B\to[0,1]$ such that $\eta_2 = 0$ in $\partial B\cup\{z\in B : \operatorname{dist}(z,\partial B)\leq\alpha_2\}$, and $\eta_2 = 1$ in $\{z\in B : \operatorname{dist}(z,\partial B)\geq\beta_2\}$ and $\|\nabla\eta_2\|\leq\frac{c}{\beta_2-\alpha_2}$.

Since

$$\lim_{j_2\to+\infty}\left|v(x_0) - \frac{1}{|B|}\int_B \eta_2(z)\tilde{v}_{j_2}(z)\,dz\right| = |v(x_0)|\left|1-\frac{1}{|B|}\int_B\eta_2(z)\,dz\right|,$$

for $j(2)\in\{j_2\}$ sufficiently large we have

$$\frac{\left|v(x_0)-\frac{1}{|B|}\int_B\eta_2(z)\tilde{v}_{j(2)}(z)\,dz\right|}{\left|1-\frac{1}{|B|}\int_B\eta_2(z)\,dz\right|} \leq |v(x_0)|+1$$

and

$$\frac{1}{|S_2|}\int_{S_2}\left|\tilde{w}_{j(2)}(z)-w_0(z)\right|\,dz \leq \frac{1}{2}\,.$$

Repeating the procedure in the layer L_3 (now working with three sublayers) and so on for the next layers, we get $j(k) \in \mathbb{N}$ increasing with k, $S_k := \{z \in B : \alpha_k < \operatorname{dist}(z, \partial B) < \beta_k\}$ layer of diameter $\frac{1}{k^2}$, and η_k cutoff function on B such that $\eta_k = 0$ in $\partial B \cup \{z \in B : \operatorname{dist}(z, \partial B) \leq \alpha_k\}$, and $\eta_k = 1$ in $\{z \in B : \operatorname{dist}(z, \partial B) \geq \beta_k\}$

$$\frac{\left|v(x_0) - \frac{1}{|B|}\int_B \eta_k(z)\tilde{v}_{j(k)}(z)\,dz\right|}{\left|1 - \frac{1}{|B|}\int_B \eta_k(z)\,dz\right|} \leq |v(x_0)| + 1, \tag{30}$$

$$\int_{S_k} \left(|\tilde{v}_{j(k)}(z)|^p + \left|\nabla \tilde{w}_{j(k)}(z)\right|\right) dz \leq \frac{c}{k} \tag{31}$$

and

$$\frac{1}{|S_k|}\int_{S_k} \left|\tilde{w}_{j(k)}(z) - w_0(z)\right|\,dz \leq \frac{1}{k}\,. \tag{32}$$

Then, defining

$$\bar{v}_k(z) := (1 - \eta_k(z))\,\frac{v(x_0) - \frac{1}{|B|}\int_B \eta_k(z)\tilde{v}_{j(k)}(z)\,dz}{1 - \frac{1}{|B|}\int_B \eta_k(z)\,dz} + \eta_k(z)\tilde{v}_{j(k)}(z) \tag{33}$$

and

$$\bar{w}_k(z) := (1 - \eta_k(z))w_0(z) + \eta_k(z)\tilde{w}_{j(k)}(z) \tag{34}$$

we have $\bar{v}_k \in L^p(B;\mathbb{R}^m) \cap C_0^\infty(B;\mathbb{R}^m)$ with

$$\frac{1}{|B|}\int_B \bar{v}_k(z)\,dz = v(x_0)$$

and $\bar{w}_k \in w_0 + W_0^{1,\infty}(B;\mathbb{R}^n)$. Therefore, since

$$\begin{aligned}
&\int_B f\big(x_0, u(x_0), \bar{v}_k(z), \nabla \bar{w}_k(z)\big)\,dz \\
&= \int_{\{z\in B:\operatorname{dist}(z,\partial B)\geq \beta_k\}} f\big(x_0, u(x_0), \tilde{v}_{j(k)}(z), \nabla \tilde{w}_{j(k)}(z)\big)\,dz \\
&\quad + \int_{S_k} f\big(x_0, u(x_0), \bar{v}_k(z), \nabla \bar{w}_k(z)\big)\,dz \\
&\quad + \int_{\{z\in B:\operatorname{dist}(z,\partial B)\leq \alpha_k\}} f\big(x_0, u(x_0), \bar{v}_k(z), \nabla \bar{w}_k(z)\big)\,dz \\
&\leq \int_B f\big(x_0, u(x_0), \tilde{v}_{j(k)}(z), \nabla \tilde{w}_{j(k)}(z)\big)\,dz + \int_{S_k} f\big(x_0, u(x_0), \bar{v}_k(z), \nabla \bar{w}_k(z)\big)\,dz \\
&\quad + \int_{\{z\in B:\operatorname{dist}(z,\partial B)\leq \alpha_k\}} f\big(x_0, u(x_0), \bar{v}_k(z), \nabla \bar{w}_k(z)\big)\,dz,
\end{aligned}$$

from Step 5, using the convex-quasiconvexity of f, assumption (H1$_p$), the definition of η_k, (30), (31), (32), (33) and (34),

$$
\begin{aligned}
g(x_0) &\geq \limsup_{k\to+\infty} \frac{1}{|B|} \int_B f\big(x_0, u(x_0), \tilde{v}_{j(k)}(z), \nabla \tilde{w}_{j(k)}(z)\big)\,dz \\
&\geq \limsup_{k\to+\infty} \frac{1}{|B|} \bigg[\int_B f\big(x_0, u(x_0), \bar{v}_k(z), \nabla \bar{w}_k(z)\big)\,dz \\
&\quad - \int_{S_k} f\big(x_0, u(x_0), \bar{v}_k(z), \nabla \bar{w}_k(z)\big)\,dz \\
&\quad - \int_{\{z\in B:\operatorname{dist}(z,\partial B)\leq \alpha_k\}} f\big(x_0, u(x_0), \bar{v}_k(z), \nabla \bar{w}_k(z)\big)\,dz \bigg] \\
&\geq \limsup_{k\to+\infty} \frac{1}{|B|} \bigg[|B| f\big(x_0, u(x_0), v(x_0), \nabla u(x_0)\big) \\
&\quad - \int_{S_k} C(1 + |\tilde{v}_k|^p + |v(x_0)|^p + |\nabla \bar{w}_k(z)|)\,dz \\
&\quad - \int_{\{z\in B:\operatorname{dist}(z,\partial B)\leq \alpha_k\}} C(1 + |v(x_0)|^p + |\nabla u(x_0)|)\,dz \bigg] \\
&= f\big(x_0, u(x_0), v(x_0), \nabla u(x_0)\big).
\end{aligned}
$$

In this way we achieved the desired inequality. ∎

4. Relaxation in $W^{1,1} \times L^\infty$. This section is devoted to characterizing the relaxed functional $\overline{J}_\infty$ introduced in (3). Indeed we prove the following relaxation result

THEOREM 12. *Let Ω be a bounded open set of $\mathbb{R}^N$, and let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0, +\infty)$ be a continuous function. If f and CQf satisfy hypotheses* (H1$_\infty$) *and* (H2$_\infty$) *then*

$$\overline{J}_\infty(u, v) = \int_\Omega CQf\big(x, u(x), v(x), \nabla u(x)\big)\,dx,$$

for every $(u, v) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^\infty(\Omega; \mathbb{R}^m)$.

REMARK 13.

1) We recall that if the hypotheses (H1$_\infty$) and (H2$_\infty$) are replaced by (8)–(10), Propositions 6 and 8 guarantee the validity of Theorem 12 under the assumption that only f satisfies (8)–(10).

2) We also observe that Theorem 12 can be proven also imposing (H1$_\infty$) and (H2$_\infty$) only on the function f but with the further requirement that f satisfies (5).

To see this it is enough to argue as in [5] in analogy with the proof of formula (6.1) therein, namely one can reproduce the proof of Theorem 12 until the introduction of the Yosida transform in page 202. Then, recalling that standard arguments (as those exploited in [10]) allow to replace test functions in (7), by periodic ones, with 0 average. Thus in order to estimate as in (39) one replaces the sequence u_n by $w_n := u * \varrho_n + \varphi_n$ (as in formula (6.11) in [5]), where φ is a periodic function almost realizing the infimum in (39) (say with an error $\eta > 0$, where clearly φ depends on η), and $\varphi_n(x) := r_n \varphi(\frac{x}{r_n})$, with

$r_n \to 0$. Clearly also $w_n \to u$ in $L^1(\Omega;\mathbb{R}^n)$. Then the integral in (39) can be estimated in an analogous way to Theorem 12.

3) We also stress that if f satisfies ($\mathrm{H1}_p$) and ($\mathrm{H2}_p$) then clearly $\overline{J}_p(u,v) \le \overline{J}_\infty(u,v)$ for every $(u,v) \in \mathrm{BV}(\Omega;\mathbb{R}^n)\times L^\infty(\Omega;\mathbb{R}^m)$, where the extension to $L^1(\Omega;\mathbb{R}^n)\times L^\infty(\Omega;\mathbb{R}^m)$ of $\overline{J}_p$ and $\overline{J}_\infty$ can be defined as in the first part of Remark 9.

Proof of Theorem 12. The assertion will be achieved by double inequality. Clearly the lower bound can be proven as for the case $W^{1,1}\times L^p$, with a proof easier than that of Theorem 11, since it is not necessary to 'truncate' the $\{v_n\}$ which are already bounded in L^∞. For what concerns the upper bound, we divide the proof into several steps after having observed that, by virtue of Lemma 8, there is no loss of generality in assuming f already convex-quasiconvex.

Step 1. Localization. In order to provide an upper bound for $\overline{J}_\infty$ we start by localizing our functional. The following procedure is entirely similar to [4, Theorem 4.3]. We define for every open set $A \subset \Omega$ and for any $(u,v) \in \mathrm{BV}(\Omega;\mathbb{R}^n)\times L^\infty(\Omega;\mathbb{R}^m)$ (see Remark 9 for the definition of BV spaces)

$$\overline{F}_\infty(u,v,A) := \inf\bigl\{\liminf_n F(u_n,v_n,A) : u_n \to u \text{ in } L^1(A;\mathbb{R}^n),\ u_n \in L^1(A;\mathbb{R}^n),\\ v_n \overset{*}{\rightharpoonup} v \text{ in } L^\infty(A;\mathbb{R}^m),\ v_n \in L^\infty(A;\mathbb{R}^m)\bigr\}$$

where

$$F(u,v,A) = \begin{cases} \int_A f\bigl(x,u(x),v(x),\nabla u(x)\bigr)\,dx & \text{if } (u,v) \in W^{1,1}(A;\mathbb{R}^n)\times L^\infty(A;\mathbb{R}^m),\\ +\infty & \text{if } (u,v) \in (L^1(A;\mathbb{R}^n)\setminus W^{1,1}(A;\mathbb{R}^n))\times L^\infty(A;\mathbb{R}^m). \end{cases}$$

We start by remarking that ($\mathrm{H1}_\infty$) implies that for every $u \in \mathrm{BV}(\Omega;\mathbb{R}^n)$ and for every $v \in L^\infty(\Omega;\mathbb{R}^m)$ such that $\|v\|_{L^\infty} \le M$, there exists a constant C_M such that $\overline{F}_\infty(u,v,A) \le C_M(|A| + |Du|(A))$. Moreover, one has

1) $\overline{F}_\infty$ is local, i.e. $\overline{F}_\infty(u,v,A) = \overline{F}_\infty(u',v',A)$, for every $A\subset\Omega$ open, $(u,v),(u',v') \in L^1(A;\mathbb{R}^n)\times L^\infty(A;\mathbb{R}^m)$, such that $(u,v) = (u',v')$ a.e. in A.
2) $\overline{F}_\infty$ is sequentially lower semi-continuous, i.e. $\overline{F}_\infty(u,v,A) \le \liminf \overline{F}_\infty(u_n,v_n,A)$ for all $A\subset\Omega$ open, for all $u_n \to u$ in $L^1(A;\mathbb{R}^n)$ and $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty(A;\mathbb{R}^m)$;
3) $\overline{F}_\infty(u,v,\cdot)$ is the restriction to $\mathcal{A}(\Omega) := \{A\subset\Omega : A \text{ is open}\}$ of a Borel measure in $\mathcal{B}(\Omega)$ (the Borelians of Ω).

Condition 1) follows from the fact that the adopted convergence does not see sets of null Lebesgue measure. Condition 2) follows by a diagonalization argument, entirely similar to the proof of (ii) in [14]. Condition 3) follows applying De Giorgi–Letta criterion (cf. [12]) and indeed proving that for any fixed $(u,v) \in \mathrm{BV}(\Omega;\mathbb{R}^n)\times L^\infty(\Omega;\mathbb{R}^m)$,

$$\overline{F}_\infty(u,v,A) \le \overline{F}_\infty(u,v,B) + \overline{F}_\infty(u,v,A\setminus\overline{C})\ \forall A,B,C \in \mathcal{A}(\Omega),\ A \Subset B \Subset C.$$

We omit the details, since they are very similar to the proof of Theorem 4.3 in [4]. The only difference consists of the fact that one has to deal with both u's and v's and exploit the growth condition ($\mathrm{H1}_\infty$).

Step 2. Blow-up. Since $\overline{J}_\infty(u,v) = \overline{F}_\infty(u,v,\Omega)$ and $\overline{F}_\infty(u,v,\cdot)$ is the restriction of a Radon measure on the open subsets of Ω (i.e. $\mathcal{A}(\Omega)$) absolutely continuous with respect to $|Du| + \mathcal{L}^N$, it will be enough to prove the inequality

$$\frac{d\overline{F}_\infty(u,v,\cdot)}{d\mathcal{L}^N}(x) \leq f\big(x, u(x), v(x), \nabla u(x)\big),\ \mathcal{L}^N\text{-a.e. } x \in \Omega.$$

The proof of this inequality follows closely [18], [4] and [5].

Assume first that $(u,v) \in (W^{1,1}(\Omega;\mathbb{R}^n) \cap L^\infty(\Omega;\mathbb{R}^n)) \times L^\infty(\Omega;\mathbb{R}^m)$. Fix a point $x_0 \in \Omega$ such that

$$\frac{d\overline{F}_\infty(u,v,\cdot)}{d\mathcal{L}^N}(x_0) \tag{35}$$

exists and is finite, which is also a Lebesgue point of u, v and ∇u and a point of approximate differentiability for u. Clearly $\mathcal{L}^N$-a.e. $x_0 \in \Omega$ satisfy all the above requirements.

As in [18] (see formula (5.6) therein) we may also assume that

$$\lim_{\varepsilon \to 0} \frac{1}{|Q(x_0,\varepsilon)|} \int_{Q(x_0,\varepsilon)} |u(x) - u(x_0)|(1 + |\nabla u(x)|)\, dx = 0, \tag{36}$$

$$\lim_{\varepsilon \to 0} \frac{1}{|Q(x_0,\varepsilon)|} \int_{Q(x_0,\varepsilon)} |v(x) - v(x_0)|\, |\nabla u(x)|\, dx = 0, \tag{37}$$

(where we used Theorem 10 since $v \in L^1_{\text{loc}}(\Omega;\mathbb{R}^m)$ with respect to the measure $|\nabla u|\mathcal{L}^N$). Choose a sequence of numbers $\varepsilon \in (0, \operatorname{dist}(x_0, \partial\Omega))$. Then, clearly for any sequences $\{u_n\}$, $u_n \to u$ in L^1 , $\{v_n\}$, $v_n \overset{*}{\rightharpoonup} v$ in L^∞,

$$\begin{aligned}\frac{d\overline{F}_\infty(u,v,\cdot)}{d\mathcal{L}^N}(x_0) &= \lim_{\varepsilon \to 0^+} \frac{\overline{F}_\infty(u,v,B_\varepsilon(x_0))}{|B_\varepsilon(x_0)|} \\ &\leq \liminf_{\varepsilon \to 0^+} \liminf_{n \to +\infty} \frac{1}{|B_\varepsilon(x_0)|} \int_{B_\varepsilon(x_0)} f\big(x, u_n(x), v_n(x), \nabla u_n(x)\big)\, dx. \end{aligned} \tag{38}$$

By virtue of Proposition 2.2 in [2] we can replace the ball $B_\varepsilon(x_0)$ in (38) by a cube of side length ε, and in fact from now on we consider such cubes.

As in Proposition 4.6 of [4] (see also [18] and [15]), we consider the Yosida transforms of f, defined as

$$f_\lambda(x,u,v,\xi) := \sup_{(x',u') \in \Omega \times \mathbb{R}^n} \big\{ f(x',u',v,\xi) - \lambda[|x - x'| + |u - u'|](1 + |\xi| + |v|) \big\}$$

for every $\lambda > 0$. Then

(i) $f_\lambda(x,u,v,\xi) \geq f(x,u,v,\xi)$ and $f_\lambda(x,u,v,\xi)$ decreases to $f(x,u,v,\xi)$ as $\lambda \to +\infty$.
(ii) $f_\lambda(x,u,v,\xi) \geq f_\eta(x,u,v,\xi)$ if $\lambda \leq \eta$ for every $(x,u,v,\xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N}$.
(iii) $|f_\lambda(x,u,v,\xi) - f_\lambda(x',u',v,\xi)| \leq \lambda(|x - x'| + |u - u'|)(1 + |\xi| + |v|)$ for every (x,u,v,ξ), $(x',u',v,\xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N}$.
(iv) The approximation is uniform on compact sets. Precisely, let K be a compact subset of $\Omega \times \mathbb{R}^n$ and let $\delta > 0$. There exists $\lambda > 0$ such that

$$f(x,u,v,\xi) \leq f_\lambda(x,u,v,\xi) \leq f(x,u,v,\xi) + \delta(1 + |v| + |\xi|) \quad \text{for every } (x,u,v,\xi) \in K \times \mathbb{R}^m \times \mathbb{R}^{n \times N}.$$

Let x_0 be such that (35) and (36) hold, let $\{\varrho_n\}$ be a sequence of standard symmetric mollifiers and set $u_n := u * \varrho_n$, $v_n := v$. Then $u_n \to u$ in $L^1(Q(x_0,\varepsilon);\mathbb{R}^n)$, $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty(Q(x_0,\varepsilon);\mathbb{R}^m)$. Fix $\delta > 0$ and let $K := \overline{B}(x_0, \frac{\operatorname{dist}(x_0,\partial\Omega)}{2}) \times \overline{B}(0, \|u\|_\infty)$. By (i)–(iv),

$$\begin{aligned}
&f\big(x, u_n(x), v(x), \nabla u_n(x)\big) \le f_\lambda(x, u_n(x), v(x), \nabla u_n(x)) \\
&\le f_\lambda(x_0, u(x_0), v(x), \nabla u_n(x)) + \lambda(|x - x_0| + |u_n(x) - u(x_0)|)(1 + |v| + |\nabla u_n(x)|) \\
&\le f\big(x_0, u(x_0), v(x), \nabla u_n(x)\big) + \delta(1 + |\nabla u_n(x)| + |v(x)|) \\
&\qquad + \lambda\big(|x - x_0| + |u_n(x) - u(x_0)|\big)\big(1 + |\nabla u_n(x)| + |v(x)|\big).
\end{aligned}$$

Since $\nabla u_n(x) = (\nabla u * \varrho_n)(x)$,

$$\begin{aligned}
&\overline{F}_\infty(u, v, Q(x_0,\varepsilon)) \le \liminf_{n\to+\infty} \int_{Q(x_0,\varepsilon)} f\big(x, u_n(x), v(x), \nabla u_n(x)\big)\, dx \\
&\le \liminf_{n\to+\infty} \int_{Q(x_0,\varepsilon)} f\big(x_0, u(x_0), v(x), \nabla u_n(x)\big)\, dx \\
&\qquad + \limsup_{n\to+\infty} \int_{Q(x_0,\varepsilon)} \delta\big(1 + |\nabla u_n(x)| + |v(x)|\big) \\
&\qquad\qquad + \lambda\big(|x - x_0| + |u_n(x) - u(x_0)|\big)\big(1 + |\nabla u_n(x)| + |v(x)|\big)\, dx \\
&\le \liminf_{n\to+\infty} \int_{Q(x_0,\varepsilon)} f\left(x_0, u(x_0), v(x_0), \nabla u(x_0)\right)\, dx \\
&\qquad + \limsup_{n\to+\infty} \int_{Q(x_0,\varepsilon)} \beta\big(1 + |\nabla u(x_0)| + |(\nabla u * \varrho_n)(x)|\big)|v(x) - v(x_0)|\, dx \\
&\qquad + \limsup_{n\to+\infty} \int_{Q(x_0,\varepsilon)} \beta\, |(\nabla u * \varrho_n)(x) - \nabla u(x_0)|\, dx \\
&\qquad + \limsup_{n\to+\infty} \int_{Q(x_0,\varepsilon)} \delta\big(1 + |\nabla u_n(x)| + |v(x)|\big) \\
&\qquad\qquad + \lambda\big(|x - x_0| + |u_n(x) - u(x_0)|\big)\big(1 + |\nabla u_n(x)| + |v(x)|\big)\, dx
\end{aligned} \tag{39}$$

(where the constant β, depending on $\|v\|_{L^\infty}$, is the constant appearing in (5)).

Since $\nabla u * \varrho_n \to \nabla u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times N})$, we obtain

$$\limsup_{n\to+\infty} \int_{Q(x_0,\varepsilon)} \beta\, |(\nabla u * \varrho_n)(x) - \nabla u(x_0)|\, dx \le \beta \int_{Q(x_0,\varepsilon)} |\nabla u(x) - \nabla u(x_0)|\, dx. \tag{40}$$

Passing to the limit on the right hand side of (39), exploiting (40), we get

$$\begin{aligned}
&\overline{F}_\infty\big(u, v, Q(x_0,\varepsilon)\big) \le |Q(x_0,\varepsilon)|\, f\big(x_0, u(x_0), v(x_0), \nabla u(x_0)\big) \\
&\quad + \beta(1 + |\nabla u(x_0)|) \int_{Q(x_0,\varepsilon)} |v(x) - v(x_0)|\, dx \\
&\quad + \limsup_{n\to+\infty} \beta \int_{Q(x_0,\varepsilon)} |\nabla u * \varrho_n|\, |v(x) - v(x_0)|\, dx + \beta \int_{Q(x_0,\varepsilon)} |\nabla u(x) - \nabla u(x_0)|\, dx \\
&\quad + (\lambda\varepsilon + \delta)\big[(1 + C)|Q(x_0,\varepsilon)|\big] + \lambda \limsup_{n\to+\infty} \int_{Q(x_0,\varepsilon)} |u_n - u(x_0)|(1 + C + |\nabla u_n|)\, dx.
\end{aligned}$$

Recalling that x_0 is a Lebesgue point for v, ∇u and that (35) holds, we have

$$\limsup_{\varepsilon\to 0^+} \frac{1}{|Q(x_0,\varepsilon)|}\beta\big(1+|\nabla u(x_0)|\big)\int_{Q(x_0,\varepsilon)} |v(x)-v(x_0)|\,dx = 0,$$
$$\limsup_{\varepsilon\to 0^+} \frac{1}{|Q(x_0,\varepsilon)|}\int_{Q(x_0,\varepsilon)} |\nabla u(x)-\nabla u(x_0)|\,dx = 0.$$

Moreover, by virtue of (36) and arguing as in the estimate of formula (5.11) of [18] we can conclude that

$$\limsup_{\varepsilon\to 0^+} \frac{\lambda}{|Q(x_0,\varepsilon)|}\limsup_{n\to+\infty}\int_{Q(x_0,\varepsilon)} |u_n-u(x_0)|\big(1+C+|\nabla u_n|\big)\,dx = 0.$$

Then we can exploit (37) and argue again as for (5.11) in [18] in order to evaluate

$$\limsup_{\varepsilon\to 0^+} \frac{\beta}{|Q(x_0,\varepsilon)|}\limsup_{n\to+\infty}\int_{Q(x_0,\varepsilon)} |\nabla u * \varrho_n|\,|v(x)-v(x_0)|\,dx.$$

We will apply [18, Lemma 2.5] and the dominated convergence theorem with respect to the measure $|\nabla u|\,dx$, obtaining

$$\limsup_{n\to+\infty}\int_{Q(x_0,\varepsilon)} |v(x)-v(x_0)|\,|\nabla u_n(x)|\,dx$$
$$\le \limsup_{n\to+\infty}\int_{Q(x_0,\varepsilon+1/n)} \big(|v-v(x_0)| * \varrho_n\big)|\nabla u(x)|\,dx \le \int_{\overline{Q(x_0,\varepsilon)}} |v(x)-v(x_0)|\,|\nabla u(x)|\,dx.$$

One obtains from (37) that

$$\limsup_{\varepsilon\to 0^+}\limsup_{n\to+\infty} \frac{1}{|Q(x_0,\varepsilon)|}\int_{Q(x_0,\varepsilon)} |v(x)-v(x_0)|\,|\nabla u_n(x)|\,dx = 0.$$

Consequently,

$$g(x_0) = \frac{d\overline{F}_\infty(u,v)(x_0)}{d\mathcal{L}^N} \le f\big(x_0,u(x_0),v(x_0),\nabla u(x_0)\big) + (1+C)\delta$$

Finally, sending δ to 0 we conclude the proof when $(u,v)\in (W^{1,1}(\Omega;\mathbb{R}^n)\cap L^\infty(\Omega;\mathbb{R}^n))\times L^\infty(\Omega;\mathbb{R}^m)$.

Step 3. Passage from $W^{1,1}(\Omega;\mathbb{R}^n)\cap L^\infty(\Omega;\mathbb{R}^n)$ *to* $W^{1,1}(\Omega;\mathbb{R}^n)$. To conclude the proof, we can argue as in [18, Theorem 2.16, Step 4], in turn inspired by [4], introducing the following approximation.

Let $\phi_n \in C^1_0(\mathbb{R}^n;\mathbb{R}^n)$ be such that $\phi_n(y)=y$ if $y\in B_n(0)$, $\|\nabla\phi_n\|_{L^\infty}\le 1$, where $B_n(0)$ is the ball centered at 0 with radius n. By [3, Theorem 3.96] $\phi_n(u)\in W^{1,1}(\Omega;\mathbb{R}^n)\cap L^\infty(\Omega;\mathbb{R}^n)$ for every $n\in\mathbb{N}$. Since $\phi_n(u)\to u$ in L^1, by the lower semicontinuity of $\overline{J}_\infty$ we get

$$\overline{J}_\infty(u,v) \le \liminf_{n\to+\infty}\int_\Omega f\big(x,\phi_n(u),v,\nabla\phi_n(u)\big)\,dx.$$

Arguing in analogy with [4, Theorem 4.9] one can prove that

$$\limsup_{n\to+\infty}\int_\Omega f\big(x,\phi_n(u),v,\nabla\phi_n(u)\big)\,dx \le \int_\Omega f(x,u,v,\nabla u)\,dx,$$

which concludes the proof. ■

5. Relaxation in $W^{1,1} \times L^p$. This section is devoted to the proof of the following theorem. It relies on Theorem 12 and on some approximation results (see [4]).

THEOREM 14. *Let $1 < p < +\infty$. Let Ω be a bounded open set of $\mathbb{R}^N$, and let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0, +\infty)$ be a continuous function. If f satisfies* (H1$_p$) *and* (H2$_p$), *then*

$$\overline{J}_p(u, v) = \int_\Omega CQf\big(x, u(x), v(x), \nabla u(x)\big)\, dx,$$

for every $(u, v) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$.

Proof. The lower bound follows from Theorem 11. For what concerns the upper bound, without loss of generality, by virtue of Lemma 8 and Proposition 5 we may assume that f is convex-quasiconvex.

Observe first that since f fulfils (H1$_p$) and (H2$_p$), then it satisfies (H1$_\infty$) and (H2$_\infty$) in the strong form (8)–(10). Consequently, up to the extension to $L^1(\Omega; \mathbb{R}^N) \times L^\infty(\Omega; \mathbb{R}^m)$ of $\overline{J_p}$ and $\overline{J}_\infty$ proposed in Remark 9,

$$\overline{J}_p(u, v, \Omega) \le \overline{J}_\infty(u, v, \Omega) \tag{41}$$

for every $(u, v) \in \mathrm{BV}(\Omega; \mathbb{R}^n) \times L^\infty(\Omega; \mathbb{R}^m)$.

For every positive real number λ, let $\tau_\lambda : [0, +\infty) \to [0, +\infty)$ be defined as

$$\tau_\lambda(t) = \begin{cases} t & \text{if } 0 \le t \le \lambda, \\ 0 & \text{if } t \ge \lambda. \end{cases}$$

For every $v \in L^p(\Omega; \mathbb{R}^m)$, define $v_\lambda := \tau_\lambda(|v|)v$. Clearly $\int_\Omega |v_\lambda|^p\, dx \le \int_\Omega |v|^p\, dx$ and $v_\lambda \to v$ in $L^p(\Omega; \mathbb{R}^m)$, as $\lambda \to +\infty$. By the lower semicontinuity of $\overline{J}_p$, (41), and Theorem 12, for every sequence $\{\lambda\}$ such that $\lambda \to +\infty$ we have

$$\overline{J}_p(u, v) \le \liminf_{\lambda\to\infty} \overline{J}_p(u, v_\lambda) = \liminf_{\lambda\to+\infty} \int_\Omega f\big(x, u(x), v_\lambda(x), \nabla u(x)\big)\, dx.$$

Lebesgue's dominated convergence theorem entails that

$$\overline{J}_p(u, v, \Omega) = \int_\Omega f\big(x, u(x), v(x), \nabla u(x)\big)\, dx,$$

for every $(u, v) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$, and this concludes the proof. ■

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ERRATUM TO "LOWER SEMICONTINUOUS ENVELOPES IN $W^{1,1} \times L^p$"

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In this note, we correct a mistake present in [2]. The theorems stated remain valid but the proof of Theorem 11 should be slightly modified. Namely, in Step 4, the sequence $v_{s,t}^{n,\lambda}$ has to be defined in a different way to obtain the weak convergence in L^p stated in condition (17).

In detail, in the proof of Theorem 11 define

$$v_{s,t}^{n,\lambda}(x) := \tau_{L_n}(v_n)(x) + \varphi_{s,t}\left(\hat{w}_n(x;\lambda)\right)\left(v_n(x) - \tau_{L_n}(v_n(x))\right),$$

thus ensuring the second convergence in formula (17). Replace also formula (20) by

$$-C + \frac{1}{C}|\tau_{L_n}(v_n)(x)|^p \leq h_n(x, w_0(x), \tau_{L_n}(v_n(x)), \nabla w_0(x)) \leq C(1 + |\tau_{L_n}(v_n(x))|^p).$$

Here the sequence $\tau_{L_n}(v_n)$ is obtained from v_n according to the Decomposition Lemma below, whose proof can be found in [1, Lemma 8.13].

LEMMA 1. *Let $1 < p < +\infty$, and let $\{v_n\}$ be a bounded sequence in $L^p(\Omega;\mathbb{R}^d)$. For $L > 0$ consider the truncation $\tau_L : \mathbb{R}^d \to \mathbb{R}^d$ given by*

$$\tau_L(z) := \begin{cases} z & \text{if } |z| \leq L, \\ L\dfrac{z}{|z|} & \text{if } |z| > L. \end{cases}$$

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Then there exists a subsequence of $\{v_n\}$ *(not relabeled) and an increasing sequence* $\{L_n\}$*, with* $L_n \to +\infty$*, such that the truncated sequence* $\{\tau_{L_n} \circ v_n\}$ *is p-equi-integrable and* $\|\tau_{L_n} \circ v_n - v_n\|_{L^q(\Omega)} \to 0$ *for all* $1 \leq q < p$*.*

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