

HOW TO CATEGORIFY ONE-HALF OF QUANTUM $\mathfrak{gl}(1|2)$

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Abstract. We describe a collection of differential graded rings that categorify weight spaces of the positive half of the quantized universal enveloping algebra of the Lie superalgebra $\mathfrak{gl}(1|2)$.

1. Lie superalgebra $\mathfrak{gl}(1|2)$, the positive half and its quantum version. The Lie superalgebra $\mathfrak{gl}(n|m)$ is defined by partitioning $(n+m) \times (n+m)$ matrices into 4 blocks: diagonal blocks of size $n \times n$ and $m \times m$ and off-diagonal blocks of size $n \times m$ and $m \times n$. Matrices with nonzero entries only in the diagonal, respectively off-diagonal blocks, are called *even*, respectively *odd*. Elementary matrix E_{ij} is even if $i, j \leq n$ or $i, j \geq n+1$ and odd otherwise.

The superbracket $[A, B]$ of matrices is defined as the usual bracket $AB - BA$ if at least one of A and B is even and as anticommutator $AB + BA$ if both A and B are odd. With these conventions, the superbracket satisfies the super analogues [2] of the antisymmetry and the Jacobi identity:

$$[a, b] = -(-1)^{p(a)p(b)}[b, a], \tag{1}$$

$$[a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)}[b, [a, c]], \tag{2}$$

where $p(a) = 0$ (resp. 1) if a is even (resp. odd).

The universal enveloping algebra UL of a Lie superalgebra L is defined in the same way as for Lie algebras. If $L = L_0 \oplus L_1$ is the decomposition of L into the sum of its even and odd parts, UL can be identified, as a vector space, with $S(L_0) \otimes \Lambda(L_1)$, the tensor product of the symmetric algebra of L_0 and exterior algebra of L_1 , once bases of L_0 and L_1 are fixed. It is a Hopf algebra in the category of super vector spaces, with $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in L$.

The decomposition of $\mathfrak{gl}(n)$ into the direct sum of strictly upper-triangular, diagonal, and strictly lower-triangular matrices generalizes to $\mathfrak{gl}(n|m)$. We denote by $\mathfrak{gl}(n|m)^+$ the

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Lie superalgebra of strictly upper-triangular $(n|m)$ -matrices and by $U^+(n|m)$ its universal enveloping algebra (say, over \mathbb{Q}).

Note that $U^+(1|1) \cong \Lambda(E_{12})$, the exterior algebra on one generator E_{12} . In particular, $U^+(1|1)$ is two-dimensional, with basis $\{1, E_{12}\}$, and $E_{12}^2 = 0$.

The universal enveloping algebra $U^+(1|2)$ (we also denote it by U^+) has two generators $E_1 := E_{12}$ and $E_2 := E_{23}$, first odd, second even, and defining relations

$$\begin{aligned} E_1^2 &= 0, \\ 2E_2E_1E_2 &= E_1E_2^2 + E_2^2E_1. \end{aligned}$$

The Lie superalgebra $\mathfrak{gl}(1|2)^+$ has a basis $\{E_2, E_1, [E_1, E_2]\}$. The first of these generators is even, the other two are odd, so that U^+ has a basis $\{E_2^m E_1^\epsilon [E_1, E_2]^{\epsilon'}\}$ where $m \in \mathbb{Z}_+$ and $\epsilon, \epsilon' \in \{0, 1\}$. The set $\{E_1^\epsilon E_2^m E_1^{\epsilon'}\}$ is also a basis, with m, ϵ, ϵ' in the same range as above, except that $m \neq 0$ if $\epsilon = \epsilon' = 1$.

The quantum deformation $U_q^+ = U_q^+(1|2)$ of U^+ is a $\mathbb{Q}(q)$ -algebra with the same generators as U^+ and defining relations

$$\begin{aligned} E_1^2 &= 0, & (3) \\ [2]E_2E_1E_2 &= E_1E_2^2 + E_2^2E_1. & (4) \end{aligned}$$

The deformation simply transforms coefficient 2 in the second relation to quantum $[2] = q + q^{-1}$. The latter relation can be rewritten

$$E_2E_1E_2 = E_1E_2^{(2)} + E_2^{(2)}E_1, \quad E_2^{(2)} := \frac{E_2^2}{[2]}, \tag{5}$$

where $E^{(m)} := \frac{E^m}{[m]!}$ denotes m -th quantum divided power of E .

Equipped with comultiplication

$$\Delta(E_i) = E_i \otimes 1 + 1 \otimes E_i, \quad i = 1, 2,$$

U_q^+ becomes a twisted bialgebra, in the sense of [8, Chapter 1], in the category of super vector spaces.

Define the integral form $U_{\mathbb{Z}}^+ = U_{\mathbb{Z}}^+(1|2)$ to be the $\mathbb{Z}[q, q^{-1}]$ -subalgebra of U_q^+ generated by E_1 and divided powers $E_2^{(m)}$ over all $m \geq 0$. As a free $\mathbb{Z}[q, q^{-1}]$ -module, $U_{\mathbb{Z}}^+$ has a basis $\{E_1^\epsilon E_2^{(m)} E_1^{\epsilon'}\}$, with $m \in \mathbb{Z}_+$, $\epsilon, \epsilon' \in \{0, 1\}$, and $m \neq 0$ if $\epsilon = \epsilon' = 1$. The set of defining relations in $U_{\mathbb{Z}}^+$ can be taken to be

$$\begin{aligned} E_1^2 &= 0 & (6) \\ E_2^{(k)} E_2^{(m-k)} &= \begin{bmatrix} m \\ k \end{bmatrix} E_2^{(m)}, \quad 0 \leq k \leq m, & (7) \\ E_2^{(k)} E_1 E_2^{(m-k)} &= \begin{bmatrix} m-1 \\ k \end{bmatrix} E_1 E_2^{(m)} + \begin{bmatrix} m-1 \\ k-1 \end{bmatrix} E_2^{(m)} E_1, \quad 0 < k < m, & (8) \end{aligned}$$

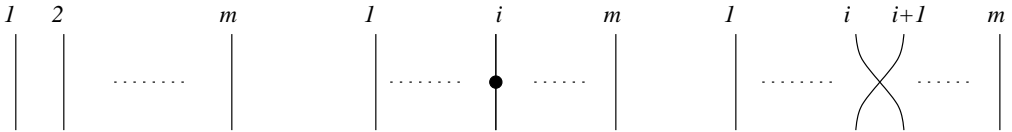
where square brackets denote quantum binomials. Notice that $E_2^{(k)} E_1 E_2^{(m-k)}$ is a linear combination of $E_1 E_2^{(m)}$ and $E_2^{(m)} E_1$ with coefficients in $\mathbb{Z}_+[q, q^{-1}]$. The weight space of U_q^+ containing these products is naturally isomorphic to the corresponding weight space of $U_q^+(\mathfrak{sl}(3))$, since the only relation that contributes to the size of this weight

space is (4) in both algebras. Moreover, $\{E_1 E_2^{(m)}, E_2^{(m)} E_1\}$ is the canonical basis of this weight space of quantum $\mathfrak{sl}(3)$, see [8, Example 14.5.4].

2. Categorification of positive half of quantum $\mathfrak{sl}(2)$. We recall categorification of $U_q^+(\mathfrak{sl}(2))$ following [4, 5]. Fix a ground field \mathbb{k} . The nilHecke algebra H_m is the algebra of endomorphisms of $\mathbb{k}[x_1, \dots, x_m]$ generated by multiplication by x_i endomorphisms (also denoted by x_i) and divided difference operators

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}},$$

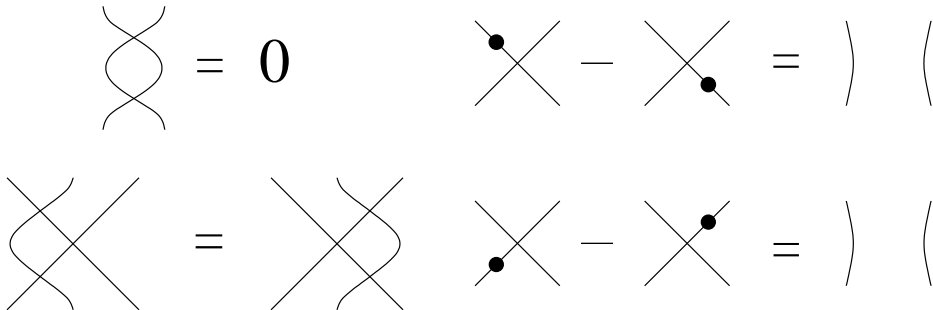
where $s_i(f)$ is f with x_i, x_{i+1} transposed. We depict the identity endomorphism of $\mathbb{k}[x_1, \dots, x_m]$ by m vertical lines, multiplication by x_i endomorphism by the dot on the i -th strand counting from left, and ∂_i by the i -th crossing:



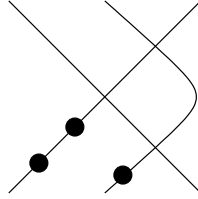
The following is a set of defining relations for H_m :

$$\begin{aligned} x_i x_j &= x_j x_i, \\ x_i \partial_j &= \partial_j x_i, \quad i \neq j, j + 1, \quad \partial_i \partial_j = \partial_j \partial_i, \quad |i - j| > 1, \\ \partial_i^2 &= 0, \quad \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}, \\ x_i \partial_i - \partial_i x_{i+1} &= 1, \quad \partial_i x_i - x_{i+1} \partial_i = 1. \end{aligned}$$

The defining relations say that far away dots and crossings can be isotoped past each other and, in addition, the following diagrammatic equalities hold



The center of H_m is isomorphic to the ring of symmetric polynomials in x_1, \dots, x_m , and H_m is isomorphic to the algebra of $m! \times m!$ matrices with coefficients in the center $Z(H_m)$. The minimal idempotent $e_m \in H_m$, given for $m = 3$ by the diagram below,



provides a Morita equivalence between H_m and its center, via the bimodules $H_m e_m$ and $e_m H_m$, the first an $(H_m, Z(H_m))$ -bimodule, the second a $(Z(H_m), H_m)$ -bimodule. Idempotent e_m is the product of the maximal permutation word in divided difference operators and $x_1^{m-1} x_2^{m-2} \dots x_{m-1}$.

Algebra H_m is graded, with $\deg(x_i) = 2$ and $\deg(\partial_i) = -2$, so that $\deg(e_m) = 0$, and the above Morita equivalence is that of graded rings.

The Grothendieck group $K_0(A)$ of a \mathbb{Z} -graded associative ring A is a $\mathbb{Z}[q, q^{-1}]$ -module with generators $[P]$, over finitely-generated graded projective A -modules P , and defining relations $[P] = [P'] + [P'']$ whenever $P \cong P' \oplus P''$ and $[P\{n\}] = q^n [P]$, where $\{n\}$ is the grading shift by n degrees up.

$K_0(H_m)$ is a free $\mathbb{Z}[q, q^{-1}]$ -module on one generator $[P_{(m)}]$, where

$$P_{(m)} := H_m e_m \left\{ \frac{m(1-m)}{2} \right\}$$

is an indecomposable graded projective H_m -module, unique up to grading shifts and isomorphisms. Placing diagrams next to each other gives inclusions $H_n \otimes H_m \subset H_{n+m}$, which lead to induction and restriction functors between categories of graded $H_n \otimes H_m$ -modules and H_{n+m} -modules. These functors preserve subcategories of finitely-generated projective modules and give us maps

$$\begin{aligned} M : K_0(H_n) \otimes K_0(H_m) &\longrightarrow K_0(H_{n+m}), \\ \Delta : K_0(H_{n+m}) &\longrightarrow K_0(H_n) \otimes K_0(H_m), \end{aligned}$$

where the tensor products are over $\mathbb{Z}[q, q^{-1}]$. Note that

$$K_0(H_n \otimes_{\mathbb{k}} H_m) \cong K_0(H_n) \otimes_{\mathbb{Z}[q, q^{-1}]} K_0(H_m), \tag{9}$$

essentially due to absolute irreducibility of graded simple modules over these rings for any field \mathbb{k} , allowing us to freely switch between the two sides of (9) and to define Δ (a similar fact for rings $R(\nu)$ was glossed over in [4]).

Summing over all n and m produces maps that turn

$$K_0(H) := \bigoplus_{m \geq 0} K_0(H_m)$$

into a twisted $\mathbb{Z}[q, q^{-1}]$ -bialgebra naturally isomorphic to $U_{\mathbb{Z}}^+(\mathfrak{sl}(2))$. This isomorphism takes $[H_m]$ to E_2^m and $[P_{(m)}]$ to the divided power $E_2^{(m)}$. Following our notation we think of $U_{\mathbb{Z}}^+(\mathfrak{sl}(2))$ as a $\mathbb{Z}[q, q^{-1}]$ -subalgebra of $U_{\mathbb{Z}}^+ = U_{\mathbb{Z}}^+(\mathfrak{gl}(1|2))$ generated by the divided powers of E_2 .

3. Lipshitz–Ozsváth–Thurston dg algebras. Continuing to work over a field \mathbb{k} , consider the \mathbb{k} -algebra H_n^- with generators $\sigma_1, \dots, \sigma_{n-1}$ and defining relations

$$\sigma_i^2 = 0, \tag{10}$$

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 0 \quad \text{if } |i - j| > 1, \tag{11}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \tag{12}$$

Algebras H_n^- can be given a graphical description [6], by considering diagrams of n lines in the strip $\mathbb{R} \times [0, 1]$ of the plane, connecting n fixed points on the bottom line $\mathbb{R} \times \{0\}$ to n matching points on the top line $\mathbb{R} \times \{1\}$. Each line projects bijectively onto $[0, 1]$ along \mathbb{R} . Diagrams are composed via concatenation, and isotopies are allowed that do not change the relative height position of crossings. In addition, the following relations are imposed

$$\begin{array}{c}
 \text{Diagram 1} \quad \dots \quad \text{Diagram 2} \quad + \quad \text{Diagram 3} \quad \dots \quad \text{Diagram 4} \quad = \quad 0 \\
 \tag{13}
 \end{array}$$

$$\begin{array}{c}
 \text{Diagram 5} \quad = \quad 0 \qquad \text{Diagram 6} \quad = \quad \text{Diagram 7}
 \end{array}$$

We draw these as thick lines, to distinguish them from the lines that enter the diagrammatics for the nilHecke algebra H_m . One can think of these lines as being fermionic, so that the far away crossing points anticommute rather than commute (change in the order of the product $\sigma_i \sigma_j \longleftrightarrow \sigma_j \sigma_i$ corresponds to the relative height change of the two far away crossings).

We make H_n^- graded, with $\text{deg}(\sigma_i) = -1$, thinking about it as cohomological grading (as opposed to the grading of H_m above, which we shall refer to as q -grading), and equip H_n^- with a differential d via the rules

$$d(\sigma_i) = 1, \quad d(ab) = d(a)b + (-1)^{\text{deg}(a)} ad(b).$$

The differential takes a diagram to the alternating sum of diagrams obtained by smoothing a crossing c of the diagram and multiplying what is left by -1 to the power the number of crossings above c .

$$\begin{array}{c}
 \text{Diagram 8} \quad \xrightarrow{\quad d \quad} \quad \text{Diagram 9} \quad \text{Diagram 10}
 \end{array}$$

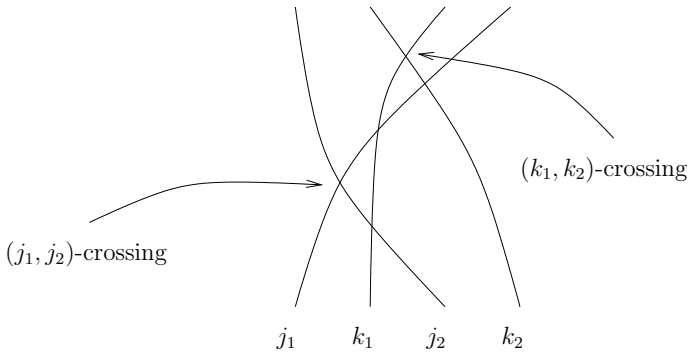
This turns H_n^- into a differential graded (dg) \mathbb{k} -algebra. Ring H_n^- is the simplest example in the family of rings introduced by Lipshitz, Ozsváth, and Thurston [6, Section 3], [7] to extend Ozsváth–Szabó 3-manifold homology to 3-manifolds with boundary and to

localize combinatorial (grid diagram) construction [9] of the Ozsváth–Szabó–Rasmussen knot Floer homology (which categorifies the Alexander polynomial). The authors of [6, 7] work over $\mathbb{Z}/2$; the above characteristic-free lifting of their dg ring is a straightforward guess.

It is obvious that the dimension of H_n^- is at most $n!$ and slightly less obvious that the dimension is exactly $n!$. Let us explain this fact. A permutation $w \in S_n$ admits (usually non-unique) reduced expression $w = s_{i_1} \dots s_{i_r}$ as a product of transpositions $s_i = (i, i+1)$, with $r = l(w)$ the length of w , also equal to the number of crossings in any minimal presentation of w via n intersecting lines in the plane. We can describe a presentation w' of w by listing the sequence of indices $w' = (i_1, \dots, i_r)$. To each presentation w' we assign the element $\sigma_{w'} = \sigma_{i_1} \dots \sigma_{i_r}$ of H_n^- . Fixing a presentation for each permutation w , we obtain a set of elements $\{\sigma_{w'}\}_{w \in S_n}$ that clearly spans H_n^- .

LEMMA 1. *This set is a basis of H_n^- as a \mathbb{k} -vector space.*

Proof. The only potential issue is that minus signs in the relations $\sigma_i \sigma_j = -\sigma_j \sigma_i$ for $|j - i| > 1$ might force the relation $\sigma_{w'} = -\sigma_{w'}$ for some permutation w , making $\sigma_{w'} = 0$ if $\text{char}(\mathbb{k}) \neq 2$. To see that this does not happen, denote by $PDI(w)$ the set of *pairs of disjoint inversions* in w . An inversion in w is a pair (j_1, j_2) of numbers such that $j_1 < j_2$ but $w(j_1) > w(j_2)$. Given a reduced presentation w' of w , inversions are in a bijection with terms of the presentation. If w' is drawn as a diagram of n intersecting lines in the plane, an inversion corresponds to a pair of intersecting lines (a crossing). A pair of disjoint inversions is a quadruple (j_1, j_2, k_1, k_2) such that (j_1, j_2) and (k_1, k_2) are inversions, $j_1 < k_1$, and all four numbers j_1, j_2, k_1, k_2 are distinct. Diagrammatically, a pair of inversions corresponds to a pair of crossings that belong to four distinct lines.



Given $x = (j_1, j_2, k_1, k_2) \in PDI(w)$ and a presentation w' of w , define $\epsilon(w', x) = 1$ if the (j_1, j_2) -crossing is located below the (k_1, k_2) -crossing in the diagram of w' and $\epsilon(w', x) = -1$ if it is located above the (k_1, k_2) -crossing. Algebraically, $\epsilon(w', x) = 1$ if the generator for the (j_1, j_2) -crossing appears in w' to the right of the generator for the (k_1, k_2) -crossing and $\epsilon(w', x) = -1$ if it appears to the left. In the above diagram, $\epsilon(w', x) = 1$.

Define $\epsilon(w') \in \{1, -1\}$ as the product of $\epsilon(w', x)$ over all $x \in PDI(w)$, and let

$$\tilde{\sigma}_{w'} = \epsilon(w') \sigma_{w'}.$$

If w'' differs from w' by a simple transposition of two consecutive terms,

$$w' = (\dots, i, j, \dots), \quad w'' = (\dots, j, i, \dots), \quad |j - i| > 1$$

(geometrically, w', w'' differ as the two diagrams in the top equation in (13)), then $\epsilon(w', x) = \epsilon(w'', x)$ for all $x \in PDI(w)$ save the one that corresponds to the permuted pair of crossings. Hence, $\epsilon(w'') = -\epsilon(w')$, matching the sign change in the equation $\sigma_j \sigma_i = -\sigma_i \sigma_j$, and $\tilde{\sigma}_{w'} = \tilde{\sigma}_{w''}$ in this case.

If w'' differs from w' by a ‘‘Reidemeister III’’ move,

$$w' = (\dots, i, i + 1, i, \dots), \quad w'' = (\dots, i + 1, i, i + 1, \dots),$$

then $\epsilon(w', x) = \epsilon(w'', x)$ for all $x \in PDI(w)$, and $\tilde{\sigma}_{w'} = \tilde{\sigma}_{w''}$.

These observations imply that if w' and w'' are two presentations of w then $\tilde{\sigma}_{w'} = \tilde{\sigma}_{w''}$, so that we can define $\sigma_w := \tilde{\sigma}_{w'}$ for any presentation w' of w , and σ_w will depend only of w and not on its presentation. That $\sigma_w \neq 0$ (consistency) follows as well. ■

REMARK. If $\text{char}(\mathbb{k}) = 2$, anticommutativity is indistinguishable from commutativity, and H_n^- turns into the nilCoxeter algebra, a subalgebra of the nilHecke algebra generated by the divided difference operators. The nilCoxeter algebras (over any field and without the differential) can be used to categorify the polynomial representation of the first Weyl algebra, as well as the bialgebra $\mathbb{Z}[E]$, $\Delta(E) = E \otimes 1 + 1 \otimes E$, see [3] (as opposed to the bigger bialgebra $\mathbb{Z}[E^{(m)}]_{m \geq 1}$ that contains divided powers $E^{(m)} = \frac{E^m}{m!}$ of E and whose categorification relies on nilHecke rings H_m , see earlier but without the q -grading).

Placing diagrams next to each other gives dg ring inclusions

$$H_n^- \otimes H_m^- \subset H_{n+m}^- \tag{14}$$

(the tensor product is taken in the category of super vector spaces, and a diagram from H_n^- is placed above and to the left of a diagram from H_m^-). We would like to form the Grothendieck group $K_0(H_n^-)$ and use these inclusions to define a multiplication and comultiplication on

$$K_0(H^-) := \bigoplus_{n \geq 0} K_0(H_n^-),$$

then identify $K_0(H^-)$ with the integral subalgebra of the positive half of quantum $\mathfrak{gl}(1|1)$. This integral subalgebra is $\mathbb{Z}[q, q^{-1}, E_1]/(E_1^2)$.

4. K_0 of a dg ring. The analogue of a projective module over a \mathbb{k} -algebra A is a projective dg module over a dg \mathbb{k} -algebra (called \mathcal{K} -projective in [1]). A (left) dg module M over A is a \mathbb{Z} -graded A -module equipped with a differential $d_M : M^i \rightarrow M^{i+1}$ such that $d_M(am) = d(a)m + (-1)^{\text{deg}(a)} ad_M(m)$. We call a dg module P over a dg \mathbb{k} -algebra A *projective* if the complex $\text{Hom}_A(P, M)$ has zero homology whenever M does; here M is a dg module over A .

For an introduction to dg modules and projective dg modules we refer the reader to [1, Section 10]. Starting with the abelian category of dg A -modules, one first produces a triangulated category $\mathcal{K}(A)$ of dg-modules by modding out by homotopic to zero morphisms, then quasi-isomorphisms are inverted to produce the derived category $D(A)$. The category $\mathcal{K}(A)$ contains the full subcategory $\mathcal{KP}(A)$ of projective dg modules.

The localization functor $\mathcal{K}(A) \rightarrow D(A)$, when restricted to $\mathcal{K}\mathcal{P}(A)$, gives an equivalence $\mathcal{K}\mathcal{P}(A) \cong D(A)$.

To define the Grothendieck group $K_0(A)$ we need to restrict the size of projective modules. An object M of $\mathcal{K}\mathcal{P}(A)$ or $D(A)$ is called *compact* if the inclusion

$$\bigoplus_{i \in I} \text{Hom}(M, N_i) \subset \text{Hom}(M, \bigoplus_{i \in I} N_i)$$

is an isomorphism for any collection $\{N_i\}_{i \in I}$ of objects indexed by a set I . This definition of a compact object makes sense in any additive category which admits arbitrary direct sums (not just finite ones). In a category of modules over a ring A , a module is compact iff it is finitely-generated as an A -module.

Let $\mathcal{P}(A) \subset \mathcal{K}\mathcal{P}(A)$ be the full subcategory of compact projective modules, for a dg algebra A . It is a triangulated category. Define $K_0(A)$ as the Grothendieck group of $\mathcal{P}(A)$. It has generators $[P]$ over all compact projectives P and relations $[P[1]] = -[P]$ (here $[1]$ is the grading shift), and $[P_2] = [P_1] + [P_3]$ for each distinguished triangle $P_1 \rightarrow P_2 \rightarrow P_3$. Note that $\mathcal{P}(A)$ is equivalent to the subcategory $\mathcal{P}'(A)$ of compact objects in $D(A)$, and we can alternatively define $K_0(A)$ as the Grothendieck group of the triangulated category $\mathcal{P}'(A)$. The following diagram summarizes our categories, inclusions, and equivalences.

$$\begin{array}{ccc}
 A\text{-dmod} & \longrightarrow & \mathcal{K}(A) \\
 & & \uparrow \searrow \\
 & & \mathcal{K}\mathcal{P}(A) \xrightarrow{\cong} D(A) \\
 & & \uparrow \uparrow \\
 \mathcal{P}(A) & \xrightarrow{\cong} & \mathcal{P}'(A)
 \end{array} \tag{15}$$

A quasi-isomorphism $A \rightarrow B$ of dg algebras induces an equivalence of derived categories $D(A) \cong D(B)$, an equivalence of subcategories of compact objects $\mathcal{P}'(A) \cong \mathcal{P}'(B)$, and an isomorphism $K_0(A) \cong K_0(B)$.

Any \mathbb{k} -algebra A is naturally a dg algebra, concentrated in degree 0 and with the trivial differential. In this case, in addition to the above definition of $K_0(A)$, there is the classical definition of $K_0(A)$ as the Grothendieck group of projective finitely-generated A -modules.

LEMMA 2. *If A is (left) Noetherian, the two definitions give naturally isomorphic groups.*

Proof. The lemma can be proved by showing that $\mathcal{P}'(A)$ is equivalent to the homotopy category of bounded complexes of finitely-generated projective left modules over the ring A . ■

For later use, we point out that the above story has a generalization if the dg algebra A has an additional grading, complementary to the cohomological grading. We call such A a graded dg ring and refer to the additional grading as q -grading. Then one can talk about the category of graded dg modules, its homotopy and derived categories, projective graded dg modules, etc. If we retain the above notation, the Grothendieck group $K_0(A)$ will be a $\mathbb{Z}[q, q^{-1}]$ -module, with q corresponding to the grading shift in the additional grading.

If A is just a graded algebra, we turn it into a graded dg algebra by placing it entirely in cohomological degree 0, so that the differential acts by 0, and form the $\mathbb{Z}[q, q^{-1}]$ -module $K_0(A)$, the Grothendieck group of the category of compact objects in $D(A)$, the derived category of the category of graded dg A -modules. The classical definition of K_0 of a graded ring, mentioned earlier (in our discussion of H_m), also produces a $\mathbb{Z}[q, q^{-1}]$ -module, with generators $[P]$, over graded finitely-generated projective A -modules, and relations coming from direct sum decompositions.

LEMMA 3. *If A is (left) graded Noetherian, the two definitions give naturally isomorphic $\mathbb{Z}[q, q^{-1}]$ -modules.*

We say that a (graded) dg algebra A is (graded) formal if it is (graded) quasi-isomorphic to its cohomology algebra $H(A)$. In this case we can identify $K_0(A) \cong K_0(H(A))$. An easy exercise shows that A is formal if $H(A)$ is concentrated in cohomological degree 0. If, furthermore, $H(A)$ is a (graded) Noetherian algebra, we can describe $K_0(A)$ via finitely-generated (graded) projective $H(A)$ -modules.

From now on all algebras and dg algebras that we consider are graded, and we work with the category of graded dg modules, its homotopy and derived categories. Corresponding K_0 -groups are $\mathbb{Z}[q, q^{-1}]$ -modules. The q -grading on H_n^- is trivial—the entire dg algebra sits in zero q -degree.

5. Categorification of positive half of quantum $\mathfrak{gl}(1|1)$. Let us compute K_0 of rings H_n^- . The ring $H_0^- = \mathbb{k}$ (the only diagram when $n = 0$ is the empty one), and $K_0(H_0^-) \cong \mathbb{Z}[q, q^{-1}]$, with the generator $[\mathbb{k}]$, since an object in the category of complexes of graded vector spaces up to chain homotopy is compact iff its total cohomology is finite-dimensional. The ring $H_1^- = \mathbb{k}$, since when $n = 1$ the diagrams have only one line and no room for interactions. Again, $K_0(H_0^-) \cong \mathbb{Z}[q, q^{-1}]$.

To treat the $n \geq 2$ case we use the following observation.

LEMMA 4. *Suppose that A is a dg ring and $x \in A$ an element of degree -1 such that $e = dx$ is an idempotent. Then the inclusion $(1 - e)A(1 - e) \subset A$ is a quasi-isomorphism. In particular, if $dx = 1$ for some $x \in A$ then $H(A) = 0$.*

Proof. An idempotent e in a ring A makes it look (superficially) like the ring of 2×2 -matrices:

$$A = eAe \oplus eA(1 - e) \oplus (1 - e)Ae \oplus (1 - e)A(1 - e).$$

For e and A as in the lemma, each of the first three summands is a contractible complex of abelian groups. The map $h : Ae \rightarrow Ae$ that takes ae to $(-1)^{|a|}axe$ satisfies $hd + dh = 1$, implying that Ae is contractible; a similar computation establishes contractibility of the second summand. ■

We call idempotents that can be written as dx , for some x , *contractible* idempotents.

The dg ring H_n^- has a special property that, for $n \geq 2$, the differential of some element equals one, for instance $d(\sigma_i) = 1$. This implies that H_n^- has trivial homology and its derived category $D(H_n^-)$ is trivial. Category $\mathcal{P}(A)$ is equivalent to $\mathcal{P}'(A)$, a full subcategory of $D(H_n^-)$, so it is trivial as well and $K_0(H_n^-) = 0$ for $n \geq 2$.

This way we do obtain $\mathbb{Z}[q, q^{-1}, E_1]/(E_1^2)$, the integral form of $U_q^+(\mathfrak{gl}(1|1))$, as the Grothendieck group

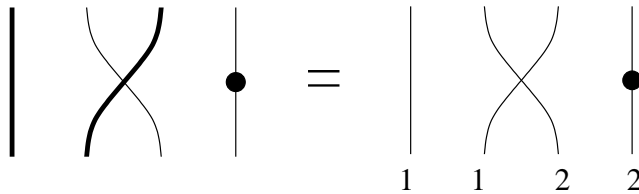
$$K_0(H^-) := \bigoplus_{n \geq 0} K_0(H_n^-).$$

Generators $1, E_1 \in \mathbb{Z}[q, q^{-1}, E_1]/(E_1^2)$ are given by the symbols $[H_0^-]$ and $[H_1^-]$ of free modules over $H_0^- \cong \mathbb{k}, H_1^- \cong \mathbb{k}$, respectively, and multiplication, comultiplication descend from the induction and restriction functors.

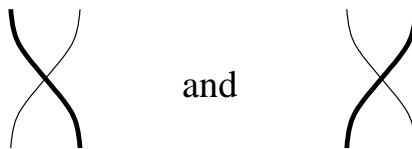
Notice that we went into a lot of trouble, only to categorify the exterior algebra on one generator, with a bit of additional structure thrown in. This will pay off momentarily, as well as teach us that to categorify algebras with nilpotent generators it helps to use categories of modules over dg rings rather than just of modules over rings.

6. Putting the two categorifications together. The Lie superalgebra $\mathfrak{gl}(1|2)$ is generated by its subalgebras $\mathfrak{gl}(1|1)$ and $\mathfrak{gl}(2)$. Likewise, $U_q^+ = U_q^+(\mathfrak{gl}(1|2))$ has generators E_1, E_2 (see the first section), first odd, second even, which generate a copy of $U_q^+(\mathfrak{gl}(1|1)), U_q^+(\mathfrak{sl}(2))$, respectively. We know categorifications of these subalgebras, via diagrammatics of braid-like pictures, and simply need to guess how to combine them. A possible answer is the following.

For each $n, m \in \mathbb{Z}_+$ let $R(n, m)$ be the dg \mathbb{k} -algebra spanned by braid-like diagrams with n lines labelled 1 and m lines labelled 2. We call lines of the first type *fermionic* or *odd*, lines of the second type *bosonic* or *even*. We draw fermionic lines thicker than bosonic. Bosonic lines can carry dots, fermionic lines cannot. Bosonic lines and dots on them interact via the diagrammatics for the nilHecke algebra, fermionic lines interact through diagrammatics for the LOT (Lipshitz–Ozsváth–Thurston) dg algebra. Alternatively, we will indicate fermionic lines by labelling their lower endpoints 1 and bosonic lines by labelling their lower endpoints 2 (as in generators E_1, E_2 of U_q^+). The following example displays the relation between the two notations:



We need to add additional generators—intersections between fermionic and bosonic lines:



and impose the following relations:

1) Far away intersections commute, unless both intersections are between fermionic lines, in which case they anticommute. We can encode these into a single relation

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} \diagup \\ \diagdown \end{array} & \cdots & \begin{array}{c} \diagdown \\ \diagup \end{array} \\
 a \quad a' & & b \quad b'
 \end{array}
 \quad = \quad (-1)^{aa'bb'} \quad
 \begin{array}{ccc}
 \begin{array}{c} \diagdown \\ \diagup \end{array} & \cdots & \begin{array}{c} \diagup \\ \diagdown \end{array} \\
 a \quad a' & & b \quad b'
 \end{array}
 \end{array}$$

where $a, a', b, b' \in \{1, 2\}$ are the labels of the four crossing lines. The only anticommuting case is the following:

$$\begin{array}{ccc}
 \begin{array}{c} \diagup \\ \diagdown \end{array} & \cdots & \begin{array}{c} \diagdown \\ \diagup \end{array} \\
 & & + \\
 \begin{array}{c} \diagdown \\ \diagup \end{array} & \cdots & \begin{array}{c} \diagup \\ \diagdown \end{array} \\
 & & = 0
 \end{array}$$

2) A dot commutes with far away intersection of any kind

$$\begin{array}{ccc}
 \begin{array}{c} \diagdown \\ \diagup \end{array} & \cdots & \begin{array}{c} \bullet \\ | \\ 1 \end{array} \\
 a \quad b & & \\
 = & & \\
 \begin{array}{c} \diagup \\ \diagdown \end{array} & \cdots & \begin{array}{c} | \\ \bullet \\ 1 \end{array} \\
 a \quad b & & \\
 & & a, b \in \{1, 2\}
 \end{array}$$

(same if the intersection is to the right of the dot). Two dots commute:

$$\begin{array}{ccc}
 \begin{array}{c} | \\ \bullet \\ \cdots \end{array} & \cdots & \begin{array}{c} \bullet \\ | \\ \cdots \end{array} \\
 = & & \\
 \begin{array}{c} \bullet \\ | \\ \cdots \end{array} & \cdots & \begin{array}{c} | \\ \bullet \\ \cdots \end{array}
 \end{array}$$

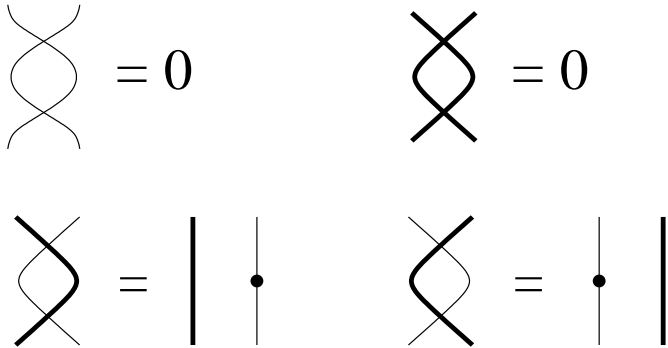
Summary of relations 1) and 2): far away generators commute if at least one of them is even and anticommute if both are odd.

3) Dot through a crossing relations:

$$\begin{array}{ccc}
 \begin{array}{c} \bullet \\ \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \diagdown \\ \bullet \end{array} & \begin{array}{c} \bullet \\ \diagdown \\ \diagup \end{array} - \begin{array}{c} \diagdown \\ \diagup \\ \bullet \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \bullet \\ | \\ \end{array} & \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \bullet \\ | \\ \end{array} \\
 \begin{array}{c} \diagdown \\ \diagup \\ \bullet \end{array} = \begin{array}{c} \diagdown \\ \diagup \\ \bullet \end{array} & \begin{array}{c} \diagdown \\ \diagup \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \bullet \\ | \\ \end{array} & \begin{array}{c} \bullet \\ \diagdown \\ \diagup \end{array} \begin{array}{c} \bullet \\ | \\ \end{array}
 \end{array}$$

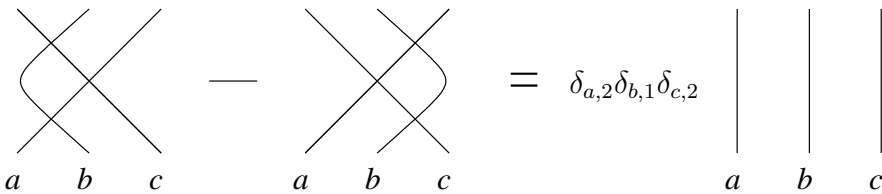
These say that a dot can freely move through an odd-even crossing, and can move through an even-even crossing at the cost of adding an extra term, as in the nilHecke algebra.

4) Two-line relations (Reidemeister II type relations):

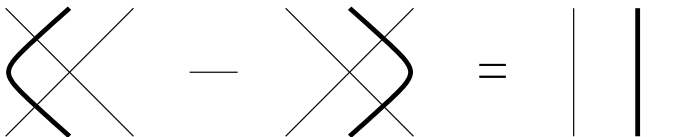


If the two lines that make a crossing are both bosonic or both fermionic, the square of the crossing is 0. A double crossing of a bosonic and a fermionic line equals to the vertical lines diagram with a dot on the bosonic line.

5) Three-line relations (Reidemeister III type relations):




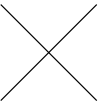
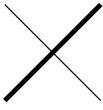

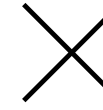
These relations say that a triple intersection homotopy is allowed, unless the three line types are even, odd, even (in this order), in which case there is an additional term:



The defining relations for $R(n, m)$ contain the nilHecke and the LOT relations. When bosonic lines are absent (case $m = 0$), they are exactly the relations in the LOT algebra, so that $R(n, 0) \cong H_n^-$. When fermionic lines are absent ($n = 0$ case), the relations are that of the nilHecke algebra, and $R(0, m) \cong H_m$.

For each sequence \mathbf{i} of n ones and m twos we have the corresponding idempotent $1_{\mathbf{i}}$ in $R(n, m)$ with the diagram of $n + m$ vertical lines, n fermionic and m bosonic, in the order \mathbf{i} . The unit element of $R(n, m)$ is the sum of these idempotents, over all sequences \mathbf{i} . Let $P_{\mathbf{i}} = R(n, m)1_{\mathbf{i}}$ be the left projective $R(n, m)$ -module corresponding to the idempotent $1_{\mathbf{i}}$.

We make $R(n, m)$ bigraded by placing each 1_i in bidegree $(0, 0)$ and listing bidegrees of other generators in the table below:

bidegree	(0,2)	(0,-2)	(0,1)	(0,1)	(-1,0)	
generator						(16)

Defining relations are homogeneous, and $R(n, m)$ is a bigraded \mathbb{k} -algebra. We call the first grading *cohomological grading*, the second grading *q-grading*. This bigrading restricts to the previously discussed gradings on the nilHecke and LOT algebras.

We turn $R(n, m)$ into a dg algebra by defining d on generators (crossings, dots, idempotents 1_i) to be 0, except that d of an odd crossing is the idempotent 1_i given by resolving the crossing:

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \xrightarrow{d} \begin{array}{c}) \\ (\end{array}$$

Bidegree of the differential is $(1, 0)$, thus d respects the q -grading, and $R(n, m)$ becomes a graded dg algebra.

Putting diagrams next to each other defines inclusions of graded dg algebras

$$R(n, m) \otimes R(n', m') \subset R(n + n', m + m')$$

(with the same caveats as for inclusions (14)) and leads to induction and restriction functors between categories of graded dg modules, and between corresponding derived categories. We claim that $K_0(R(n, m))$ can be identified with the weight $n\alpha_1 + m\alpha_2$ subspace of $U_{\mathbb{Z}}^+$,

$$K_0(R(n, m)) \cong U_{\mathbb{Z}}^+(n, m), \tag{17}$$

so that induction and restriction functors descend to multiplication and comultiplication on $U_{\mathbb{Z}}^+$:

$$\begin{aligned}
 K_0(R) &\cong U_{\mathbb{Z}}^+, \\
 K_0(R) &:= \bigoplus_{n, m \geq 0} K_0(R(n, m)), & U_{\mathbb{Z}}^+ &= \bigoplus_{n, m \geq 0} U_{\mathbb{Z}}^+(n, m).
 \end{aligned}$$

(do not confuse the notation (n, m) for the weight space of $\mathfrak{gl}(1|2)$ with the earlier notation $(n|m)$ for the parameters of $\mathfrak{gl}(n|m)$).

$P_i = R(n, m)1_i$ is a dg module, since $d(1_i) = 0$. We want isomorphism (17) to take $[P_i]$ to E_i , were $E_i = E_{i_1} \dots E_{i_{n+m}}$ for $\mathbf{i} = i_1 i_2 \dots i_{n+m}$. Also, the divided power element $E_2^{(m)}$ should correspond to the indecomposable projective $R(0, m)$ -module

$$P_{2^{(m)}} := R(0, m)e_m \left\{ \frac{m(1-m)}{2} \right\}. \tag{18}$$

The idempotent $e_m \in H_m$ was described earlier. Here we use the same notation e_m for the image of e_m in $R(0, m)$ under the canonical isomorphism $H_m \cong R(0, m)$.

Reflecting diagrams about the x -axis induces an anti-involution on $R(n, m)$. Reflecting a diagram about the y -axis and multiplying it by $(-1)^b$, where b is the number of 22-crossings, induces an involution on $R(n, m)$.

Any diagram representing an element of $R(n, m)$ can be simplified to a linear combination of diagrams which consist of fixed minimal length presentations of permutations $w \in S_{n+m}$ with some number of dots at the top of each bosonic strand. Similar to the case of rings $R(\nu)$, see [4], this set is a basis of $R(n, m)$ as a \mathbb{k} -vector space. For rings $R(\nu)$ this was shown [4] by checking that these elements act linearly independently on a certain representation Pol_ν of $R(\nu)$.

The analogue of Pol_ν is a representation Pol of $R(n, m)$ given by

$$\text{Pol} = \bigoplus_{\mathbf{i}, w} \text{Pol}(\mathbf{i}, w), \quad \text{Pol}(\mathbf{i}, w) = \mathbb{k}[x_1(\mathbf{i}, w), \dots, x_m(\mathbf{i}, w)].$$

Here \mathbf{i} ranges over all sequences of n ones and m twos, and w over elements of the symmetric group S_n . The idempotent $1_{\mathbf{i}}$ acts as identity on $\text{Pol}(\mathbf{i}, w)$ for each w and by 0 on $\text{Pol}(\mathbf{j}, w')$ for $\mathbf{j} \neq \mathbf{i}$. An element $x \in R(n, m)1_{\mathbf{i}}$ takes elements of $\text{Pol}(\mathbf{j}, w)$ to 0 for $\mathbf{j} \neq \mathbf{i}$.

A diagram of vertical lines for a sequence \mathbf{i} with a dot on the i -th bosonic strand counting from the left takes $f \in \text{Pol}(\mathbf{i}, w)$ to $x_i(\mathbf{i}, w)f$ (these diagrams are generators of the first type listed in table (16)).

A crossing diagram of the i -th and $(i + 1)$ -st bosonic strands counting from the left (assuming they are next to each other) acts as the divided difference operator, taking $f \in \text{Pol}(\mathbf{i}, w)$ to $\partial_i(f) \in \text{Pol}(\mathbf{i}, w)$. These diagrams are generators of the second type listed in table (16).

A crossing diagram of the i -th bosonic and j -th fermionic strands, assuming they are next to each other in the sequence and the bosonic strand is on the main diagonal (third type generator in the table) takes $f \in \text{Pol}(\mathbf{i}, w)$ to the same polynomial f in variables $x_1(\mathbf{i}', w), \dots, x_m(\mathbf{i}', w)$ instead of $x_1(\mathbf{i}, w), \dots, x_m(\mathbf{i}, w)$, where \mathbf{i}' is given by transposing i -th two with j -th one in the sequence \mathbf{i} :

$$\mathbf{i} = \dots 12 \dots, \quad \mathbf{i}' = \dots 21 \dots$$

A crossing diagram of the i -th bosonic and j -th fermionic strands, assuming they are next to each other in the sequence and the fermionic strand is the main diagonal (fourth type generator in the table) takes $f \in \text{Pol}(\mathbf{i}, w)$ to $x_i(\mathbf{i}', w)f'$, where f' is the same as f but with variables $x_1(\mathbf{i}', w), \dots, x_m(\mathbf{i}', w)$ substituted for $x_1(\mathbf{i}, w), \dots, x_m(\mathbf{i}, w)$. Here \mathbf{i}' is given by transposing i -th two with j -th one in the sequence \mathbf{i} :

$$\mathbf{i} = \dots 21 \dots, \quad \mathbf{i}' = \dots 12 \dots$$

The crossing diagram of the i -th and $(i + 1)$ -st fermionic strands (assuming they are next to each other; this is a fifth type generator from (16)) takes $f \in \text{Pol}(\mathbf{i}, w)$ to $\epsilon_w^i f' \in \text{Pol}(\mathbf{i}, s_i w)$ if $l(s_i w) = l(w) + 1$ (to go from f to f' change variables $x_i(\mathbf{i}, w)$ to $x_i(\mathbf{i}, s_i w)$) and to 0 if $l(s_i w) = l(w) - 1$, where l is the usual length function in the

symmetric group, and $\epsilon_w^i \in \{1, -1\}$ is determined by the formula $\sigma_{s_i w} = \epsilon_w^i \sigma_i \sigma_w$ (see the definition of σ_w earlier).

It is an easy but enlightening exercise to check that these rules give an action of $R(n, m)$ on Pol . An argument similar to the one in [4, Section 2.3] shows that the action is faithful and implies that the spanning set in $R(n, m)$ described earlier is a basis of this algebra. If desired, the rules for the action of the two types of bosonic-fermionic crossings can be interchanged, resulting in the same endomorphism algebra.

The algebra of symmetric polynomials $\text{Sym}(n, m)$ in dots on bosonic lines belongs to the center of $R(n, m)$. Since $R(n, m)$ is a free module of finite rank over $\text{Sym}(n, m)$, we conclude that $R(n, m)$ is both left and right Noetherian.

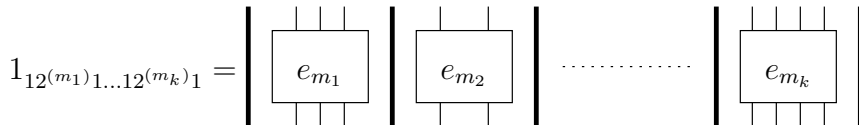
We can construct a homomorphism

$$\gamma : U_{\mathbb{Z}}^+ \longrightarrow K_0(R) \tag{19}$$

by taking generator E_1 to $[P_1]$, where $P_1 = R(1, 0)$ is the free rank one $R(1, 0)$ -module, and generator $E_2^{(m)}$ to $[P_{2^{(m)}}]$. According to our definition, $U_{\mathbb{Z}}^+$ is spanned by various products $E_1 E_2^{(m_1)} E_1 \dots E_1 E_2^{(m_k)} E_1$ and their variations given by removing the first or the last E_1 from the product, or both. γ will take the above product to the symbol of projective module

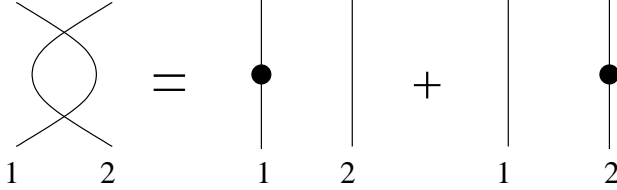
$$P_{1_{2^{(m_1)}1\dots 1_{2^{(m_k)}1}} := R(n, m) 1_{1_{2^{(m_1)}1\dots 1_{2^{(m_k)}1}} \{a\},$$

where $1_{1_{2^{(m_1)}1\dots 1_{2^{(m_k)}1}}$ is the idempotent obtained by placing e_{m_1}, \dots, e_{m_k} in parallel next to each other, separated by odd lines, one for each E_1 in the product, and $a = \sum_i \frac{m_i(1-m_i)}{2}$:



To check that γ is well-defined we consider the defining relations (3), (4). We already saw that $[P_{11}] = 0$, due to contractibility of the dg module P_{11} . More generally, for any sequences \mathbf{i}, \mathbf{j} projective module $P_{\mathbf{i}11\mathbf{j}}$ is contractible as a complex of \mathbb{k} -vector spaces, so that $[P_{\mathbf{i}11\mathbf{j}}] = 0$. In particular, the equivalent of relation (3) holds in $K_0(R)$.

7. Comparison with categorified $\mathfrak{sl}(3)$. To see that the equivalent of relation (4) holds in $K_0(R)$, we will compare $R(1, m)$ with the graded ring $R(\alpha_1 + m\alpha_2)$ that categorifies the weight space $\alpha_1 + m\alpha_2$ of $U_q^+(\mathfrak{sl}(3))$. This is a special case of rings $R(\nu)$ defined in [4]. The Dynkin diagram of $\mathfrak{sl}(3)$ consists of two vertices joined by an edge, in this way similar to the Dynkin diagram of $\mathfrak{gl}(1|2)$, which also consists of two vertices and an edge. Just like diagrams describing $R(1, m)$, diagrams for $R(\alpha_1 + m\alpha_2)$ have one line labelled 1 and m lines labelled 2. But now the line labelled one can carry dots, which freely slide through intersections with type 2 lines (there are no intersections of two lines labelled one, since in this weight there is only one such line). Another difference from $R(1, m)$ is that the double intersection of line 1 and line 2 equals the sum of two terms rather than one: a dot on line 1 plus a dot on line 2:



Other relations are the same. This gives a homomorphism

$$\tau : R(\alpha_1 + m\alpha_2) \longrightarrow R(1, m)$$

that kills any diagram which contains a dot on type 1 line and is the identity on diagrams without such dots. The kernel of τ is spanned by diagrams with at least one dot on line 1. Equivalently, the kernel is the two-sided ideal generated by diagrams of vertical lines, with a dot on the line labelled 1.

In the relations for $R(\alpha_1 + m\alpha_2)$ dot on line 1 can be slid up and down without obstacles. Let J be the graded Jacobson radical of the graded ring $R(\alpha_1 + m\alpha_2)$. We know from [4] that J has finite codimension in $R(\alpha_1 + m\alpha_2)$ and the quotient $R(\alpha_1 + m\alpha_2)/J$ is a finite-dimensional semisimple \mathbb{k} -algebra. The ideal $(\ker(\tau))^N$ is spanned by diagrams with at least N dots on line 1, and its easy to see that, for degree reasons, $(\ker(\tau))^N \subset J$ for sufficiently large N . This in turn implies that $\ker(\tau) \subset J$, so that the induced map of Grothendieck groups of graded rings

$$K_0(\tau) : K_0(R(\alpha_1 + m\alpha_2)) \longrightarrow K_0(R(1, m))$$

is an isomorphism.

Given a sequence \mathbf{i} of divided powers of symbols 1 and 2, we associate to it an idempotent $1_{\mathbf{i}} \in R(n\alpha_1 + m\alpha_2)$ and graded projective $R(n\alpha_1 + m\alpha_2)$ -module

$$P'_{\mathbf{i}} = R(n\alpha_1 + m\alpha_2)1_{\mathbf{i}}\{a\}$$

for n and m equal to total weight of 1 and 2 in the sequence (in [4] this module was denoted by $P_{\mathbf{i}}$), with grading shift as in [4]. We only treat the case $n = 1$, and then sequence \mathbf{i} also defines graded projective $R(1, m)$ -module, denoted by

$$P_{\mathbf{i}} \cong P'_{\mathbf{i}}/\ker(\tau)P'_{\mathbf{i}} = R(1, m)1_{\mathbf{i}}\{a\},$$

where $a = \sum \frac{m_i(1-m_i)}{2}$ and m_i 's are divided powers of 2 that appear in \mathbf{i} .

Proposition 2.13 in [4] implies that, in our notation,

$$P'_{212} \cong P'_{12^{(2)}} \oplus P'_{2^{(2)}1}.$$

Applying τ , we see that

$$P_{212} \cong P_{12^{(2)}} \oplus P_{2^{(2)}1},$$

and, more generally,

$$P_{\mathbf{i}212\mathbf{j}} \cong P_{\mathbf{i}12^{(2)}\mathbf{j}} \oplus P_{\mathbf{i}2^{(2)}1\mathbf{j}}, \quad [P_{\mathbf{i}212\mathbf{j}}] = [P_{\mathbf{i}12^{(2)}\mathbf{j}}] + [P_{\mathbf{i}2^{(2)}1\mathbf{j}}],$$

for any sequences \mathbf{i}, \mathbf{j} of ones and twos. Therefore, relations (4), (5) hold in $K_0(R)$.

From [4, Section 3.3] we know that $K_0(R(\alpha_1 + m\alpha_2))$ is a free $\mathbb{Z}[q, q^{-1}]$ -module with basis elements $[P'_{12^{(m)}}], [P'_{2^{(m)}1}]$ (this is implied by $\{E_1E_2^{(m)}, E_2^{(m)}E_1\}$ being the canonical basis of weight $\alpha_1 + m\alpha_2$ subspace of $U_q^+(\mathfrak{sl}(3))$). Moreover, any finitely-generated

graded projective $R(\alpha_1 + m\alpha_2)$ -module is isomorphic to a direct sum of (graded shifts of) modules $P'_{12(m)}, P'_{2(m)1}$, with multiplicities determined by the image of the module in the Grothendieck group. In particular,

$$P'_{2^{(k)}12^{(m-k)}} \cong (P'_{12(m)})^{s'} \oplus (P'_{2(m)1})^{s''},$$

where

$$s' = \begin{bmatrix} m-1 \\ k \end{bmatrix}, \quad s'' = \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}, \quad s', s'' \in \mathbb{Z}_+[q, q^{-1}],$$

and P^s , for a graded module P and $s \in \mathbb{Z}_+[q, q^{-1}]$, denotes the direct sum of $s(1)$ copies of P with grading shifts:

$$P^s = \bigoplus_{i \in \mathbb{Z}} P^{s_i} \{i\}, \quad s = s(q) = \sum s_i q^i.$$

Polynomials s', s'' are determined by equation (8). Applying τ , we obtain

$$P_{2^{(k)}12^{(m-k)}} \cong (P_{12(m)})^{s'} \oplus (P_{2(m)1})^{s''},$$

so that

$$[P_{2^{(k)}12^{(m-k)}}] = s'[P_{12(m)}] + s''[P_{2(m)1}],$$

and the equivalent of relation (8) holds in $K_0(R)$. That (7) holds follows from the corresponding result for the nilHecke algebra [4, 5]. Therefore, γ is well-defined. Why γ is an isomorphism will be explained next.

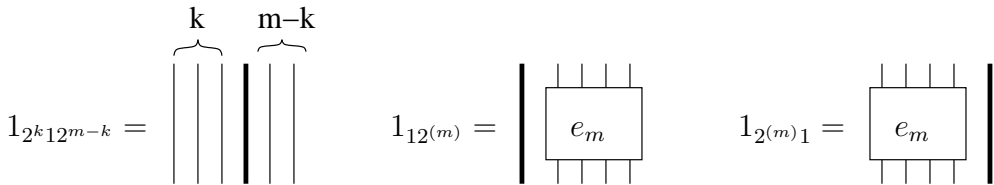
We have

$$P'_{2^k 12^{m-k}} = R(\alpha_1 + m\alpha_2)1_{2^k 12^{m-k}} \cong (P'_{12(m)})^{r'} \oplus (P'_{2(m)1})^{r''} \tag{20}$$

as a graded left $R(\alpha_1 + m\alpha_2)$ -module, where

$$r' = [k]![m-k]!s', \quad r'' = [k]![m]!s''.$$

Projective modules in the above equation correspond to the following three idempotents:



Direct sum decomposition (20) is equivalent to a choice of homogeneous elements

$$\begin{aligned} \alpha'_a &\in 1_{2^k 12^{m-k}} R(\alpha_1 + m\alpha_2) 1_{12(m)}, & 1 \leq a \leq r'(1), \\ \beta'_a &\in 1_{12(m)} R(\alpha_1 + m\alpha_2) 1_{2^k 12^{m-k}}, & 1 \leq a \leq r'(1), \\ \alpha''_a &\in 1_{2^k 12^{m-k}} R(\alpha_1 + m\alpha_2) 1_{2(m)1}, & 1 \leq a \leq r''(1), \\ \beta''_a &\in 1_{2(m)1} R(\alpha_1 + m\alpha_2) 1_{2^k 12^{m-k}}, & 1 \leq a \leq r''(1), \end{aligned}$$

such that

$$\beta'_b \alpha'_a = \delta_{a,b} 1_{12^{(m)}}, \tag{21}$$

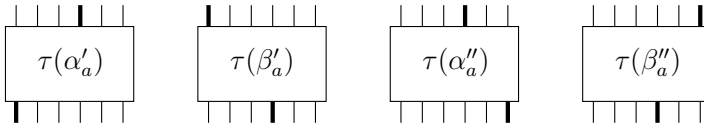
$$\beta''_b \alpha''_a = \delta_{a,b} 1_{2^{(m)}_1}, \tag{22}$$

$$\beta''_b \alpha'_a = 0, \tag{23}$$

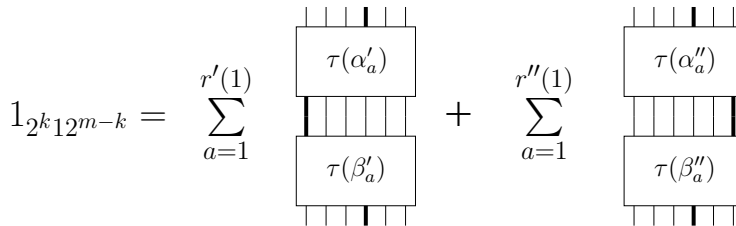
$$\beta'_b \alpha''_a = 0, \tag{24}$$

$$1_{2^k 12^{m-k}} = \sum_{a=1}^{r'(1)} \alpha'_a \beta'_a + \sum_{a=1}^{r''(1)} \alpha''_a \beta''_a. \tag{25}$$

From the standard representation theory of graded rings and the observation that $\ker(\tau)$ belongs to the graded Jacobson radical of $R(\alpha_1 + m\alpha_2)$ it follows that any finitely-generated graded projective $R(1, m)$ -module is a direct sum of indecomposable projectives $P_{12^{(m)}}$ and $P_{2^{(m)}_1}$ with grading shifts, that these two projectives are not isomorphic, and $K_0(R(1, m))$ is a free $\mathbb{Z}[q, q^{-1}]$ -module with basis elements $[P_{12^{(m)}}]$ and $[P_{2^{(m)}_1}]$. The above homogeneous elements of $R(\alpha_1 + m\alpha_2)$ descend via τ to homogeneous elements of $R(1, m)$ satisfying the same identities. We depict these elements by boxes:



Upon applying τ equation (25) becomes



To establish that γ is an isomorphism, we recall from the beginning of the paper that

- $U_{\mathbb{Z}}^+(0, m)$ is a free $\mathbb{Z}[q, q^{-1}]$ -module with generator $E_2^{(m)}$,
- $U_{\mathbb{Z}}^+(1, m)$ is a free rank two $\mathbb{Z}[q, q^{-1}]$ -module with generators $E_1 E_2^{(m)}$, $E_2^{(m)} E_1$,
- $U_{\mathbb{Z}}^+(2, m)$ is a free $\mathbb{Z}[q, q^{-1}]$ -module generated by $E_1 E_2^{(m)} E_1$,
- $U_{\mathbb{Z}}^+(n, m) = 0$ for $n \geq 3$.

We now look at the size of $K_0(R(n, m))$.

Case $n = 0$. The ring $R(0, m)$ is graded Noetherian and concentrated in cohomological degree 0 with trivial differential. By Lemma 3, $K_0(R(0, m))$ can be computed via finitely-generated graded projectives. Any such projective is isomorphic to a finite direct sum of shifts of the indecomposable projective $P_{2^{(m)}}$ defined in (18). Hence,

$$K_0(R(0, m)) \cong \mathbb{Z}[q, q^{-1}] \cdot [P_{2^{(m)}}],$$

giving us a match with $U_{\mathbb{Z}}^+(0, m)$, see above.

Case $n = 1$. The ring $R(1, m)$ is also graded Noetherian concentrated in cohomological degree 0 with trivial differential. We have established that the graded Grothendieck group $K_0(R(1, m))$ is a free rank two $\mathbb{Z}[q, q^{-1}]$ -module with the basis given by symbols $[P_{12^{(m)}}]$, $[P_{2^{(m)}_1}]$. This matches with the above basis for $U_{\mathbb{Z}}^+(1, m)$.

Case $n \geq 3$. We will prove that dg algebra $R(n, m)$ has trivial homology when $n \geq 3$. Consider the element $y_k \in R(3, m)$:

$$y_k = \sum_{a=1}^{r'(1)} \left(\begin{array}{c} \tau(\alpha'_a) \\ \tau(\beta'_a) \end{array} \right) + \sum_{a=1}^{r''(1)} \left(\begin{array}{c} \tau(\alpha''_a) \\ \tau(\beta''_a) \end{array} \right)$$

Applying d and using the previous diagrammatic equation, we get

$$d(y_k) = 1_{12^k 12^{m-k} 1}.$$

Therefore, for any sequence \mathbf{i} which contains at least three ones we can find $y_{\mathbf{i}} \in R(n, m)$ such that $d(y_{\mathbf{i}}) = 1_{\mathbf{i}}$. We can write $1 \in R(n, m)$ as the sum of $1_{\mathbf{i}}$ over all sequences \mathbf{i} with n ones ($n \geq 3$) and m twos. Then

$$d\left(\sum_{\mathbf{i}} y_{\mathbf{i}}\right) = 1$$

implying that $H(R(n, m)) = 0$ and $K_0(R(n, m)) = 0$.

Case $n = 2$. Observe that the unit element of $R(2, m)$ decomposes

$$1 = \sum_{k, \ell} 1_{2^k 12^\ell 12^{m-k-\ell}}$$

and that projective module $P_{2^k 12^\ell 12^{m-k-\ell}}$ is isomorphic to the direct sum of shifts of modules $P_{12^{(k+\ell)} 12^{m-k-\ell}}$ and $P_{2^{(k+\ell)} 112^{m-k-\ell}}$. The latter are contractible, as complexes of vector spaces (since their sequences contain two consecutive ones), while modules of the first kind decompose as direct sum of shifts of modules $P_{112^{(m)}}$ and $P_{12^{(m)}_1}$. We can discard $P_{112^{(m)}}$ because of contractibility and conclude that the K_0 groups of graded dg rings $R(2, m)$ and

$$R_1 := 1_{12^{(m)}_1} R(2, m) 1_{12^{(m)}_1}$$

are naturally isomorphic. To make this argument more accurate, introduce additional dg rings defined via idempotents $e(4), e(3), e(2)$:

$$R_4 = e(4)R(2, m)e(4), \quad e(4) := \sum_{t=0}^m (1_{12^{(t)} 12^{m-t}} + 1_{2^{(t)} 112^{m-t}}),$$

$$R_3 = e(3)R_4e(3), \quad e(3) := \sum_{t=0}^m 1_{12^{(t)} 12^{m-t}},$$

$$R_2 = e(2)R_3e(2), \quad e(2) := 1_{112^{(m)}} + 1_{12^{(m)}_1}.$$

Graded dg rings R_4 and $R(2, m)$ are graded dg Morita equivalent in the strongest sense, via dg bimodules $R(2, m)e(4)$ and $e(4)R(2, m)$. These bimodules produce an equivalence of abelian categories of graded dg modules over R_4 and $R(2, m)$, as well as all the other categories of dg modules and their graded counterparts that appear in the diagram (15). Hence, there is a canonical isomorphism $K_0(R(2, m)) \cong K_0(R_4)$. Next, R_3 is a graded dg subring of R_4 obtained by removing chunks of R_4 corresponding to contractible idempotents $1_{2^{(t)}112^{(m-t)}}$. In view of Lemma 4 and invariance of K_0 under quasi-isomorphisms of graded dg rings there is a natural isomorphism $K_0(R_4) \cong K_0(R_3)$. Graded dg rings R_3 and R_2 are graded dg Morita equivalent, via bimodules

$$R_3(1_{112^{(m)}} + 1_{12^{(m)}1}) \quad \text{and} \quad (1_{112^{(m)}} + 1_{12^{(m)}1})R_3,$$

giving us an isomorphism $K_0(R_3) \cong K_0(R_2)$. Finally, reducing R_2 via Lemma 4 applied to the contractible idempotent $1_{112^{(m)}}$ results in R_1 . Nonunital inclusions

$$R_1 \subset R_2 \subset R_3 \subset R_4 \subset R(2, m)$$

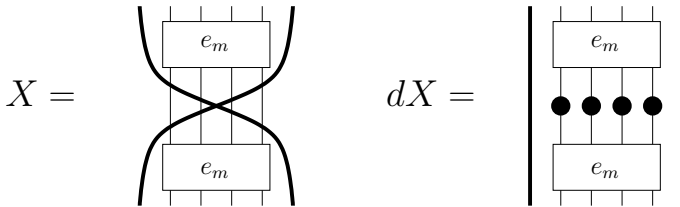
are quasi-isomorphisms, giving rise to canonical isomorphisms of Grothendieck groups

$$K_0(R_1) \cong K_0(R_2) \cong K_0(R_3) \cong K_0(R_4) \cong K_0(R(2, m)). \tag{26}$$

Diagrams representing elements of R_1 have two fermionic lines that start and end at the leftmost and rightmost top and bottom endpoints and m bosonic lines, capped off on both sides by the projector $e_m = 1_{2^{(m)}}$. We can decompose R_1 into the direct sum of the 2-sided ideal I spanned by diagrams in which the two fermionic lines intersect and the subring R'_1 spanned by diagrams with the disjoint fermionic lines, $R_1 = R'_1 \oplus I$. The ring R'_1 is isomorphic to $e_m H_m e_m$, where H_m is the nilHecke algebra (note that nonintersecting fermionic lines can be pulled away to be disjoint from the m bosonic lines). In turn, $e_m H_m e_m$ is isomorphic to the ring of symmetric polynomials in x_1, \dots, x_m , corresponding to the dots on the bosonic lines, so that $R'_1 \cong \text{Sym}(x_1, \dots, x_m)$. Ideal I is isomorphic to R'_1 when viewed as an R'_1 -bimodule, with the generator X depicted below and on the left. The differential in the dg ring R_1 is zero on R'_1 and takes I injectively to R'_1 ,

$$0 \longrightarrow I \xrightarrow{d} R'_1 \longrightarrow 0,$$

with the generator X taken to $x_1 x_2 \dots x_m$. In the diagram below each box denotes idempotent e_m .



Therefore, the quotient map $R_1 \longrightarrow R'_1/d(I)$ is a quasi-isomorphism of dg rings, inducing an isomorphism $K_0(R_1) \cong K_0(R'_1/d(I))$. The ring $R'_1/d(I)$ has trivial differential and is concentrated in cohomological degree 0. It is isomorphic to the quotient of the graded ring of symmetric polynomials $\text{Sym}(x_1, \dots, x_m)$ by the ideal $(x_1 x_2 \dots x_m)$. This

quotient is a graded local ring, with K_0 isomorphic to $\mathbb{Z}[q, q^{-1}]$. The above arguments tell us that

$$K_0(R(2, m)) \cong K_0(R_1) \cong \mathbb{Z}[q, q^{-1}],$$

with the generator being $[P_{12(m)_1}]$.

We conclude that the map γ is an isomorphism. It is straightforward to check that γ respects comultiplication of twisted bialgebras in (19). Hence, we have a canonical isomorphism of twisted bialgebras

$$U_{\mathbb{Z}}^+ \cong K_0(R)$$

taking weight spaces of $U_{\mathbb{Z}}^+$ to $K_0(R(n, m))$ for various n, m .

8. Perspectives. It is not hard to guess how one should couple the above diagrammatics for categorified $U_q^+(\mathfrak{gl}(1|2))$ to the diagrammatics [4] for categorified $U_q^+(\mathfrak{g})$ to produce a graphical calculus for categorified $U_q^+(\mathfrak{gl}(1|n))$ and some other classical Lie superalgebras in place of $\mathfrak{gl}(1|n)$. Technical obstacles related to switching from algebras to dg algebras and their representations were largely avoided in the present paper due to small size of Grothendieck groups of dg rings $R(n, m)$ and manual case-by-case considerations. We plan to discuss categorification of $U_q^+(\mathfrak{gl}(1|n))$ in a follow-up paper. Categorification of $U_q^+(\mathfrak{gl}(n|m))$ for $n, m \geq 2$ should require additional ideas.

Varagnolo and Vasserot [11] established that rings $R(\nu)$ from [4], [10] that categorify weight spaces of $U_q^+(\mathfrak{g})$ are isomorphic to equivariant ext groups of sheaves on Lusztig quiver varieties. It is an interesting problem to find a similar interpretation for the dg rings $R(n, m)$ described above that categorify weight spaces of $U_q^+(\mathfrak{gl}(1|2))$ and for their generalizations.

The problem of categorifying the entire (suitably idempotent) quantum group $U_q(\mathfrak{gl}(1|1))$ rather than just its positive half is open as well. With Lauda's categorification [5] of $U_q(\mathfrak{sl}(2))$ in mind, we expect that one should generalize the Lipshitz–Ozsváth–Thurston rings to allow the lines to travel in all directions rather than just up. Another challenging problem is to extend the work of Webster [12, 13] to the superalgebra case.

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