

COCYCLE INVARIANTS OF CODIMENSION 2 EMBEDDINGS OF MANIFOLDS

JÓZEF H. PRZYTYCKI

*Department of Mathematics, The George Washington University
Washington, DC 20052, U.S.A.,
University of Maryland CP, and University of Gdańsk
E-mail: przytyck@gwu.edu*

WITOLD ROSICKI

*Institute of Mathematics, University of Gdańsk
E-mail: wrosicki@mat.ug.edu.pl*

Abstract. We consider the classical problem of a position of n -dimensional manifold M^n in \mathbb{R}^{n+2} . We show that we can define the fundamental $(n+1)$ -cycle and the shadow fundamental $(n+2)$ -cycle for a fundamental quandle of a knotting $M^n \rightarrow \mathbb{R}^{n+2}$. In particular, we show that for any fixed quandle, quandle coloring, and shadow quandle coloring, of a diagram of M^n embedded in \mathbb{R}^{n+2} we have $(n+1)$ - and $(n+2)$ -(co)cycle invariants (i.e. invariant under Roseman moves).

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2010 *Mathematics Subject Classification*: 57Q45, 57M25.

The paper is in final form and no version of it will be published elsewhere.

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1. Introduction. We consider the classical problem of a position of n -dimensional manifold M^n , in an $(n + 2)$ -manifold W^{n+2} . The classical case deals with $W^{n+2} = \mathbb{R}^{n+2}$, however our method is also well suited for a more general case of W^{n+2} being a product of an oriented $(n + 1)$ -manifold F^{n+1} and the interval or the twisted interval bundle over an unorientable $(n + 1)$ -manifold F^{n+1} , as we have, in these cases, the natural projection of W^{n+2} onto F^{n+1} (see Section 7).

Historically, the main tool to study an n -knotting, $f : M^n \rightarrow \mathbb{R}^{n+2}$, was the fundamental group of the knotting complement in \mathbb{R}^{n+2} . This was greatly extended by applying quandle colorings and (co)cycle invariants. We follow, to some extent, the exposition by Carter, Kamada, and Saito in [CKS-3], generalizing on the way the case of surfaces in \mathbb{R}^4 to general n -knottings. The paper is organized as follows: at the beginning of the first section we recall the definition of a rack and quandle and their (co)homology. Then we give a short introduction to diagrams, D_M , of knottings, and rack and quandle colorings of D_M . Furthermore, we analyze shadow rack and quandle colorings. In the second part of the first section we define $(n + 1)$ - and $(n + 2)$ -chains associated to rack and quandle colorings. In the second section we prove that our chains are, in fact, cycles. In the third section we show that the set of colorings by a given quandle is a topological invariant. The first step in this direction is given by comparing two definitions of the fundamental rack and quandle of a knotting (one from the diagram and one abstract). In the fourth section we show that the homology classes represented by cycles constructed for diagrams of knottings are topological invariants. Here we carefully consider Roseman’s pass move (generalized third Reidemeister move). We offer various versions of cycle invariants of knottings in particular taking into account the fact that a quandle acts on the space of quandle colorings. We complete Section 4 by expressing invariants in the language of cohomology (cocycle invariants). In the fifth section we generalize previous results to twisted homology and cohomology. In the sixth section we discuss in detail general position projection of n -knotting and Roseman moves; this is a service section to the previous considerations.

Finally, in Section 7 we give a short overview of a knotting in $F^{n+1} \bar{\times} [0, 1]$, and in Section 8 we discuss possibility of working with a Yang–Baxter operator in place of a right self-distributive operation.

1.1. Quandles and quandle homology. We give here a short historical introduction to distributive structures and to homology based on distributivity. The word *distributivity* was coined in 1814 by François Servois. C. S. Peirce in 1880 emphasized the importance of (right) self-distributivity in algebraic structures [Peir]. The first explicit example of a

non-associative self-distributive system was given by Ernst Schröder in 1887 [Schr, Deh]. The detailed study of distributive structures started with the 1929 paper by C. Burstin and W. Mayer¹ [B-M]. The basic example they give is the same as Schröder’s, and was later used for Fox 3-colorings:

*	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

The first book partially devoted to distributivity is by Anton Sushkevich, 1937 [Sus].

DEFINITION 1.1. Let $(X; *)$ be a magma, that is a set with binary operation, then:

- (i) If $*$ is right self-distributive, that is, $(a * b) * c = (a * c) * (b * c)$, then $(X; *)$ is called a RDS or a shelf (the term coined by Alissa Crans in her Ph.D. thesis [Cra]).
- (ii) If a shelf $(X; *)$ satisfies the idempotent² condition, $a * a = a$ for any $a \in X$, then it is called a *right spindle*, or just a spindle (again the term coined by Crans).
- (iii) If a shelf $(X; *)$ has $*$ invertible,³ that is the map $*_b : X \rightarrow X$ given by $*_b(a) = a * b$ is a bijection for any $b \in X$, then it is called a *rack*⁴.
- (iv) If a rack $(X; *)$ satisfies the idempotent condition, then it is called a *quandle* (the term coined in Joyce’s Ph.D. thesis of 1979 [Joy-1]).
- (v) If a quandle $(X; *)$ satisfies $(a * b) * b = a$ then it is called *kei* or an involutive quandle. The term kei (圭) was coined in a pioneering paper by M. Takasaki in 1942 [Tak]⁵.

Three axioms of a quandle arise (see [Joy-2, Matv]) as an algebraic reflection of three Reidemeister moves on link diagrams. Idempotent condition corresponds to the first move, invertibility to the second, and right self-distributivity to the third move. In Fig. 1.2 we illustrate how right self-distributivity is arising from the third Reidemeister move, R_3 ,

¹Walter Mayer is well known for Mayer–Vietoris sequence and for being assistant to Einstein at Institute for Advanced Study, Princeton. Celestin Burstin (1888–1938) born in Tarnopol, finished high school there in 1907, moved to Vienna, then to Minsk where he was a member of the Belarussian National Academy of Sciences, and a Director of the Institute of Mathematics of the Academy. In 1937, he was arrested on suspicion of activity as a spy for the Poland and Austria. He died in 1938, when interrogated in a prison (in Minsk); rehabilitated March 2, 1956 [B-Mat, Mal, Mio].

²The term coined in 1870 by Benjamin Peirce [Pei], the father of Charles Sanders Peirce.

³If X is a set then the set $Bin(X)$ of all binary operations on X forms a monoid with composition $*_1 *_2$ given by $a *_1 *_2 b = (a *_1 b) *_2 b$ and the identity element $*_0$ given by $a *_0 b = a$. Then the condition (ii) is equivalent to invertibility of $*$ in $Bin(X)$. If $*$ is invertible, we write $\bar{*}$ for $*^{-1}$.

⁴The term wrack, like in “wrack and ruin”, of J. H. Conway from 1959, was modified to rack in [F-R]). The main example considered in 1959 by Conway and Wraith was a group G with a $*$ operation given by conjugation, that is, $a * b = b^{-1}ab$ [C-W].

⁵Mituhisa Takasaki worked at Harbin Technical University in 1940, likely as an assistant to Kôshichi Toyoda. Both perished when Red army entered Harbin in 1945. Takasaki was considering keis associated to abelian groups, that is the Takasaki kei (or quandle) of an abelian group H , denoted by $T(H)$ satisfies $a * b = 2b - a$.

when we color (label) arcs of the diagram by elements of X according to the following rule (Fig. 1.1):

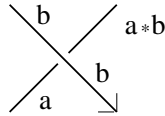


Fig. 1.1. Magma coloring of a crossing

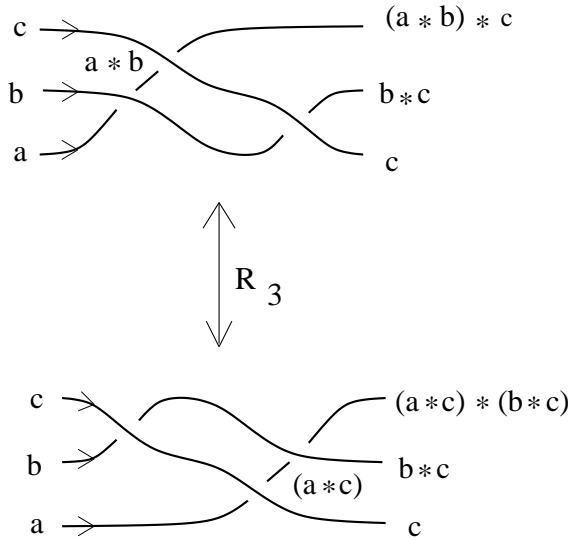


Fig. 1.2. Distributivity from R_3

(Co)homology of racks was introduced by Fenn, Rourke and Sanderson between 1990 and 1995, [FRS-1, Fenn]. Quandle (co)homology was constructed by Carter, Kamada, and Saito (compare [CKS-3]). Their motivation was to associate to any link diagram and its quandle coloring, elements (cocycles) of quandle cohomology. The case of $n = 1$ was first implicitly considered in the Michael Greene’s Ph.D. thesis of 1997 [Gr]. In [CKS-3] the detailed construction of (co)cycles is given for a knotting $f : M^n \rightarrow \mathbb{R}^{n+2}$ for $n = 1$ or $n = 2$, and we start our paper from the definition for any n (essentially following [CKS-3]). First we give after [CKS-3] the definition of rack, degenerate and quandle (co)homology.

DEFINITION 1.2.

- (i) For a given rack X let $C_n^R(X)$ be the free abelian group generated by n -tuples (x_1, x_2, \dots, x_n) of elements of a rack X ; in other words $C_n^R(X) = \mathbb{Z}X^n = (\mathbb{Z}X)^{\otimes n}$. Define a boundary homomorphism $\partial : C_n^R(X) \rightarrow C_{n-1}^R(X)$ by

$$\begin{aligned} \partial(x_1, x_2, \dots, x_n) &= \partial^{(*_0)} - \partial^{(*)} = \sum_{i=1}^n (-1)^i (d_i^{(*_0)} - d_i^{(*)}) \\ &= \sum_{i=1}^n (-1)^i ((x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) - (x_1 * x_i, x_2 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n)). \end{aligned}$$

$(C_*^R(X), \partial)$ is called a rack chain complex of X . Here we assume that $C_0^R(X) = 0$ so $\partial_1 : C_1 \rightarrow C_0$ is a zero map.

- (ii) Assume that X is a quandle, then we have a subchain complex $C_n^D(X) \subset C_n^R(X)$ generated by n -tuples (x_1, \dots, x_n) with $x_{i+1} = x_i$ for some i . The subchain complex $(C_n^D(X), \partial)$ is called a degenerated chain complex of a quandle X .
- (iii) The quotient chain complex $C_n^Q(X) = C_n^R(X)/C_n^D(X)$ is called the quandle chain complex. We have the short exact sequence of chain complexes:

$$0 \rightarrow C_n^D(X) \rightarrow C_n^R(X) \rightarrow C_n^Q(X) \rightarrow 0.$$

- (iv) The homology of rack, degenerate and quandle chain complexes are called rack, degenerate and quandle homology, respectively. We have the long exact sequence of homology of quandles:

$$\dots \rightarrow H_n^D(X) \rightarrow H_n^R(X) \rightarrow H_n^Q(X) \rightarrow H_{n-1}^D(X) \rightarrow \dots$$

R. Litherland and S. Nelson [L-N] proved that the short exact sequence from (iii) splits respecting the chain maps. $\alpha : C_n^Q(X) \rightarrow C_n^R(X)$ is given, in the notation introduced in [N-P], by

$$\alpha(x_1, x_2, x_3, \dots, x_n) = (x_1, x_2 - x_1, x_3 - x_2, \dots, x_n - x_{n-1})$$

(recall that in our notation $(x_1, x_2 - x_1, x_3 - x_2, \dots, x_n - x_{n-1}) = x_1 \otimes (x_2 - x_1) \otimes (x_3 - x_2) \otimes \dots \otimes (x_n - x_{n-1}) \in C_n^R(X)$). In particular, α is a chain complex monomorphism and $H_n^R(X) = H_n^D(X) \oplus \alpha_*(H_n^Q(X))$.

In a recent paper [P-P-2] it is demonstrated that degenerate homology of a quandle can be reconstructed from the quandle (normalized) homology of the quandle by a version of a Künneth formula.

We define cohomology in a standard way (we follow [CKS-3]).

DEFINITION 1.3. For an abelian group A define the cochain complexes $C_W^*(X, A) = Hom(C_*^W, A)$. Here, $W = D, R, Q$, so we describe all cases (degenerate, rack and quandle). We define $\partial^n : C^n \rightarrow C^{n+1}$ in the usual way, that is for $c \in C_W^n(X; A)$ we have

$$\partial^n(c)(x_1, \dots, x_n, x_{n+1}) = c(\partial_n(x_1, \dots, x_n, x_{n+1})).$$

Cohomology groups are defined as usual as $H_W^n(X, A) = \ker \partial^n / \text{im}(\partial^{n-1})$.

Another useful definition is that of right X -set, in particular the set of colorings of knotting diagram by a quandle X will be a right X -quandle-set.

DEFINITION 1.4. Let E be a set, $(X; *)$ a magma and $* : E \times X \rightarrow E$ an action of X on E (we can use the same symbol $*$ for operation in X and the action as it unlikely leads to confusion). Then

- (i) If $(X; *)$ is a shelf and $(e * x_1) * x_2 = (e * x_2) * (x_1 * x_2)$ then E is a right X -shelf-set.
- (ii) If $(X; *)$ is a rack and E a right X -shelf-set and additionally the map $*_b : E \rightarrow E$ given by $*_b(e) = e * b$ is invertible then we say that E is a right X -rack-set. In the case X is a quandle we will say that E is a right X -quandle-set.

The basic example of a quandle (resp. rack or shelf) right X -quandle- (resp. rack-, shelf-) -set is $E = X^n$ with $(x_1, \dots, x_n) * x = (x_1 * x, \dots, x_n * x)$. One should also

add that for a given X -rack-set E one can define homology (as before) by assuming $C_n(X, E) = \mathbb{Z}(E \times X^n)$, and $d_i^{(*)}(e, x_1, \dots, x_n) = (e * x_i, x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n)$. If E has one point, so the action is trivial, we reach exactly homology of Definition 1.2(i). In the standard definition we assume $C_0(X, E) = 0$, but here more natural is an augmented version $\tilde{C}_0(X, E) = \mathbb{Z}E$, and $\partial_1^{(*)}(e, x) = e * x$, and $\partial_1^{(*0)}(e, x) = e$, with $\tilde{H}_0(X, E)$ a free abelian group generated by orbits of the action of X on E .

OBSERVATION 1.5. *The chain groups $C_n(X) = ZX^n$ of a rack chain complex are X -rack-sets. Furthermore, the action $*_x : C_n(X) \rightarrow C_n(X)$, given by $*_x(x_1, \dots, x_n) = (x_1, \dots, x_n) * x = (x_1 * x, \dots, x_n * x)$, is a chain map for any x , inducing the identity on homology. It is a well know but important fact (see e.g. [CJKS, N-P]) and we use it in Theorems 4.4, 4.8, and 5.5. To prove this fact we use chain homotopy $(-1)^{n+1}h_x : C_n \rightarrow C_{n+1}$, where $h_x(x_1, \dots, x_n) = (x_1, \dots, x_n, x)$. We check directly that $\partial_{n+1}(-1)^{n+1}h_x + (-1)^n h_x \partial_n = \text{Id} - *_x$. If we consider $\partial^T = t\partial^{(*0)} - \partial^{(*)}$, as is the case in twisted rack or quandle homology, we see that $*_x$ induces $t \cdot \text{Id}$ on homology. We use it in Definition 5.3.*

We need yet another observation that if X is a quandle and E is an X -quandle-set then $X \sqcup E$ has also a natural quandle structure.

OBSERVATION 1.6.

- (i) *Let $(X; *)$ be a shelf and E an X -shelf-set (with a right action of X on E also denoted by $*$), then $X \sqcup E$ is also a shelf with $*$ operation $a * e = a$ for any $a \in X \sqcup E$, and $e \in E$. Furthermore, if $(X; *)$ is a rack (resp. spindle, or quandle) then $(X \sqcup E; *)$ is also a rack (resp. spindle, or quandle). Then we observe that a chain complex $C_n(X, E)$ is a subchain complex of $C_n(X \sqcup E)$.*
- (ii) *If X is a one element set, say $X = \{x\}$, then E instead of being a trivial shelf can be a shelf (or a rack) with $e_1 * e_2 = e_1 * x$ which complemented by the rule $x * x = x = x * e$ makes $E \sqcup \{x\}$ into a shelf (resp. rack).*

Proof. (i) It is a direct exercise so we demonstrate only that $(E \sqcup X; *)$ is a shelf. We have

- (1) $(a * b) * c = (a * c) * (b * c)$ for $a, b, c \in X$ as $(X; *)$ is a shelf.
- (2) $ = (a * c) * (b * c)$ for $a \in E, b, c \in X$ as E is an X -shelf-set.
- (3) $ = a * c$ for $b \in E$ and also we have $(a * c) * (b * c) = a * b$ as $b * c \in E$.
- (4) $ = a * b$ for $c \in E$ and also we have $(a * c) * (b * c) = a * b$.

(ii) Let $f : E \sqcup \{x\} \rightarrow E \sqcup \{x\}$ where $f(u) = u * x$ (in particular $f(x) = x$, then

$$(u_1 * u_2) * u_3 = f(f(u_1)) = (u_1 * u_3) * (u_2 * u_3) \text{ for any } u_1, u_2, u_3 \in E \sqcup \{x\}. \blacksquare$$

1.2. Presimplicial module and a weak simplicial module. We follow here [Lod, Prz-1] and introduce the notion of a presimplicial and weak simplicial module. This will simplify our calculation and provide a language for visualization.

DEFINITION 1.7. A *weak simplicial module* (M_n, d_i, s_i) is a collection of R -modules M_n , $n \geq 0$, together with face maps, $d_i : M_n \rightarrow M_{n-1}$ and degenerate maps $s_i : M_n \rightarrow M_{n+1}$,

$0 \leq i \leq n$, which satisfy the following properties:

- (1) $d_i d_j = d_{j-1} d_i$ for $i < j$.
- (2) $s_i s_j = s_{j+1} s_i$, $0 \leq i \leq j \leq n$,
- (3) $d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ s_j d_{i-1} & \text{if } i > j + 1, \end{cases}$
- (4') $d_i s_i = d_{i+1} s_i$.

(M_n, d_i) satisfying (1) is called a *presimplicial module* and leads to the chain complex (M_n, ∂_n) with $\partial_n = \sum_{i=0}^n (-1)^i d_i$.

If (4') is replaced by a stronger condition

- (4) $d_i s_i = d_{i+1} s_i = \text{Id}_{M_n}$

then (M_n, d_i, s_i) is a (*classical*) *simplicial module*.

The following basic lemma will be used later:

LEMMA 1.8. *Let (M_n, d_i) be a presimplicial module then the map $d_0 d_0 : C_n \rightarrow C_{n-2}$ is a chain map, chain homotopic to zero. In particular, if $d_0 d_0 = 0$ then $(-1)^n d_0$ is a chain map.*

Proof. We think of $d_0 : C_n \rightarrow C_{n-1}$ as a chain homotopy and we have:

$$\begin{aligned} d_0 \partial_n + \partial_{n-1} d_0 &= d_0 \sum_{i=0}^n (-1)^i d_i + \sum_{i=0}^{n-1} (-1)^i d_i d_0 \\ &= d_0 d_0 + \sum_{i=1}^n (-1)^i (d_0 d_i - d_{i-1} d_0) \stackrel{(1)}{=} d_0 d_0. \end{aligned}$$

In particular, if $d_0 d_0 = 0$ we have $(-1)^n d_0 \partial_n = (-1)^{n-1} \partial_{n-1} d_0$. ■

For us it is important that rack and quandle homology can be described in the language of weak simplicial modules:

PROPOSITION 1.9 ([Prz-1]).

- (i) *Let $(X; *)$ be a rack, $C_n = ZX^n$, $d_i^{(*0)} : C_n \rightarrow C_{n-1}$ be given by $d_i^{(*0)}(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, and $d_i^{(*)} : C_n \rightarrow C_{n-1}$ be given by $d_i^{(*)}(x_1, \dots, x_n) = (x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n)$, and furthermore $d_i = d_i^{(*0)} - d_i^{(*)}$, then $(C_n(X), d_i^{(*0)})$, $(C_n(X), d_i^{(*)})$, and $(C_n(X), d_i)$ are presimplicial modules. (We have here shift by one comparing to Definition 1.7, that is we start from 1 not from 0, but it is not important in our considerations).*
- (ii) *Assume now that $(X; *)$ is a quandle and degeneracy maps $s_i : C_n(X) \rightarrow C_{n+1}(X)$ are given, as before, by $s_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_i, x_i, x_{i+1}, \dots, x_n)$. Then $(C_n(X), d_i^{(*0)}, s_i)$, $(C_n(X), d_i^{(*)}, s_i)$, and $(C_n(X), d_i, s_i)$ are weak simplicial modules.*

REMARK 1.10.

- (i) The homology related to $(C_n(X), d_i^{(*)})$ is called one term distributive homology and it is studied in [Prz-1, P-S, P-P-1, P-P-2, CPP].

(ii) We define the trivial quandle $(X; *_0)$ by $a *_0 b = a$. Then indeed we have

$$d_i^{(*_0)}(x_1, \dots, x_n) = (x_1 *_0 x_i, \dots, x_{i-1} *_0 x_i, x_{i+1}, \dots, x_n) \\ = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

(iii) Notice that $d_1^{(*)} = d_1^{(*_0)}$ so $d_1 = d_1^{(*_0)} - d_1^{(*)} = 0$, and this is the reason why we could start summation in Definition 1.2 from $i = 2$ (but ideologically it is better to start summation from $i = 1$).

(iv) Let $\gamma_n = d_0^{(*)} : C_n \rightarrow C_{n-1}$, that is $\gamma(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n)$, then $(-1)^n \gamma_n$ is a chain map in (C_n, ∂_n) , by Lemma 1.8 (e.g. [CJKS, N-P]).

1.3. Codimension 2 embedding, lower decker set. We introduce here, following [CKS-3, Kam-1], the language needed to define quandle colorings and (co)cycle invariants in codimension 2. Let $M = M^n$ be a closed smooth n -dimensional manifold and $f : M \rightarrow \mathbb{R}^{n+2}$ its smooth embedding which is called a smooth knotting (or just knotting). Define $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$ by $\pi(x_1, \dots, x_{n+1}, x_{n+2}) = (x_1, \dots, x_{n+1})$ to be a projection on the first $n + 1$ coordinates. The projection of the knotting is the set $M^* = \pi f(M)$. Crossing set (or singularity set) D^* of the knotting, is the closure in M^* of the set of all points $x^* \in M^*$ such that $(\pi f)^{-1}(x^*)$ contains at least two points, that is

$$D^* = \text{closure}(\{y \in \mathbb{R}^{n+1} \mid |\pi^{-1}(y) \cap M| > 1\}).$$

We define the double point set $D = (\pi f)^{-1}(D^*)$ (or sometimes as $(\pi)^{-1}(D^*)$ if we need it to be a subspace of \mathbb{R}^{n+2}). Let $f : M \rightarrow \mathbb{R}^{n+2}$ be a knotting which is in general position with respect to the projection $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$. The precise definition is in Section 6 (Definition 6.1), here we only use the basic notions:

D^* is $(n - 1)$ -dimensional stratified complex. Its $(n - 1)$ -dimensional strata consist of transverse double points (double point set strata Δ^1). The crossing set D^* divides $\pi f(M)$ into pieces. Each piece (connected component of $\pi f(M) - D^*$) is an open n -manifold embedded in \mathbb{R}^{n+1} consisting of regular points of $\pi f(M)$, which is called open regular sheet or n -semi-region of M^* . Regular sheets are 2-sided (even if we allow M to be nonorientable [Kam-1]).

The lower decker set D_- is the closure of the subset of pure double points which are lower in the projection (that is with respect to the last coordinate of \mathbb{R}^{n+2}). Similarly, the upper decker set D_+ is the closure of the subset of pure double points which are higher in the projection. M is cut by D_- into the set of n -dimensional regions (n -regions) denoted by \mathcal{R} , that is \mathcal{R} is the set of connected components of $M - D_-$ (notice that πf restricted to $M - D_-$ is an embedding and that the image of an n -region can contain several open regular sheets (semi-regions)).

The diagram D_M of a knotting with a general position projection is the knotting projection M^* together with “over under” information for the crossing set. In other words, it is M^* with D_+ and D_- given.

1.4. Quandle colorings, and quandle shadow coloring. We define here, after [CKS-3], the notion of quandle coloring and quandle shadow coloring of diagrams of knottings ([CKS-3] gives only definition in dimension $n \leq 2$ but generalization is natural and it is implicit in the work of Fenn, Rourke and Sanderson [F-R, FRS-2]). In [P-R] the

core coloring was considered for any n . We assume in the paper (unless otherwise stated) that the considered n -dimensional manifold M is oriented thus the normal orientation (co-orientation) of every open sheet of M^* is well defined⁶.

DEFINITION 1.11 (magma coloring). Fix a magma $(X; *)$. Let $f : M \rightarrow \mathbb{R}^{n+2}$ be an n -knotting, $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$ a regular projection, and D_M the knotting diagram. Let \mathcal{R} be the set of n -regions of M cut by lower decker set. We define a magma coloring of a diagram D_M (or a pair (M, π)) as a function $\phi : \mathcal{R} \rightarrow X$ satisfying the following condition: if R_1 and R_2 are two regions separated by n -dimensional upper decker region R_3 and the orientation normal to R_3 points from R_1 to R_2 , then $\phi(R_1) * \phi(R_3) = \phi(R_2)$; compare Fig. 1.3. Coloring of n -regions leads also to coloring of open sheets of the diagram D_M of the knotting. Through the paper we often refer to this as coloring of a knotting diagram. We denote by $\text{Col}_X(D_M)$ the set of colorings of D_M by X , and by $\text{col}_X(D_M)$ its cardinality. Note that the definition is not using any properties of $*$; only when we will demand invariance of $\text{col}_X(D_M)$ under various moves on D_M , we will need some specific properties of $*$.

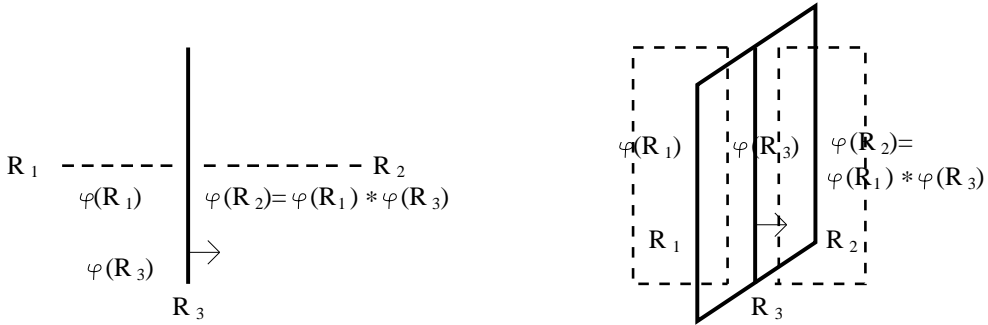


Fig. 1.3. Rules for magma (e.g. quandle) coloring for $n = 1, 2$

If we assume that $(X; *)$ is a shelf then the set $\text{Col}_X(D_M)$ is a right X -shelf-space with an action of X on $\text{Col}_X(D_M)$ given by $(\phi * x)(R) = \phi(R) * x$ for any region $R \in \mathcal{R}$. By the right self-distributivity of $*$ we have $(\phi * x)(R_1) * (\phi * x)(R_3) = (\phi(R_1) * x) * (\phi(R_3) * x) \stackrel{\text{distr}}{=} (\phi(R_1) * \phi(R_3)) * x = \phi(R_2) * x = (\phi * x)(R_2)$.

Before we define shadow coloring it is useful to notice that rack coloring of sheets of D_M allows unique coloring of any closed path in \mathbb{R}^{n+1} in a general position to D_M , as long as a base point is colored (Lemma 1.12). In a preparation for the lemma we need the following:

Fix a rack $(X, *)$ and an element $q_0 \in X$. Let $t_0 < t_1 < \dots < t_k < t_{k+1}$ be points on the line \mathbb{R} . Each point t_i ($1 \leq i \leq k$) is equipped with a ± 1 framing, according

⁶In the case of M unorientable, we can work with involutive quandle (kei) X and develop the theory of colorings and (co)-cycle invariants. I was informed by Seiichi Kamada (at Lafayette, November 7, 2013) that his symmetric quandle can be used to define fundamental coloring of a knotting $M^n \rightarrow \mathbb{R}^{n+2}$ with unorientable M^n and then it can be used to construct the (co)-cycle invariants stronger from that given by a kei. The symmetric quandle is a quandle with an involution $\rho : X \rightarrow X$ such that $\rho(x * y) = \rho(x) * y$ and $x * \rho(y) = x \bar{y}$. For a kei the involution is the identity [Kam-2].

to the convention: $\bullet \xrightarrow{+} \mathbf{t}_i$, $\mathbf{t}_i \xleftarrow{-} \bullet$. Then any function $\phi : \{t_1, \dots, t_k\} \rightarrow X$ extends uniquely to the function $\tilde{\phi} : [t_0, t_k] \rightarrow X$ with $\tilde{\phi}(t_0) = q_0$ by the following rule. If $a \in [t_{i-1}, t_i]$ and $b \in [t_i, t_{i+1}]$ then

$$\tilde{\phi}(b) = \begin{cases} \tilde{\phi}(a) * \phi(t_i) & \text{if the framing at } t_i \text{ is positive} \\ \tilde{\phi}(a) \bar{*} \phi(t_i) & \text{if the framing at } t_i \text{ is negative.} \end{cases}$$

In particular, $\tilde{\phi}(t_{k+1}) = (q_0 * \phi(t_1)) * \dots * \phi(t_k)$, where $*_i = *$ if the framing of t_i is positive and $*_i = \bar{*}$ if the framing of t_i is negative. Finally we can apply the above to an arc $\alpha : [t_0, t_{k+1}] \rightarrow \mathbb{R}^{n+1}$ in a general position with respect to D_M with some X -coloring ϕ , and which cuts D_M at k points $\alpha(t_1), \dots, \alpha(t_k)$. The framing of points t_i ($1 \leq i \leq k$) is yielded by co-orientation of D_M and points $\alpha(t_i)$. We can identify $\phi(t)$ with $\phi(\alpha(t_i))$, thus by above ϕ can be extended to the function $\tilde{\phi} : [t_0, t_{k+1}] \rightarrow X$. Now we are ready to prove that:

LEMMA 1.12. *If $\alpha : [t_0, t_{k+1}] \rightarrow \mathbb{R}^{n+1}$ is a closed path, that is $\alpha(t_0) = \alpha(t_{k+1})$, then $\tilde{\phi}(\alpha(t_0)) = \tilde{\phi}(\alpha(t_{k+1}))$.*

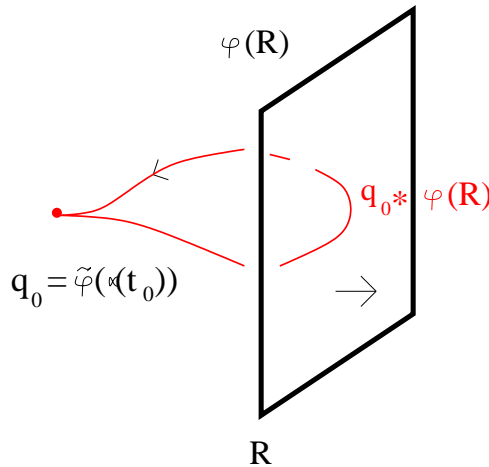


Fig. 1.4. A path moving for and back through the n -sheet $(q_0 * \phi(R)) \bar{*} \phi(R) = q_0$

Proof. Using the fact that \mathbb{R}^{n+1} is simple connected, we can contract α to a base point $\alpha(t_0)$, and we can put contracting homotopy in a general position with respect to D_M . The proof is by induction on the number of critical points of contracting homotopy. The critical points are either cancelling a piece of the path going forth and back, or crossing a double point strata. In the first case if we start from the color a and cross color b , forth and back, thus we get a color $(a * b) \bar{*} b$ or $(a \bar{*} b) * b$ which is a by invertibility of $*$. In the case when isotopy is crossing a double point set, we use the fact that double point crossing looks like classical crossing multiplied by \mathbb{R}^{n-1} , and the interesting case is when the closed path is below sheets it crosses. The situation can be illustrated by using classical crossing and coherence of coloring follows from right distributivity and invertibility of $*$, see Figures 1.4 and 1.5. ■

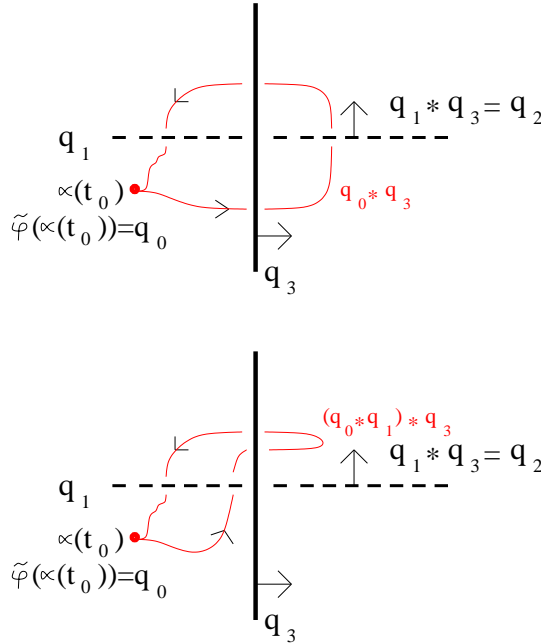


Fig. 1.5. A path is crossing double point stratum: $\left(\left(\left(q_0 * q_3\right) * \left(q_1 * q_3\right)\right) * q_3\right) * q_1 = \left(\left(\left(q_0 * q_1\right) * q_3\right) * q_3\right) * q_1 = \left(q_0 * q_1\right) * q_1 = q_0$

DEFINITION 1.13 (magma shadow coloring). A *shadow coloring* of a knotting diagram (extending the given coloring ϕ) is a function $\tilde{\phi} : \tilde{\mathcal{R}} \cup \mathcal{R} \rightarrow X$, where $\tilde{\mathcal{R}}$ is the set of $(n+1)$ -dimensional regions(chambers⁷) of $\mathbb{R}^{n+1} - \pi f(M)$ satisfying the following condition. If R_1 and R_2 are $n+1$ regions (chambers) separated by n -dimensional region (regular sheet) α where the orientation normal of α points from R_1 to R_2 , then $\tilde{\phi}(R_1) * \tilde{\phi}(\alpha) = \tilde{\phi}(R_2)$ and $\tilde{\phi}$ restricted to the set of n -dimensional regions is a given coloring ϕ (compare [CKS-3] and Fig. 1.6 for $n = 1$ or 2). Again the definition works for any binary operation but if $(X; *)$ is a rack, then any coloring ϕ and a constant q_0 chosen for a fixed $(n+1)$ -chamber, R_0 , yield the unique extension to shadow coloring $\tilde{\phi}$ so that $\tilde{\phi}(R_0) = q_0$; this follows from Lemma 1.12.⁸ We denote by $\text{Col}_{sh,X}(D_M)$ the set of all shadow colorings of (\mathbb{R}^{n+1}, D_M) by X and by $\text{col}_{sh,X}(D_M)$ its cardinality.

⁷We should appreciate here, not that accidental, analogy to Weyl chambers in representation theory of Lie algebras. Thus we use the term *chamber* throughout the paper.

⁸We can also see this important property, as follows: consider a small trivial circle T_c in a chosen chamber R_b of $\mathbb{R}^{n+1} - \pi f(M)$ let T be the boundary of a regular neighborhood of T_c , thus $T = S^1 \times S^{n-1}$. We can extend coloring ϕ to $M \cup T$ by coloring T by a fixed color q_0 . Then we isotope the circle T_c (and T) so that it is always below $f(M)$ part but it touches every chamber of $\mathbb{R}^{n+1} - \pi f(M)$. Now having initial chamber colored by q_0 any other chamber is now colored by an appropriate color of part of T in the chamber. Unlike in approach using Lemma 1.12, we use here the fact that Roseman moves on a diagram are preserving colorings by a rack (see Theorem 3.4).

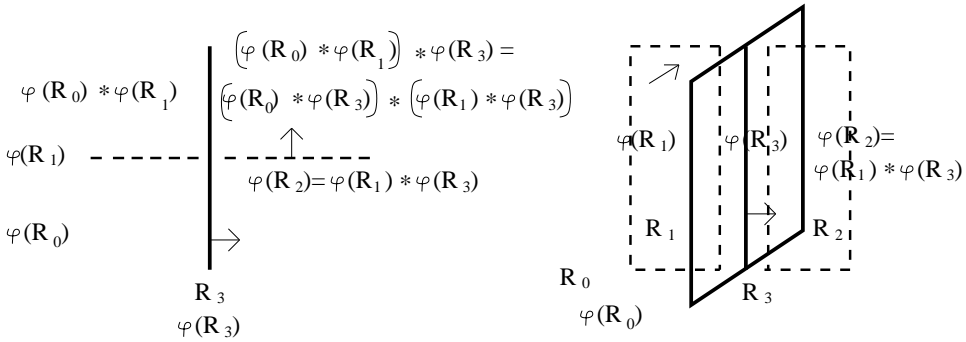


Fig. 1.6. Quandle shadow coloring for $n = 1, 2$

Similarly as in non-shadow case if we assume that $(X; *)$ is a shelf then the set $\text{Col}_{sh, X}(D_M)$ is a right X -shelf-space with an action of X on $\text{Col}_{sh, X}(D_M)$ given by $(\tilde{\phi} * x)(R) = \tilde{\phi}(R) * x$ for any region or chamber R .

REMARK 1.14. A shadow coloring of n -knotting diagram can be interpreted as a special case of coloring in dimension $n + 1$. To this aim we consider the $(n + 1)$ -dimensional manifold $\tilde{M} = (M^n \times \mathbb{R}) \sqcup \mathbb{R}^{n+1}$ embedded in \mathbb{R}^{n+3} as follows: Let $\tilde{f} : \tilde{M} \rightarrow \mathbb{R}^{n+3}$ with $\tilde{f}(m, x) = (f(m), x)$ and $\tilde{f}(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{n+1}, h, 0)$, where for $f(m) = (f_1(m), \dots, f_{n+2}(m))$ we assume $h \leq f_{n+2}(m)$ for any $m \in M^n$ (in other words, \mathbb{R}^{n+1} is embedded below M^n). The projection $\tilde{\pi} : \mathbb{R}^{n+3} \rightarrow \mathbb{R}^{n+2}$ is defined as $\pi \times \text{Id}$. We get the diagram $D_{\tilde{M}} = \tilde{\pi}\tilde{f}(\tilde{M}) = (D_M \times \mathbb{R}) \cup (\mathbb{R}^{n+1} \times \{0\})$ in \mathbb{R}^{n+2} . The points of multiplicity $n + 1$ in D_M gives rise to points of multiplicity $n + 2$ in $D_{\tilde{M}}$, and each shadow coloring of (\mathbb{R}^{n+1}, D_M) gives rise to a coloring of $D_{\tilde{M}}$ (also the (shadow) chain $c_{n+2}(D_M)$ gives rise to the chain $c_{n+2}(D_{\tilde{M}})$ as will be clear in the next subsection). As we do not use this remark later, we should not worry that the resulting manifold \tilde{M} is not compact (otherwise we need to consider knotting up to isotopy with compact support).

OBSERVATION 1.15. *As noted by S. Kamada, one can consider shadow coloring $\tilde{\phi}$ also in the case of $(X; *)$ a rack and E an X -rack-set. In this case we color the chambers of $\mathbb{R}^{n+1} - D_M$ by elements of E with a natural convention that if R_1 and R_2 are chambers separated by n -dimensional region α where the orientation normal of α points from R_1 to R_2 , then $\tilde{\phi}(R_1) * \tilde{\phi}(\alpha) = \tilde{\phi}(R_2)$. One can also show, using Lemma 1.12 or footnote 8, that if we take $\phi'(R_0) = q_0$ for some $q_0 \in E$ then $\tilde{\phi}$ is uniquely extendable from coloring of D_M . The reason is that in place of X and E we can consider a rack $X \sqcup E$ as in Observation 1.6(i).*

1.5. $(n + 1)$ - and $(n + 2)$ -chains of diagrams of knotting. We show in this part how to any knotting diagram D_M given by $\pi f : M^n \xrightarrow{f} \mathbb{R}^{n+2} \xrightarrow{\pi} \mathbb{R}^{n+1}$ and a shelf $(X; *)$ associate two chains of dimension $(n + 1)$, and $(n + 2)$ in rack (and quandle) chain groups $C_{n+1}^W(X)$ and $C_{n+2}^W(X)$ respectively ($W = R$ or D or Q , that is Rack, Degenerate or Quandle).

We start from the classical theory in dimensions $n = 1$.

Carter, Kamada, and Saito noticed in 1998 that if we color a classical oriented link diagram, D , by elements of a given quandle X and consider a sum over all crossings of D

of pairs in X^2 , $\pm(q_1, q_2)$ according to the convention of Fig. 1.7 then the sum has an interesting behavior under Reidemeister moves.

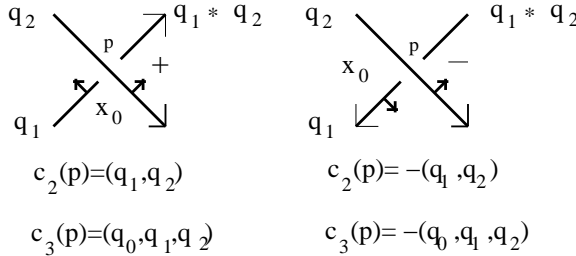


Fig. 1.7. The contribution to the 2-chain is (q_1, q_2) for a positive crossing and $-(q_1, q_2)$ for a negative crossing; in the case of the 3-chain for a shadow coloring, we have (q_0, q_1, q_2) and $-(q_0, q_1, q_2)$, respectively

This led them to define in 1998 a 2-cocycle invariant and relate it to rack homology defined between 1990 and 1995 by Fenn, Rourke, and Sanderson. Because the first Reidemeister move is changing the sum by $\pm(x, x)$ they were assuming that (x, x) should be equivalent to zero (so (x, x) should represent degenerate element). The 3-cocycle of the shadow X -colorings was motivated by [R-S] and developed in [CKS-1] (compare [CKS-3], page 154). In particular, they noted that the 3-chain constructed with convention of Fig. 1.7, that is $c_3(D) = \sum_{p \in \text{crossings}} \text{sgn}(p)c_3(p)$, is a 3-cycle and Reidemeister moves preserve homology class of $c_3(D)$.

We define, in this subsection, the chains $c_{n+1}(D_M, \phi)$ and $c_{n+2}(D_M, \tilde{\phi})$ for any diagram D_M of a knotting M and chosen colorings ϕ and $\tilde{\phi}$ (the fact that they are cycles is proven in Section 2, Theorem 2.1, and topological invariance of their homology via Roseman moves is proven in Section 4, Theorem 4.1). To make the general definition we need some conventions and notation concerning a crossing of multiplicity $(n + 1)$ in a diagram of an n -knotting.

The sign of a crossing of multiplicity $(n + 1)$ is chosen so that it agrees with the definition of the sign of a crossing in a classical knot theory. We say that the sign of p is positive if $n + 1$ normal vectors to $n + 1$ hyperplanes intersecting at p listed starting from the top (that is the normal vector to the highest hyperplane is first) form a positive orientation of \mathbb{R}^{n+1} ; otherwise the sign of p is equal to -1 . The source chamber R_0 of $\mathbb{R}^{n+1} - \pi f(M)$ adjacent to p is the region from which normals of hyperplanes points (compare page 151 of [CKS-3]).

DEFINITION 1.16 (($n+1$)-chain). Let $f : M \rightarrow \mathbb{R}^{n+2}$ be an n -knotting, $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$ a regular projection, and D_M the knotting diagram.

- (i) Fix a quandle X and a coloring $\phi : \mathcal{R} \rightarrow X$. Let p be a crossing of multiplicity $n + 1$ of D_M . We define a chain $c_{n+1}(p, \phi) \in C_{n+1}(X)$ by

$$c_{n+1}(p) = \text{sgn}(p)(q_1, \dots, q_{n+1}) \text{ where } (q_1, \dots, q_{n+1}) \text{ is obtained as follows:}$$

We consider the source region, say R_0 , around p and (q_1, \dots, q_{n+1}) are colors of hyperplanes intersecting at p around R_0 listed in the order of hyperplanes from the

lowest to the highest (see Fig. 1.8 for the case of $n = 2$). We often write $c_{n+1}(p)$ for $c_{n+1}(p, \phi)$ if ϕ is fixed.

- (ii) The chain associated to the diagram and fixed coloring is the sum of above chains taken over all crossings of multiplicity $n + 1$ of D_M :

$$c_{n+1}(D_M, \phi) = \sum_{p \in \text{crossings}} c_{n+1}(p, \phi).$$

- (iii) Finally, if X is finite, we sum over all X colorings of D_M so the result is in the group ring over $C_{n+1}(X)$ (in fact, it is in the group ring of $H_{n+1}(X)$ but this will be proven later). It is convenient here to use multiplicative notation for chains so that $c_{n+1}(D_M, \phi) = \prod_p (q_1, \dots, q_{n+1})^{\text{sgn}(p)}$ and then

$$c_{n+1}(D_M) = \sum_{\phi} c_{n+1}(D_M, \phi) = \sum_{\phi} \prod_p (q_1, \dots, q_{n+1})^{\text{sgn}(p)}.$$

- (iv) If X is possibly infinite, in place of a sum we consider the set with multiplicity

$$c_{n+1}^{\text{set}}(D_M) = \{c_{n+1}(D_M, \phi)\}_{\phi \in \text{Col}_X(D_M)}.$$

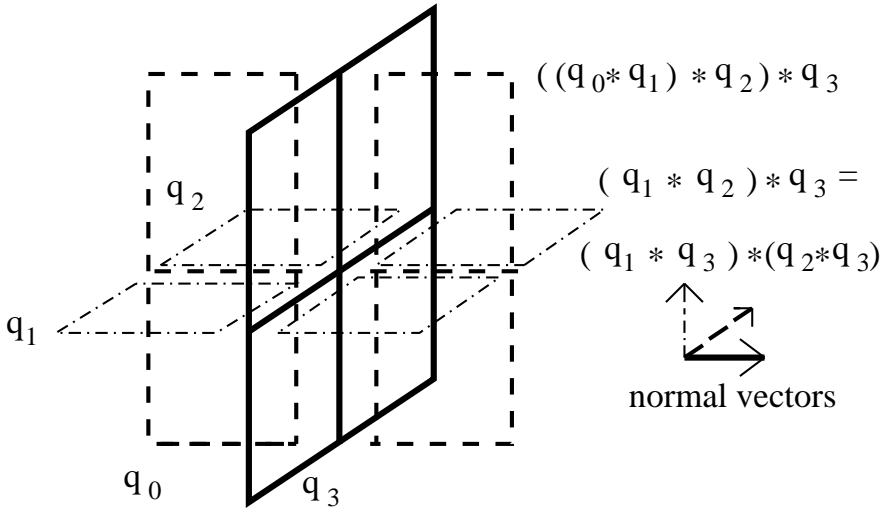


Fig. 1.8. Multiplicity three point in M^3 knotting and quandle coloring; the point yields the 3-chain (q_1, q_2, q_3) and the 4-chain (q_0, q_1, q_2, q_3) ; normal vectors yield here positive orientation

DEFINITION 1.17 ($(n + 2)$ -(shadow) chain for D_M). Here we generalize the previous definition to construct $(n + 2)$ -chains from shadow colorings related to link diagram. We color not only regions of M (as in Definitions 1.11, 1.16) but also $(n + 1)$ -chambers of \mathbb{R}^{n+1} cut by $\pi f(M)$. As before we take the product of signed chains associated to every multiplicity $(n + 1)$ -crossing point, p and sum these products over all shadow colorings. Thus we start from $c_{n+2}(p) = \text{sgn}(p)(q_0, q_1, \dots, q_{n+1})$, where q_0 is the color of the source chamber R_0 . In effect, if X is finite then

$$c_{n+2}(D_M) = \sum_{\tilde{\phi}} c_{n+2}(D_M, \tilde{\phi}) = \sum_{\tilde{\phi}} \prod_p (q_0, q_1, \dots, q_{n+1})^{\text{sgn}(p)}.$$

If X is possibly infinite, in place of a sum we consider the set with multiplicity

$$c_{n+2}^{\text{set}}(D_M) = \{c_{n+2}(D_M, \tilde{\phi})\}_{\tilde{\phi} \in \text{Col}_{sh, X}(D_M)}.$$

We illustrate the case of $n = 1$ in Fig. 1.7, and the case of $n = 2$ in Fig. 1.8.

For M which is not connected we can use a trick of [CENS] to have more delicate chains. They behave nicely under Roseman moves but they are not cycles. Thus we can use them to produce cocycle invariants but not cycle invariants of knottings (Theorem 4.8(iii)).

DEFINITION 1.18. Let $M = M_1 \cup M_2 \cup \dots \cup M_k$. We define an $(n + 1)$ -chain $c_{n+1}(D_M, \phi, i)$ by considering only those crossings of multiplicity $(n + 1)$ whose bottom sheet belongs to M_i . We denote the set of such $(n + 1)$ -crossings by \mathcal{T}_i . Then we define

$$c_{n+1}(D_M, \phi, i) = \sum_{p \in \mathcal{T}_i} c_{n+1}(p, \phi).$$

2. $(n + 1)$ - and $(n + 2)$ -cycles for n -knotting diagrams. We show in this section the two chains $c_{n+1}(D_M, \phi)$ and $c_{n+2}(D_M, \tilde{\phi})$ constructed in Subsection 1.5 are, in fact, cycles.

THEOREM 2.1. *The chains $c_{n+1}(D_M, \phi)$ and $c_{n+2}(D_M, \tilde{\phi})$ are cycles in $C_{n+1}^Q(X)$ and $C_{n+2}^Q(X)$ respectively.*

The main idea of the proof is to analyze points of multiplicity $n + 1$ and $n + 2$ in $M^* \in \mathbb{R}^{n+1}$, and associated $(n + 1)$ - and $(n + 2)$ -chains in $\mathbb{Z}X^{n+1}$ and $\mathbb{Z}X^{n+2}$, and to identify face maps of the chains, $d_i^{(*0)}$ and $d_i^{(*)}$, as associated to arcs of multiplicity n . Then by Roseman theory (see Section 6), such an arc starts at a point of multiplicity $n + 1$ (say p_1) and ends either at another point of multiplicity $n + 1$ (say p_2) or at a singular point (of multiplicity less than n). In the first case, we prove that there are proper cancellations of face maps associated to the arc. In the second case (which may happen for $n > 1$), the colorings of the arc give degenerate chains which can be ignored in $C_{n+1}^Q(X)$ and $C_{n+2}^Q(X)$. Details are given in the following subsections, starting from the classical case of $n = 1$.

2.1. Shadow 3-cycle for $n = 1$. We start from the known case of $n = 1$ but present our proof in a way which will allow natural generalization for any n . We show in detail in this subsection that the 3-chain constructed with convention of Fig. 1.7, that is $c_3(D) = \sum_{p \in \text{crossings}} \text{sgn}(p)c_3(p)$, is a 3-cycle.

We start from 2 crossings p_1 and p_2 connected by an arc colored by the pair $q_{\text{sh}}(\text{arc}) = (q_0, b)$ in our convention (that is the color of the arc is b and the shadow color of the source region (chamber) close to the arc is q_0 ; see Definition 2.2). In our examples the horizontal line is first above and then below the other arcs, and it will be denoted as pair of type $(2, 1)$ later in generalization. For a reader who would like to visualize here the general case, we stress that the arc connecting crossings p_1 and p_2 will be an arc of points of multiplicity n (intersection of n hyperplanes in \mathbb{R}^{n+1}) connecting points of multiplicity $n + 1$.

Now consider our four cases.

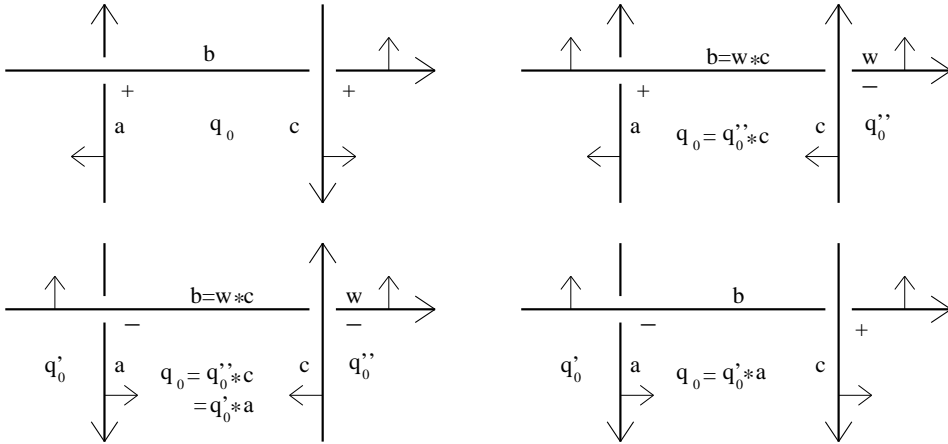


Fig. 2.1. In all cases the connecting arc has label $q_{sh}(arc)$ equal to (q_0, b) ; the multi-labelling q_{sq} is explained in full generality in Definition 2.2.

To describe precisely the outcome of our pictures we denote the shadow 3-chain of our diagram by $c_3(D)$, the contribution of the first crossing p_1 by $c_3(p_1)$, the contribution of the second crossing, p_2 , by $c_3(p_2)$, and $c_3(p_1, p_2) = c_3(p_1) + c_3(p_2)$ denotes the contribution of both crossings. We also use the notation of Fig. 2.2 which is the special case of Definition 2.2.

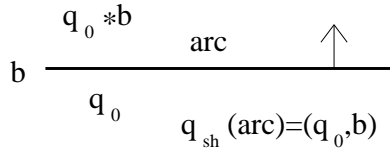


Fig. 2.2. Convention for arc coloring, $q_{sh}(arc) = (q_0, b)$

Thus in the first case we have

$$c_3(p_1) = (q_0, a, b), \quad c_3(p_2) = (q_0, b, c), \quad \text{and} \quad c_3(p_1, p_2) = (q_0, a, b) + (q_0, b, c).$$

We have then

$$d_2^{(*0)}(c_3(p_1)) = d_2^{(*0)}(q_0, a, b) = d_3^{(*0)}(q_0, b, c) = (q_0, b) = q_{sh}(arc).$$

Thus proper pieces of $\partial(p_1)$ and $\partial(p_2)$ cancel out in $c_3(p_1, p_2)$. This will be the case always, and below we shortly analyze other cases:

In the second case we have

$$c_3(p_1) = (q_0, a, b), \quad c_3(p_2) = -(q''_0, w, c), \quad \text{and} \quad c_3(p_1, p_2) = (q_0, a, b) - (q''_0, w, c),$$

where $b = w * c$. Then

$$d_2^{(*0)}(q_0, a, b) = d_3^{(*)}(q''_0, w, c) = (q_0, b) = q_{sh}(arc).$$

In the third case

$$c_3(p_1) = -(q'_0, a, b), \quad c_3(p_2) = -(q''_0, w, c), \quad \text{and} \quad c_3(p_1, p_2) = -(q'_0, a, b) - (q''_0, w, c).$$

Then

$$d_2^*(q'_0, a, b) = d_3^*(q''_0, w, c) = (q_0, b) = q_{sh}(arc).$$

Finally, in the fourth case

$$c_3(p_1) = -(q'_0, a, b), \quad c_3(p_2) = +(q_0, b, c), \quad \text{and} \quad c_3(p_1, p_2) = -(q'_0, a, b) + (q_0, b, c).$$

Then

$$d_2^{(*)}(q'_0, a, b) = d_3^{(*)}(q_0, b, c) = (q_0, b) = q_{\text{sh}}(\text{arc}).$$

This proves that $c_3(D)$ is a cycle in $C_3(X)$, as $\partial = \partial^{(*)_0} - \partial^{(*)} = \sum_{i=2}^3 (-1)^i d_i^{(*)_0} - \sum_{i=2}^3 (-1)^i d_i^{(*)}$. We did not consider all cases as we will argue in the general case that all cases follows at once, however we illustrate one more case, of type (2, 2), that is the horizontal line is above both crossings. We have

$$c_3(p_1) = -(q'_0, a, b), \quad c_3(p_2) = -(q_0, c, b), \quad \text{and} \quad c_3(p_1, p_2) = -(q'_0, a, b) - (q_0, c, b).$$

Then:

$$d_2^{(*)}(q'_0, a, b) = d_2^{(*)_0}(q_0, c, b) = (q_0, b) = q_{\text{sh}}(\text{arc}); \quad \text{as illustrated in Fig. 2.3.}$$

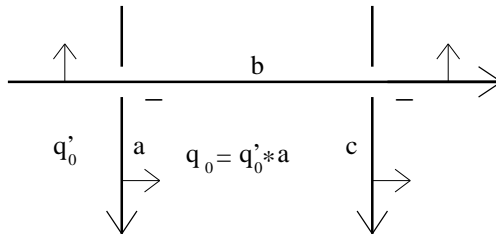


Fig. 2.3. Two crossings of type (2, 2)

We also can work with a tangle diagram T in place of a link diagram D ; then $c_3(T)$ is not an absolute cycle, but we can work in the setting of relative chain (of $(T, \partial T)$). We do not follow this idea here but it may be useful in many situations.

2.2. The general case of $(n + 1)$ -and $(n + 2)$ -cycles. For the general case we need a notation for coloring of strata of a neighborhood of a crossing of multiplicity $(n + 1)$ in \mathbb{R}^{n+1} generalizing coloring and shadow coloring.

For a given vector w in \mathbb{R}^{n+1} let V_w be an $(n + 1)$ -dimensional linear subspace orthogonal to w . For basic vectors $e_1 = (1, 0, \dots, 0), \dots, e_i = (0, \dots, 0, 1, 0, \dots, 0), \dots, e_{n+1} = (0, \dots, 0, 1)$, /we write V_i for V_{e_i} . We have $\bigcap_{i=1}^{n+1} V_i = (0, \dots, 0) = p$ and it is our “model” singularity (crossing of multiplicity $n + 1$). In a standard way we associate to this crossing the signum $+1$. If our system of hypersurfaces $\bigcup V_i$ is a part of a knotting diagram, the sign $+1$ would correspond to the situation $V_1 > V_2 > \dots > V_{n+1}$, that is V_i above V_{i+1} at the crossing. However, our convention, motivated by right self-distributivity, makes more convenient assumption $V_1 < V_2 < \dots < V_{n+1}$ and then with our convention the crossing p has the signum $\text{sgn}(p) = (-1)^{n(n+1)/2}$.

We introduce two labellings (generalizing colorings and shadow colorings):

- (i) $q : \bigcup V_i \rightarrow X \cup X^2 \cup \dots \cup X^{n+1}$;
- (ii) $q_{\text{sh}} : \mathbb{R}^{n+1} \rightarrow X \cup X^2 \cup \dots \cup X^{n+2}$.

The strata of the labelling of a point x depend on the “order of singularity”, that is, $q(x)$ (respectively $q_{\text{sh}}(x)$) is an element of X^k (respectively X^{k+1}), where X is a fixed shelf

and $k = k(x)$ is the number of hyperplanes V_i to which x belongs (if $x \in \mathbb{R}^{n+1} - \bigcup V_i$ then $k(x) = 0$). Both colorings are coherent because of right self-distributivity law in X . The idea of coloring is that we choose a color, say q_0 , for a source region, R_0 , of $\mathbb{R}^{n+1} - \bigcup_{i=1}^{n+1} V_i$ (this will be part of q_{sh} coloring, that is $q_{sh}(x) = q_0$ for $x \in R_0$). Furthermore, we choose colors q_1, q_2, \dots, q_{n+1} and if $x \in R_i^{(s)}$ where $R_i^{(s)}$ is the source sheet of V_i , then $q(x) = q_i$ and $q_{sh}(x) = (q_0, q_i) \in X^2$.

With an assumption that V_i is always below V_{i+1} in our considerations. we propagate our colors according to our rules of coloring. Notice that we use only $*$ (never $\bar{*}$) in our coloring, so assumption that X is a shelf suffices here.

We describe this idea formally below.

DEFINITION 2.2. Choose a shelf X and $n + 2$ elements $(q_0, q_1, \dots, q_n, q_{n+1})$ in X .

- (i) For $x = (x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} - \bigcup V_i$ the label $q_{sh}(x)$ is defined to be $q_0 * q_{i_1} * \dots * q_{i_s}$ where $0 < i_1 < \dots < i_s$ are precisely these indexes for which $x_{i_j} > 1$. In particular, if for all $i, x_i < 0$ (i.e. x is in the source sector), then $q_{sh}(x) = q_0$.
- (ii) If $x = (x_1, \dots, x_n, x_{n+1})$ has only one coordinate, say i -th, equal to zero (that is $x \in V_i$ but $x \notin V_j$ for $j \neq i$) then $q(x)$ is defined to be $q_i * q_{i_1} * \dots * q_{i_s}$ where $i < i_1 < \dots < i_s$ are precisely these indexes for which $i_j > i$ and $x_{i_j} > 1$. In particular, if for all j such that $j > i$ we have $x_j < 0$, then $q(x) = q_i$.
- (iii) Let $x = (x_1, \dots, x_n, x_{n+1})$ belong to exactly k hyperplanes, $x \in \bigcap_{j=1}^k V_{i_j}$, then $q(x) = (q^{(i_1)}, \dots, q^{(i_k)}) \in X^k$, where $q^{i_j}(x) = q(x^{i_j})$ where x^{i_j} is obtained from x by replacing all coordinates equal to 0, but x_{i_j} , by -1 (recall that $x_{i_1} = x_{i_2} = \dots = x_{i_k} = 0$ and other coordinates are different from 0).
- (iv) If $x \in \bigcup V_i$ then $q_{sh}(x)$ is obtained from $q(x)$ by $q_{sh}(x) = (q_{sh}(x'), q(x))$ where x' is obtained from $x = (x_1, \dots, x_n)$ by replacing all 0 in the sequence by -1 (i.e. x' is a point in a source chamber). In particular, if $q(x) = q_i$ then $q_{sh}(x) = (q_0, q_i)$.

We can complete with this notation, the proof that $c_{n+1}(D_M)$ and $c_{n+2}(D_M)$ are cycles.

Very schematic visualization of the general case is shown in Fig. 2.4 where the vertical lines represent n -dimensional sheets of $M^* = f\pi(M^n)$ with source colors a and c respectively and horizontal line representing the line of intersection of n sheets (in \mathbb{R}^{n+1}) with the coloring $b = q(\text{arc}) = (q_1, \dots, q_n)$.

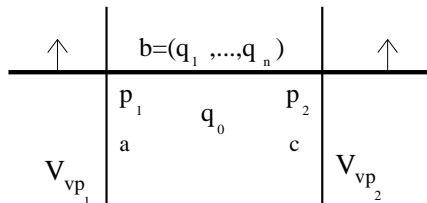


Fig. 2.4. Connecting arc has label $q_{sh}(\text{arc}) = (q_0, q(\text{arc})) = (q_0, b) = (q_0, q_1, \dots, q_n)$

We denote the position of the first vertical sheet as $i(p_1)$ and the second vertical sheet by $i(p_2)$. The $n + 2$ chains corresponding to p_1 and p_2 depend on direction of vertical vectors, n_{p_1} and n_{p_2} to vertical sheets V_{vp_1} and V_{vp_2} . We write $\epsilon(p_1, p_2) = 1$ if

the vector n_{p_1} points from p_1 to p_2 and 0 otherwise. Similarly $\epsilon(p_2, p_1) = 1$ if the vector n_{p_2} points from p_2 to p_1 , and it is 0 otherwise. This notation is used to identify operation $*_0$ and $* = *_1$. Then we have:

$$\begin{aligned} & d_{i(p_1)}^{(*\epsilon(p_1, p_2))}(q_0, q_1, \dots, q_{i(p_1)-1}, a, q_{i(p_1)}, \dots, q_n) \\ &= d_{i(p_2)}^{(*\epsilon(p_2, p_1))}(q_0, q_1, \dots, q_{i(p_2)-1}, b, q_{i(p_2)}, \dots, q_n). \end{aligned}$$

Thus $\text{sgn}(p_1)(-1)^{i(p_1)} = -\text{sgn}(p_2)(-1)^{i(p_2)}$ and in $c_{n+2}(p_1, p_2)$ the terms $d_{i(p_1)}^{(*\epsilon(p_1, p_2))}$ and $d_{i(p_2)}^{(*\epsilon(p_2, p_1))}$ cancel out. In conclusion, the $(n + 2)$ -chain $c_{n+2}(D_M, \tilde{\phi})$ is a cycle.

The similar proof works for $c_{n+1}(D_M, \phi)$. The fact that $c_{n+1}(D_M, \phi)$ is a cycle follows also from the following observation:

OBSERVATION 2.3. *Consider the map $\gamma_n : C_n(X) \rightarrow C_{n-1}(X)$ given by cutting the first coordinate, that is, $\gamma_n(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n)$. Then by Lemma 1.8 (compare Remark 1.10(iv)) the map $(-1)^n \gamma$ is a chain map for $\partial^{(*_0)}$ and $\partial^{(*)}$ thus also for $\partial = \partial^{(*_0)} - \partial^{(*)}$. From definitions of c_{n+1} and c_{n+2} , we have $c_{n+1}(D_M, \phi) = \gamma_{n+2} c_{n+2}(D_M, \tilde{\phi})$. From this we conclude that if $c_{n+2}(D_M, \tilde{\phi})$ is an $(n + 2)$ -cycle then $c_{n+1}(D_M, \phi)$ is an $(n + 1)$ -cycle.*

In the next two sections we show that our cycles (and their sums) are topological invariants. We will start from the fact that the spaces of colorings (and shadow colorings) are topological invariants.

3. Topological invariance of colorings. The logic of the section is as follows: We introduce here, following [Joy-2, F-R], two definitions of a fundamental rack or quandle of a knotting, the abstract one and the concrete definition. The abstract definition of the fundamental rack or quandle of a knotting is independent of any projection in the similar way, as the fundamental group. In the concrete definition, for a given projection, we get the concrete presentation of a fundamental rack or quandle from the diagram using generators and relations in a way reminiscent of the Wirtinger presentation of the fundamental group of a classical link complement. It was observed in [F-R, FRS-2] that these definitions are equivalent using a general position argument. As a consequence we see that a concrete rack and quandle colorings are topological invariants (independent of a diagram) because in both cases abstract and concrete colorings are obtained from a homomorphism from the fundamental object to X .

3.1. The fundamental rack and quandle of an n -knotting. The first definition we give uses a knotting diagram. We follow Definitions 1.11 and 1.13, except that in place of concrete chosen $(X; *)$ we build a universal (called fundamental) object (magma, shelf, rack or quandle).

DEFINITION 3.1 (fundamental magma of a knotting diagram). Let $f : M \rightarrow \mathbb{R}^{n+2}$ be a knotting, $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$ a regular projection and D_M related knot diagram.

- (i) The *fundamental magma* $X(D_M) = (X; *)$ is given by the following finite presentation. The generators of X are in bijection with the set of regions of M cut by lower

decker set. Relations in $(X; *)$ are given as follows: if R_1 and R_2 are two regions separated by n -dimensional upper decker region R_3 (with variables, respectively, q_1, q_2 and q_3) and the orientation normal to R_3 points from R_1 to R_2 , then $q_1 * q_3 = q_2$.⁹

- (ii) If $(X; *)$ is required to be a shelf we get a *fundamental shelf* of D_M .
- (iii) If $(X; *)$ is required to be a rack we get a *fundamental rack* of D_M .
- (iv) If $(X; *)$ is required to be a quandle we get a *fundamental quandle* of D_M .
- (v) Fix a quandle $(X; *)$ and the knotting $f : M \rightarrow \mathbb{R}^{n+2}$, with regular projection $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$. The *quandle* (resp. *rack*) *coloring* of a diagram D_M is a quandle homomorphism from the fundamental quandle (resp. rack) $X(D_M)$ to X .

The quandle (or rack) coloring described in Definition 3.1 is equivalent to Definition 1.11.

The fundamental shadow magma, shelf, rack and quandle of a knotting diagram are defined analogously to that of coloring; we give a full definition so it is easy to refer to it.

DEFINITION 3.2 (fundamental shadow magma of a knotting diagram). Let $f : M \rightarrow \mathbb{R}^{n+2}$ be a knotting, $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$ a regular projection and D_M related knot diagram.

- (i) The *fundamental shadow magma* $X_{\text{sh}}(D_M) = (X; *)$ is given by the following finite presentation. The generators of X are in bijection with the set $R \cup R_{\text{cha}}$ where R is the set of regions of M cut by lower decker set and R_{cha} is the set of chambers of $\mathbb{R}^{n+1} - \pi f(M)$. Relations in $(X; *)$ are given as follows: if R_1 and R_2 are two regions separated by n -dimensional upper decker region R_3 (with variables, respectively, q_1, q_2 and q_3) and the orientation normal to R_3 points from R_1 to R_2 , then $q_1 * q_2 = q_3$. Furthermore, if \tilde{R}_1 and \tilde{R}_2 are $n + 1$ chambers separated by n -dimensional region α where the orientation normal of α points from \tilde{R}_1 to \tilde{R}_2 , and $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$ are colors of \tilde{R}_1, \tilde{R}_2 and α , respectively, then $\tilde{q}_1 * \tilde{q}_3 = \tilde{q}_2$.¹⁰
- (ii) If $(X; *)$ is required to be a shelf we get a *fundamental shelf* of D_M .
- (iii) If $(X; *)$ is required to be a rack we get a *fundamental rack* of D_M .
- (iv) If $(X; *)$ is required to be a quandle we get a *fundamental quandle* of D_M .
- (v) Fix a quandle (or a rack) $(X; *)$ and the knotting $f : M \rightarrow \mathbb{R}^{n+2}$, with regular projection $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$. The *quandle* (resp. *rack*) *shadow coloring* of a diagram D_M is a quandle homomorphism from the fundamental quandle (resp. rack) $X_{\text{sh}}(D_M)$ to X .

Again, the quandle (or rack) shadow coloring described in Definition 3.2 is equivalent to Definition 1.13.

REMARK 3.3. If we assume that $X(D_M)$ is the fundamental rack (or quandle) of a diagram D_M , then the presentation of the fundamental shadow rack $X_{\text{sh}}(D_M)$ can be obtained by adding one generator and no new relations (except that of rack (or quandle)

⁹Notice that however our definition of $X(D_M)$ is not using any properties of $*$ but still for $n > 1$ and a projection with a triple crossing point, colors involved in the crossing satisfy right self-distributivity (see Fig. 1.8 where $(q_1 * q_2) * q_3 = (q_1 * q_3) * (q_2 * q_3)$).

¹⁰Notice that however our definition of $X_{\text{sh}}(D_M)$ is not using any properties of $*$ but still if there is a double crossing in the projection then colors involved in the crossing satisfy right self-distributivity (see Fig. 1.6).

relations). That is: $X_{\text{sh}}(D_M) = \{X(D_M), w \mid \}$. The new generator w is a color of an arbitrary, but fixed, chamber of $\mathbb{R}^{n+1} - D_M$. The presentation can be justified using Lemma 1.12 or by a method described in footnote 8 to Definition 1.13.

We recall in the next subsection a projection free approach to the fundamental rack and quandle of a knotting and use it to notice that $X(D_M)$ and $X_{\text{sh}}(D_M)$ are independent of the concrete diagram.

3.2. Abstract definitions of a fundamental rack and quandle. Joyce, Fenn, and Rourke [Joy-2, F-R] gave an abstract definition of the fundamental rack of a knotting, independent of a projection and they noted that it is equivalent to the concrete definition given in Subsection 3.1.

We follow here [F-R] in full generality, however we are concerned mostly with the case of the ambient manifold $W = \mathbb{R}^{n+2}$.

(i) Let $L : M \rightarrow W$ be a knotting (codimension two embedding). We shall assume that the embedding is proper at the boundary if $\partial M \neq \emptyset$, that W is connected and that M is transversely oriented in W . In other words we assume that each normal disk to M in W has an orientation which is locally and globally coherent. The link is said to be *framed* if there is given cross section (called framing) $\lambda : M \rightarrow \partial N(M)$ of the normal disk bundle (the total space of the bundle is a tubular neighborhood of $L(M)$ in W). Denote by M^+ the image of M under λ . We call M^+ the parallel manifold to M .

(ii) We consider homotopy classes Γ of paths in $W_0 = \text{closure}(W - N(M))$ from a point in M^+ to a base point. During the homotopy the final point of the path at the base point is kept fixed and the initial point is allowed to wander at will in M^+ .

(iii) The set Γ is a right $\pi_1(W_0)$ -group-set, that is the fundamental group of a knotting complement acts on Γ as follows: let γ be a loop in W_0 representing an element g of the fundamental group. If $a \in \Gamma$ is represented by the path α then define $a \cdot g$ to be the class of the composition path $\alpha \circ \gamma$. We can use this action to define a rack structure on Γ . Let $p \in M^+$ be a point on the framing image. Then p lies on a unique meridian circle of the normal circle bundle. Let m_p be the loop based at p which follows round the meridian in a positive direction. Let $a, b \in \Gamma$ be represented by the paths α, β respectively. Let $\partial(b)$ be the element of the fundamental group determined by the loop $\bar{\beta} \circ m_\beta \circ \beta$. (Here $\bar{\beta}$ represents the reverse path to β and m_β is an abbreviation for $m_{\beta(0)}$ the meridian at the initial point of β .) The *fundamental rack* of the framed link L is defined to be the set $\Gamma = \Gamma(L)$ of homotopy classes of paths as above with operation

$$a * b = a \cdot \partial(b) = [\alpha \circ \bar{\beta} \circ m_\beta \circ \beta].$$

(iv) A rack coloring, by a given rack $(X; *)$ is a rack homomorphism $f : \Gamma(L) \rightarrow X$. In a case of $W = \mathbb{R}^{n+2}$ we give also down to earth definition from the link projection (initially depending on the projection) (see Definition 1.11).

(v) If L is an unframed link then we can define its *fundamental quandle*: let $\Gamma_q = \Gamma_q(L)$ be the set of homotopy classes of paths from the boundary of the regular neighborhood $(N(M))$ to the base point where the initial point is allowed to wander during the course of the homotopy over the whole boundary. The rack structure on $\Gamma_q(L)$ is

defined similarly to that of $\Gamma(L)$. Thus the fundamental quandle of L is the quotient of the fundamental rack of L by relations generated by idempotency $x * x = x$.

(vi) A quandle coloring, by a given quandle $(X; *)$, is a quandle homomorphism $f : \Gamma_q(L) \rightarrow X$.

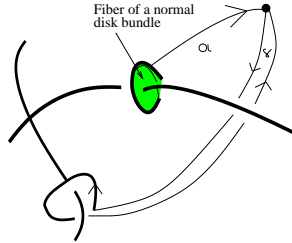


Fig. 3.1. Composition $a \cdot g$ where a is a class of an arc α and g is a class of a loop γ

For $W = \mathbb{R}^{n+2}$ (or S^{n+2}) our two definitions of the fundamental rack (or quandle) are equivalent. If D_M is a diagram of a knotting $L : M \rightarrow \mathbb{R}^{n+1}$ with the regular projection $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$ then we have a natural epimorphism $F_q : X_q(D_M) \rightarrow \Gamma_q(L)$ given as follows: Let x_H be a generator of $X_q(D_M)$ corresponding to a sheet (region) H of D_M . Choose a base point of $\mathbb{R}^{n+1} - \pi L(M)$ very high (call it ∞) and project it to a point of H cutting ∂V_M at some point m_H , then we define $F_q(x_H)$ to be the class of a straight line from m_H to ∞ . Similarly we define a rack epimorphism $F : X(D_M) \rightarrow \Gamma(L)$, by extending $F_q(x_H)$ by starting from the point of M^+ being on the same fiber disk of V_M as m_H and connecting along the boundary of the disk to m_H .

THEOREM 3.4 ([F-R, FRS-2]).

- (i) *Two definitions of a fundamental rack (resp. quandle) of a (framed) n -link $L : M \rightarrow \mathbb{R}^{n+2}$ coincide, the map $F : X(D_M) \rightarrow \Gamma(L)$ is a quandle isomorphism. In particular, Definition 3.1 (and equivalent 1.11) for racks and quandles are independent of regular projection¹¹ and give a finite presentation of the fundamental rack $\Gamma(L)$ (resp. quandle $\Gamma_q(L)$).*
- (ii) *The fundamental shadow rack (resp. quandle) is independent of regular projection thus it is well defined for a knotting n -link $L : M \rightarrow \mathbb{R}^{n+2}$; we denote it by $\Gamma^{sh}(L)$ (resp. $\Gamma_q^{sh}(L)$).*
- (iii) *The sets (X -quandle-sets) $\text{Col}_X(D_M)$ and $\text{Col}_{X,sh}(D_M)$ do not depend on the diagram of a given linking M .*

Proof. The statement and a sketch of a proof is given in [F-R] (Remarks (2) p. 375¹²) and [FRS-2] (Lemma 3.4; p. 718). One can give also a proof using Roseman moves (starting with the pass move $S(c, n + 2, 0)$ discussed in detail later in this paper).

¹¹In quandle case we consider knottings up to (smooth) ambient isotopy (equivalently up to Roseman moves), and in a rack case up to framed (smooth) ambient isotopy.

¹²Fenn and Rourke write in Remarks (2): *A similar analysis can be carried out for an embedding of M^n in S^{n+2} : we obtain a “diagram” by projecting onto \mathbb{R}^{n+1} in general position and regarding top dimensional strata (n -dimensional sheets) as “arcs” to be labelled by generators and $(n - 1)$ -dimensional strata (simple double manifolds) as “crossings” to be labelled by relations. In general position a homotopy between paths only crosses the $(n - 1)$ -strata and a proof along the lines of the theorem can be given that this determines a finite presentation of the fundamental rack.*

We also can conclude that the set $X_{\text{sh}}(D_M)$ (in fact, X quandle set) of shadow colorings is a topological invariant. We use here Remark 3.3, namely, as $X(D_M)$ is a topological invariant, so is $X_{\text{sh}}(D_M) = \{X(D_M), w \mid \}$. ■

REMARK 3.5. Theorem 3.4 should be understood as follows: If R is a Roseman move on a diagram of n -knotting D_M resulting in RD_M , then there are natural X -rack isomorphisms (i.e. bijections preserving right action by X), $R_{\#} : \text{Col}_X(D_M) \rightarrow \text{Col}_X(RD_M)$ and $\tilde{R}_{\#} : \text{Col}_{X,sh}(D_M) \rightarrow \text{Col}_{X,sh}(RD_M)$. Natural means here that outside a ball (tangle) in which the move R takes place, the bijections $R_{\#}$ and $\tilde{R}_{\#}$ are identity. This raises an interesting question: assume that after using a finite number of Roseman moves we come back to the diagram D_M . What automorphism of X -quandle-sets $\text{Col}_X(D_M)$ and $\text{Col}_{X,sh}(D_M)$ we performed? Is it always an inner automorphism¹³. When it is the identity?

COROLLARY 3.6. *The homology, $H_*^W(X(D_M))$ and $H_*^W(X_{\text{sh}}(D_M))$ of the fundamental rack and the fundamental shadow rack are topological knotting invariants. M. Eisermann proved that in the classical case a knot is nontrivial if and only if $H_2^Q(X(K)) = \mathbb{Z}$, [Eis-1]. We can ask what we can say in a general case about $H_{n+1}^Q(X(D_M))$.*

The presentation of $X(D_M)$ gives the coloring of D_M by the quandle $X(D_M)$; we call this the fundamental coloring and denote by ϕ_{fund} . Similarly, The presentation of $X_{\text{sh}}(D_M)$ gives the shadow coloring of D_M by the quandle $X_{\text{sh}}(D_M)$; we call this the fundamental shadow coloring and denote by $\tilde{\phi}_{\text{fund}}(D_M)$.

COROLLARY 3.7. *The homology classes of cycles $c_{n+1}(D_M, \phi_{\text{fund}})$ ($c_{n+2}(D_M, \tilde{\phi}_{\text{fund}})$, resp.) are knotting invariants up to isomorphism of homology groups generated by an automorphism of a fundamental quandle (fundamental shadow quandle, resp.). M. Eisermann proved that in the classical case, the homology class of the fundamental cycle¹⁴ of a nontrivial knot is a generator of $H_2^Q(X(K)) = \mathbb{Z}$, [Eis-1].*

4. Roseman moves are preserving homology classes of fundamental cycles.

We deal in this section with the main result of our paper, about (co)cycle invariants of knottings. We start by describing precisely the case of a pass move, R , (generalization of the third Reidemeister move), that is a move of type $S(c, n + 2, 0)$ in notation of [Ros-1, Ros-2]; see Section 6 (we write $R \in S(c, n + 2, 0)$).

4.1. Colorings and homology under Roseman moves. Fix a quandle $(X; *)$. We already established bijection, for any Roseman move R between sets of colorings of D_M and the set of colorings of RD_M , Theorem 3.4. We denote this bijection by $R_{\#}$, so $R_{\#} : \text{Col}_X(D_M) \rightarrow \text{Col}_X(RD_M)$. In fact, $R_{\#}$ is X -quandle-sets isomorphism, that is it preserves the right multiplication by elements of X (i.e. $R_{\#}(\phi * x) = R_{\#}(\phi) * x$).

¹³If we change a base point in the definition of the fundamental rack we make an inner automorphism on it (i.e. generated by X action) reflecting similarity with fundamental group), [Joy-2].

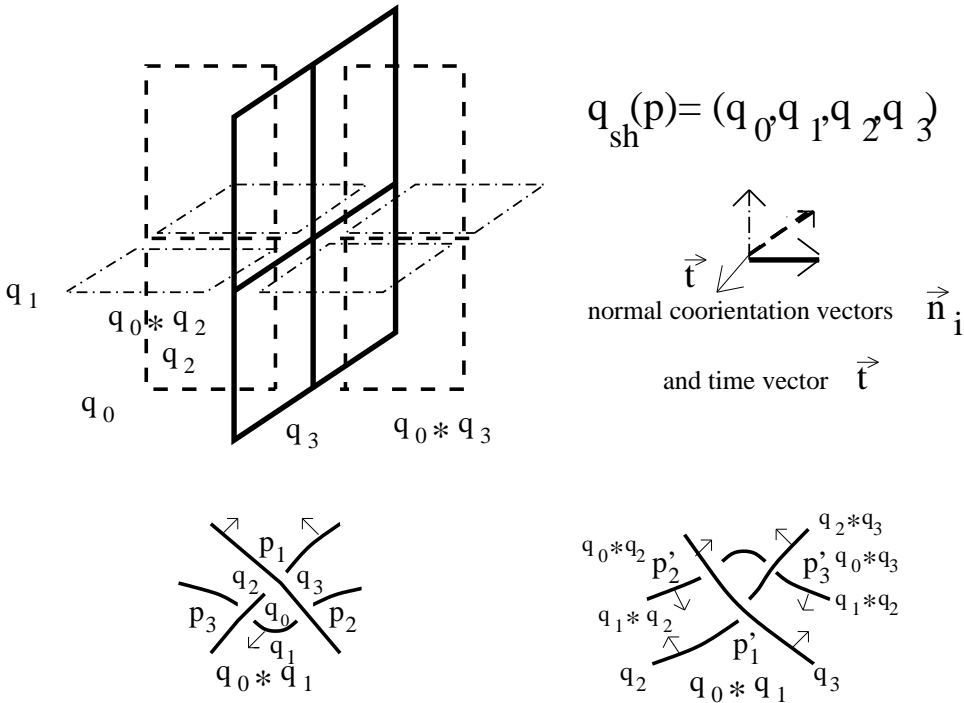
¹⁴This class is called in [Eis-1] the *orientation class* of K .

THEOREM 4.1. *For a fixed $\phi \in \text{Col}_X(D_M)$ the cycles $c_{n+1}(D_M, \phi)$, $c_{n+1}(RD_M, R_\#(\phi))$ are homologous in $H_{n+1}^Q(X)$. Similarly for $\tilde{\phi} \in \text{Col}_{X,sh}(D_M)$, the cycles $c_{n+2}(D_M, \tilde{\phi})$, $c_{n+2}(RD_M, R_\#(\tilde{\phi}))$ are homologous in $H_{n+2}^Q(X)$.*

The main, and, as we see later, essentially the only one nontrivial to check is the pass move R of type $S(c, n + 2, 0)$.

LEMMA 4.2. *The pass move $R \in S(c, n + 2, 0)$ preserves the homology class of $c_{n+1}(L)$ and $c_{n+2}(L)$. To be precise, let ϕ be a fixed coloring of D_M and $R_\#(\phi)$ the corresponding coloring of RD_M then the cycles $c_{n+1}(D_M, \phi)$ and $c_{n+1}(RD_M, R_\#(\phi))$ are homologous in $H_{n+1}^Q(X)$. Similarly for $\tilde{\phi} \in \text{Col}_{sh,X}(D_M)$, the cycles $c_{n+2}(D_M, \tilde{\phi})$, $c_{n+2}(RD_M, R_\#(\tilde{\phi}))$ are homologous in $H_{n+2}^Q(X)$.*

Proof. We show the result for all possible types of pass moves (including all possible co-orientation of sheets) at once. We start from $n + 2$ sheets (hypersurfaces) in \mathbb{R}^{n+2} with arbitrary co-orientation, intersecting in a point p , and the direction of time \vec{t} in general position to co-orientation vectors.



Projections before and after move

Fig. 4.1. From isotopy to pass move ($n = 1$ case)
 $c_3(p_1, p_2, p_3) = (q_0, q_2, q_3) - (q_0, q_1, q_3) + (q_0, q_1, q_2) = \partial^{(*)0}(q_0, q_1, q_2, q_3)$
 $c_3(p'_1, p'_2, p'_3) = (q_0 * q_1, q_2, q_3) - (q_0 * q_2, q_1 * q_2, q_3) + (q_0 * q_3, q_1 * q_3, q_2 * q_3) = \partial^{(*)1}(q_0, q_1, q_2, q_3)$

Before we give technical details, we first use a simple visualization of our proof:

On each side of the move, say for $t = -1$ and $t = 1$, we have $n + 2$ crossings (p_1, \dots, p_{n+2}) and (p'_1, \dots, p'_{n+2}) respectively. Each crossing represents the intersection

of $n + 1$ sheets in \mathbb{R}^{n+1} , that is, p_i is the intersection of all sheets V_1, \dots, V_{n+2} but V_i at $t = -1$, and p'_i is the intersection of all sheets but V_i at $t = 1$. Furthermore, we have $\text{sgn}(p_i) = \text{sgn}(p'_i)$ for any $i \geq 1$. We concentrate on the case of a shadow coloring $\tilde{\phi}$ of D_M (the non-shadow case being similar). The weights associated to p_i and p'_i are $\text{sgn}(p_i)d_i^{(*0)} q_{\text{sh}}(p)$ and $\text{sgn}(p'_i)d_i^{(*)} q_{\text{sh}}(p)$ (the order depends on co-orientation \vec{n}_i of V_i). Furthermore, the sign of p_i is $(-1)^{n-i} \text{sgn}(\vec{n}_i \cdot \vec{t})$ that is the sign depend on whether \vec{n}_i agrees or disagrees with \vec{t} (that is the scalar product $\vec{n}_i \cdot \vec{t}$ is positive or negative). In effect $c_{n+2}(p_i) - c_{n+2}(p'_i) = \epsilon(-1)^{-i}(d_i^{(*0)} - d_i^{(*)})(q_{\text{sh}}(p)$, where $\epsilon = \pm 1$, and in effect $c_{n+2}(D_M) - c_{n+2}(RD_M) = \pm \partial_{n+3}(q_{\text{sh}}(p))$. Therefore $c_{n+2}(D_M) - c_{n+2}(RD_M)$ is homol-
ogous to zero in $H_{n+2}(X)$ as needed¹⁵. Similarly $c_{n+1}(D_M, \phi) - c_{n+1}(RD_M, R_{\#}(\phi)) = \pm \partial_{n+2}(q(p))$. This follows also directly by using Observation 2.3. ■

Proof of Theorem 4.1. Now we can complete the proof of Theorem 4.1.

We use Roseman moves in more substantial way than before. Details of Roseman theory are given in Section 6 where we follow [Ros-1, Ros-2, Ros-3]. Here we give a short description referring often to that section. D. Roseman proved that for any n there are a finite number of moves on link diagrams in \mathbb{R}^{n+1} so that if two diagrams F_1^n and F_2^n represent ambient isotopic links in \mathbb{R}^{n+2} then our diagrams are related by a finite number of Roseman moves. For $n = 1, 2$ and 3 , the moves of Roseman were explicitly given (for $n = 1$ these are classical Reidemeister moves).

We are showing that any Roseman move is preserving the homology class of $c_{n+1}(D_M)$ and $c_{n+2}(D_M)$. Because only crossings of multiplicity $n + 1$ are contributing to cycles $c_{n+1}(D_M)$ and $c_{n+2}(D_M)$, thus it suffices to consider only those Roseman moves which involve singularities of multiplicity $n + 1$ before or after the move. A precise definition of Roseman moves and their properties is given in Section 6 and here we need only the fact that there are exactly three types of moves of interest:

- (i) A move of type $S(c, n + 2, 0)$ which we analyzed in Lemma 4.2 called the pass move or maximal crossing move or the generalized third Reidemeister move.
- (ii) A move of type $S(c, n + 1, 0)$ (or its inverse a move of type $S(c, n + 1, 1)$). This move describes two cancelling crossing points of a knotting diagram and can be called the generalized second Reidemeister move (in the case of $n = 2, 3$ they are moves (e) in [Ros-1]).

In the isotopy the arc of points of multiplicity $n+1$ joins these crossing points, and they have opposite signs. Furthermore, up to sign, these crossings have the same contributions to $c_{n+1}(D_M, \phi)$ (and $c_{n+2}(D_M, \tilde{\phi})$). Thus in the state sum of Definitions 1.16 and 1.17 they do cancel.

¹⁵We were informed by Scott Carter that this observation was crucial in the definition of rack homology by Fenn, Rourke and Sanderson. In particular, the relation of the generalized Reidemeister move can be read from the boundary of singularity of one dimension higher. We deal then with a point \hat{p} of multiplicity $n + 2$ and we choose any time vector \vec{t} in general position to normal vectors of $(n + 1)$ -dimensional hyperplanes. We shadow color neighborhood of \hat{p} so that $q_{\text{sh}}(\hat{p}) = (q_0, q_1, \dots, q_{n+2})$. Then we analyze face maps $d_i^{*0}(q_{\text{sh}}(\hat{p}))$ and $d_i^*(q_{\text{sh}}(\hat{p}))$ and recognize q_{sh} of points p_1, \dots, p_{n+2} and p'_1, \dots, p'_{n+2} at cross section at $t = -1$ and $t = 1$. Fig. 4.1 illustrates it for $n = 1$.

Notice that we can interpret our situation as a special case of considerations in Subsection 2.2 (consider Fig. 2.4 with $V_{vp_1} = -V_{vp_2}$ and $a = c$ as describing the knotting diagram before the move).

(iii) A move of type $S(m, (1, n-1), 0, p)$ (with $p = 0$ or 1), where one side of the isotopy has a point of multiplicity $n + 1$ (compare the move (f) in the case of M^2 in \mathbb{R}^4 and the move (l) in the case of M^3 in \mathbb{R}^5 [Ros-1]).¹⁶ Then the branch set B is the boundary of the lower decker set D_- so the lower decker set does not separate the regions; thus both sides of this set have the same color. Therefore the chains corresponding to the crossing of multiplicity $n + 1$ are degenerate in $C_{n+1}^R(X)$ and $C_{n+2}^R(X)$, thus these chains do not contribute to quandle homology $H_{n+1}^Q(X)$ and $H_{n+2}^Q(X)$, respectively.

This completes our proof of Theorem 4.1. ■

If $M = M_1 \cup M_2 \cup \dots \cup M_k$ we can generalize Theorem 4.1 for a non-shadow coloring of D_M (we use notation of Definition 1.18). Our proof of Theorem 4.1 also works in this case.

COROLLARY 4.3. *For a fixed $\phi \in \text{Col}_X(D_M)$ and a Roseman move R the difference of chains before and after the move, $c_{n+1}(D_M, \phi, i) - c_{n+1}(RD_M, R_\#(\phi, i))$ is a boundary (so homologically trivial).*

4.2. Cycle invariants of knottings. To obtain invariants of knottings using Theorem 4.1 we can either sum over all colorings of the cycles $c_{n+1}(D_M, \phi)$ or take them as a set with multiplicity (in order not to lose an information that some colorings have the same cycle):

THEOREM 4.4. *Let $(X; *)$ be a fixed quandle, $f : M \rightarrow \mathbb{R}^{n+2}$ be an n -knotting, $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$ a regular projection, and D_M the knotting diagram. We use the notation $[c]$ for a homology class of a cycle c .*

- (i) *Let $[c_{n+1}(D_M), \phi]$ denote the homology class of the cycle $c_{n+1}(D_M, \phi)$. For a finite X the state sum, defined below*

$$[c_{n+1}(D_M)] = \sum_{\phi \in \text{Col}_X(D_M, \phi)} [c_{n+1}(D_M), \phi]$$

in the group ring $ZH_{n+1}(X)$ is a topological invariant of a knotting M . Thus we can denote this invariant by $c_{n+1}(M)$ and call it the (non-shadow) cycle invariant of a knotting $f : M \rightarrow \mathbb{R}^{n+2}$ (or shortly of M).

- (ii) *The reduced (non-shadow) cycle invariant of the knotting $f : M \rightarrow \mathbb{R}^{n+2}$*

$$c_{n+1}^{\text{red}}(M) = [c_{n+1}^{\text{red}}(D_M)] = \sum_{\phi \in \text{Col}_{\text{red}, X}(D_M)} [c_{n+1}(D_M, \phi)]$$

*is a topological invariant. Notation is explained as follows. We sum here over smaller number of colorings using the fact that set of colorings $\text{Col}_X(D_M)$ is an X -quandle-set and, as proven in Observation 1.5, $c_{n+1}(D_M, \phi)$ is homologous to $c_{n+1}(D_M, \phi * x)$, for any $x \in X$. Thus we take $\text{Col}_{\text{red}, X}(D_M)$ to be the subset of all*

¹⁶There is a misprint in [Ros-1] page 353; it should be $S(m, (1, 1), 0, 0)$ or $S(m, (1, 1), 0, 1)$ in place of $S(m, (1, 2), 0, 0)$ or $S(m, (1, 2), 0, 1)$.

X coloring, one coloring from every orbit. Even if we can have various choices for $\text{Col}_{\text{red},X}(D_M)$ the resulting $c_{n+1}^{\text{red}}(M) = [c_{n+1}^{\text{red}}(D_M)]$ is well defined.

- (iii) Let $[c_{n+2}(D_M), \tilde{\phi}]$ denote the homology class of the cycle $c_{n+2}(D_M, \tilde{\phi})$. For a finite X the state sum

$$[c_{n+2}(D_M)] = \sum_{\tilde{\phi} \in \text{Col}_{\text{sh},X}(D_M)} [c_{n+2}(D_M, \tilde{\phi})]$$

in the group ring $ZH_{n+2}(X)$ is a topological invariant of a knotting M . Thus we denote this invariant by $c_{n+2}(M)$ and call it the shadow cycle invariant of a knotting M .

- (iv) The reduced shadow cycle invariant of the knotting M is a topological invariant

$$c_{n+2}^{\text{red}}(M) = [c_{n+2}^{\text{red}}(D_M)] = \sum_{\tilde{\phi} \in \text{Col}_{\text{red,sh},X}(D_M)} [c_{n+2}(D_M, \tilde{\phi})].$$

- (v) If in any sum of (i)–(iv) we replace sum by a set with multiplicity, we obtain topological invariants, $c_{n+1}^{\text{set}}(M)$, $c_{n+1}^{\text{red,set}}(M)$, $c_{n+2}^{\text{set}}(M)$, and $c_{n+2}^{\text{red,set}}(M)$ respectively (we allow X to be infinite here).

4.3. Cocycle invariants of knottings. In this subsection we reformulate our main result in the language of cocycles and cohomology. For a fixed quandle $(X; *)$ and fixed cocycles in $C^{n+1}(X, A)$ and $C^{n+2}(X, A)$ we obtain directly cocycle invariants of n -knotting. It generalizes the case of $n = 1, 2$ (see [CKS-3] for a summary), and the case of $n = 3$ checked in [Rosi-2]. We start from the definition which involves diagrams.

DEFINITION 4.5. Let $(X; *)$ be a fixed quandle, $f : M \rightarrow \mathbb{R}^{n+2}$ an n -knotting, $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$ a regular projection, and D_M the knotting diagram.

- (i) For a fixed coloring $\phi \in \text{Col}_X(D_M)$ and $(n + 1)$ -cocycle $\Phi : \mathbb{Z}X^{n+1} \rightarrow A$ we define the value $\Phi(D_M, \phi) \in A$ by

$$\Phi(D_M, \phi) = \Phi(c_{n+1}(D_M, \phi)) = \sum_p \Phi(c_{n+1}(p, \phi)),$$

where $\Phi(c_{n+1}(p, \phi))$ is a Boltzmann weight of the crossing p of multiplicity $n + 1$, and the sum is taken over all crossings of D_M .

- (ii) We can also take into account the fact that M is not necessary connected (following [CENS, CKS-3] in the case $n = 1$). That is if $M = M_1 \cup M_2 \cup \dots \cup M_k$, we can take the sum from (i) not over all crossings p of multiplicity $n + 1$ but only those which have M_i on the bottom of the crossings. Let us denote such a set of crossings by \mathcal{T}_i . Then we define

$$\Phi(D_M, \phi, i) = \sum_{p \in \mathcal{T}_i} \Phi(c_{n+1}(p, \phi))$$

- (iii) For a fixed shadow coloring $\tilde{\phi} \in \text{Col}_{X,\text{sh}}(D(M))$ and $(n+2)$ -cocycle $\tilde{\Phi} : \mathbb{Z}X^{n+1} \rightarrow A$, we define the value $\tilde{\Phi}(D_M, \tilde{\phi}) \in A$ by the formula

$$\tilde{\Phi}(D_M, \tilde{\phi}) = \tilde{\Phi}(c_{n+2}(D_M, \tilde{\phi})) = \sum_p \tilde{\Phi}(c_{n+2}(p, \tilde{\phi})).$$

THEOREM 4.6 (Cocycle invariants). Consider a knotting $f : M \rightarrow \mathbb{R}^{n+2}$ and fix a quandle $(X; *)$ and $(n + 1)$ -cocycle $\Phi : \mathbb{Z}X^{n+1} \rightarrow A$ and $(n + 2)$ -cocycle $\tilde{\Phi} : \mathbb{Z}X^{n+2} \rightarrow A$.

- (i) For a fixed colorings $\phi \in \text{Col}_X(D_M)$, the element $\Phi_X(D_M, \phi) \in A$ is preserved by any Roseman move R that is $\Phi_X(D_M, \phi) = \Phi_X(RD_M, R_\#(\phi))$ in A .
- (ii) If $M = M_1 \cup M_2 \cup \dots \cup M_k$ then the conclusion of (i) holds also for $\Phi(D_M, \phi, i)$, that is $\Phi_X(D_M, \phi, i) = \Phi_X(RD_M, R_\#(\phi), i)$ in A .
- (iii) For a fixed shadow coloring $\tilde{\phi} \in \text{Col}_{X,sh}(D_M)$, the element $\tilde{\Phi}_X(D_M, \tilde{\phi}) \in A$ is preserved by any Roseman move, that is, $\tilde{\Phi}_X(D_M, \tilde{\phi}) = \tilde{\Phi}_X(RD_M, R_\#(\tilde{\phi}))$ in A .

Proof. Theorem 4.6 follows directly from the analogous result for homology. We should stress that we do not need here the property that $c_{n+1}(D_M, \phi, i)$ are cycles, as we can evaluate a cocycle on any chain. Furthermore, we can work also with tangles not only with knotting diagrams. ■

To produce invariant of a knotting we should make our invariants of diagram independent of a choice of a coloring (and use Theorems 4.4 and 4.6). As before, two natural solutions are to take a set with multiplicity of invariants over all colorings or, for finite X , sum invariants over all colorings as usually is done in statistical mechanics.

DEFINITION 4.7. Let $(X; *)$ be a fixed quandle, $f : M \rightarrow \mathbb{R}^{n+2}$ be an n -knotting, $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$ a regular projection, and D_M the knotting diagram.

(1) Let Φ be a fixed cocycle in $C_Q^{n+1}(X)$. Then:

$$(i) \quad \Phi_X(D_M) = \sum_{\phi \in \text{Col}_X(D_M)} \Phi_X(D_M, \phi) = \sum_{\phi} \prod_p \Phi(c_{n+1}(p, \phi)),$$

where X is a finite quandle. Here we have classical cocycle invariant in ZA , written as a state sum (A in a multiplication notation) and generalizing cocycle invariants of [CKS-3].

$$(ii) \quad \Phi_X(D_M, i) = \sum_{\phi \in \text{Col}_X(D_M)} \Phi_X(D_M, \phi, i),$$

where $M = M_1 \cup M_2 \cup \dots \cup M_k$ and we consider only the crossings which have M_i on the bottom of the crossings. For $n = 1, 2$ this invariant of M with ordered components was described in [CENS, CKS-3].

$$(iii) \quad \Phi_X^{\text{red}}(D_M) = \sum_{\phi \in \text{Col}_{X,\text{red}}(D_M)} \Phi_X(D_M, \phi).$$

Here we use the fact that X acts on $\text{Col}_X(D_M)$ and we can choose in the sum one representative from any orbit. Any choice is good (see Observation 1.5) and we write for the chosen subset $\text{Col}_{X,\text{red}}(D_M)$,¹⁷

$$(iv) \quad \Phi_X^{\text{red}}(D_M, i) = \sum_{\phi \in \text{Col}_{X,\text{red}}(D_M)} \Phi_X(D_M, \phi, i);$$

here we reduce crossings as in (ii) and colorings as in (iii).

¹⁷For example, for the trivial knotting $S^n \subset \mathbb{R}^{n+2}$, finite X and any cocycle, we have $\Phi_X(S^2) = |X| \cdot 1$ and $\Phi_X^{\text{red}}(S^2) = |\mathcal{O}_r| \cdot 1$, where \mathcal{O}_r is the set of orbit of the action of X on X on the right. Furthermore, in our notation 1 is a zero of an abelian group written multiplicatively, and $|\mathcal{O}_r| \cdot 1$ is an element of a group ring $\mathbb{Z}(\mathbb{Z}X^{n+1})$.

(v) Without restriction to finite X we can repeat all definitions of (i)–(iv) by considering, in place of the sum over colorings, the set of invariants indexed by colorings (thus we have a set with multiplicities, or better cardinalities of elements if X is infinite). We get $\Phi_{\text{set}}(D_M)$, $\Phi_{\text{set}}(D_M, i)$, $\Phi_{\text{set}}^{\text{red}}(D_M)$, and $\Phi_{\text{set}}^{\text{red}}(D_M, i)$, respectively.

(2) For a fixed $(n + 2)$ -cocycle $\tilde{\Phi} : \mathbb{Z}X^{n+2} \rightarrow A$, we define:

$$(i) \quad \tilde{\Phi}_X(D_M) = \sum_{\tilde{\phi} \in \text{Col}_{X,sh}(D_M)} \tilde{\Phi}_X(D_M, \tilde{\phi}) = \sum_{\tilde{\phi}} \prod_p \tilde{\Phi}(c_{n+2}(p, \tilde{\phi})),$$

where X is a finite quandle. Here we have classical shadow cocycle invariant in $\mathbb{Z}A$.

$$(ii) \quad \tilde{\Phi}_{X,\text{red}}(D_M) = \sum_{\tilde{\phi} \in \tilde{\text{Col}}_{X,\text{red}}(D_M)} \tilde{\Phi}_{X,\text{red}}(D_M, \tilde{\phi}).$$

(iii) Without restriction on X to be finite, we can repeat definitions of (i) and (ii) by considering, in place of the sum over colorings, the set with multiplicity of invariants indexed by colorings. We get $\tilde{\Phi}_{\text{set}}(D_M)$ and $\tilde{\Phi}_{\text{set},\text{red}}(D_M)$.

THEOREM 4.8 (cocycle invariants). *Consider a knotting $f : M \rightarrow \mathbb{R}^{n+2}$ and for a fixed quandle $(X; *)$, quandle cocycles $\Phi : \mathbb{Z}X^{n+1} \rightarrow A$ and $\tilde{\Phi} : \mathbb{Z}X^{n+2} \rightarrow A$. Then:*

- (1) *If X is finite then $\Phi_X(M)$, $\Phi_X^{\text{red}}(M)$, $\tilde{\Phi}_X(M)$, and $\tilde{\Phi}_X^{\text{red}}(M)$ are topological invariants of the knotting (i.e. independent of a diagram, invariant under Roseman moves). They are called cocycle, and shadow cocycle invariants of a knotting M .*
- (2) *For any X , $\Phi_{\text{set}}(M)$, $\Phi_{\text{set}}^{\text{red}}(M)$, $\tilde{\Phi}_{\text{set}}(M)$, and $\tilde{\Phi}_{\text{set}}^{\text{red}}(M)$ are topological invariants of the knotting M .*
- (3) *If $M = M_1 \cup \dots \cup M_n$ and X is finite then $\Phi_X(M, i)$ and $\Phi_X^{\text{red}}(M, i)$ are topological invariants of the knotting M with ordered components. Similarly, for any X the sets with multiplicity $\Phi_X^{\text{set}}(M, i)$ and $\Phi_{X,\text{red}}^{\text{set}}(M, i)$ are topological invariants of the knotting M with ordered components.*
- (4) *We can make invariants of (3) to be independent of the order of components if we take the set with multiplicity of invariants over all i .*

Proof. It follows directly from Theorems 4.4 and 4.6. Notice here that for shadow coloring the idea of considering $M = M_1 \cup M_2 \cup \dots \cup M_k$ and only crossings where M_i is on the bottom will not work as $d_2^{(*0)}$ usually differs from $d_2^{(*)}$, thus $c_{n+2}^{\text{red}}(p_i) - c_{n+2}^{\text{red}}(p'_i)$ is not necessary homological to zero.. In the non-shadow case we only needed $d_1^{(*0)} = d_1^{(*)}$. ■

5. Twisted (co)cycle invariants of knottings. Twisted homology (and cohomology) was introduced in [CENS]. Most of the results of the paper generalize, without many changes, to the twisted case so we give a concise explanation.

DEFINITION 5.1.

- (i) The twisted chain complex of a shelf $(X; *)$ is given by the chain modules $C_n^T(X) = \mathbb{Z}[t^{\pm 1}]X^n$ (that is, free modules with basis X^n and with coefficients in a ring of Laurent polynomials in variable t), and the chain map $\partial^T = t\partial^{(*0)} - \partial^{(*)}$. Recall

that

$$\partial^{(*0)}(x_1, \dots, x_n) = \sum_{i=1}^n (-1)^i (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

$$\partial^{(*)}(x_1, \dots, x_n) = \sum_{i=1}^n (-1)^i (x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n).$$

- (ii) If $(X; *)$ is a spindle (e.g. a quandle) we define as in the untwisted case the degenerate and quandle homology. Thus as before we consider $H_n^{TW}(X)$ for $W = R, D$ and Q .
- (iii) The cohomology $H_{TW}^n(X, A)$ is defined in a standard way with A being an $Z[t^{\pm 1}]$ -module.

The theory of cocycle invariants was introduced in [CES-1] for $n = 1, 2$. We give definition for any n -knotting below. Our description follows [CKS-3], the important tool we use is the classical Alexander numbering of chambers in $(\mathbb{R}^{n+1}, \pi f(M))$ (see [CKS-0, CKS-3]). Our version of the definition refers to shadow colorings by an (extended) shift rack structure on integers with infinity $(\mathbb{Z} \cup \infty; *_s)$ where $a *_s b = a + 1$ (in particular $\infty *_s b = \infty$).

DEFINITION 5.2.

- (i) Let X be a set and $f : X \rightarrow X$ a bijection with a fixed point b . We define a rack $(X; *_f)$ by $a *_f u = f(a)$, where $u \in X$. Then for a given knotting diagram D_M the shadow rack coloring of chambers of the knottings, is called the generalized Alexander numbering. More precisely, we color regions of the diagram trivially by b , choose one chamber and color it by an element of $X - b$ and the resulting shadow coloring of chambers is a generalized Alexander numbering (compare Observation 1.6(ii)).
- (ii) The Alexander numbering of chambers (e.g. [CKS-3]) starts from the rack $(\mathbb{Z} \cup \infty; *_s)$ and the unbounded chamber is colored by 0.

DEFINITION 5.3 (twisted chains of knotting). Let $f : M \rightarrow \mathbb{R}^{n+2}$ be an n -knotting, $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$ a regular projection, and D_M the knotting diagram. Furthermore, fix a rack or quandle X , a coloring $\phi : \mathcal{R} \rightarrow X$, and a shadow coloring $\tilde{\phi}$.

- (i) If p is a crossing of multiplicity $(n + 1)$ then we define the chain (twisted Boltzmann weight) associated to p as $c_{n+1}^T(p, \phi) = t^{-k(R_0)} c_{n+1}(p, \phi)$, where $k(R_0)$ is the Alexander numbering of the source region in the neighborhood of p and $c_{n+1}(p, \phi)$ is the untwisted Boltzmann weight.
- (ii) In the case $\tilde{\phi}$ is the shadow coloring we define a twisted shadow Boltzmann weight by $c_{n+2}^T(p, \tilde{\phi}) = t^{-k(R_0)} c_{n+2}(p, \tilde{\phi})$, where $c_{n+2}(p, \tilde{\phi})$ is the untwisted shadow Boltzmann weight associated to p .
- (iii) The twisted chain associated to the diagram D_M is

$$c_{n+1}^T(D_M, \phi) = \sum_{p \in \text{crossings}} c_{n+1}^T(p, \phi).$$

- (iv) Finally, we sum over all X colorings of D_M so the result is in the group ring over $C_{n+1}^T(X)$ (in fact, it is in the group ring of $H_{n+1}^T(X)$). It is convenient to use multi-

plicative notation for chains so that $c_{n+1}^T(D_M, \phi) = (\prod_p(q_1, \dots, q_{n+1})^{\text{sgn}(p)})^{t^{-k(R_0)}}$ and then

$$c_{n+1}^T(D_M) = \sum_{\phi} c_{n+1}^T(D_M, \phi).$$

- (v) We define the twisted shadow chain associated to the diagram D_M in an analogous manner:

$$c_{n+2}^T(D_M, \tilde{\phi}) = \sum_{p \in \text{crossings}} c_{n+2}^T(p, \tilde{\phi}).$$

Then we sum over all colorings to get

$$c_{n+2}^T(D_M) = \sum_{\tilde{\phi}} c_{n+2}^T(D_M, \tilde{\phi}).$$

- (vi) As in untwisted version we can consider smaller sum by taking into account only one element from each orbit of action by X on the space of colorings. However we should be careful here about which action we consider because the action $(x_1, \dots, x_n) * x$ is equal on homology to $t \cdot \text{Id}$ according to Observation 1.5. Thus we should change this action to $(x_1, \dots, x_n) \mapsto t^{-1}(x_1, \dots, x_n) * x$. We obtain then reduced versions of (iv) and (v).
- (vii) Each of the above has its cocycle version as long as we choose a twisted $(n + 1)$ - and $(n + 2)$ -cocycles in $C_T^{n+1}(X)$ and $C_T^{n+2}(X)$, respectively.

Most of the results as in Theorems 4.1, 4.6, and 4.8 generalize without any problem to twisted (co)homology. We give two examples below.

THEOREM 5.4. *For a fixed $\phi \in \text{Col}_X(D_M)$ the chain $c_{n+1}^T(D_M, \phi)$ is a cycle and it is homologous to $c_{n+1}^T(RD_M, R_{\#}(\phi))$ in $H_{n+1}^{TQ}(X)$, where R is any Roseman move on a diagram D_M . Similarly for a fixed shadow coloring $\tilde{\phi} \in \text{Col}_{X,sh}(D_M)$, the chain $c_{n+2}^T(D_M, \tilde{\phi})$ is a cycle and it is homologous to $c_{n+2}^T(RD_M, R_{\#}(\tilde{\phi}))$ in $H_{n+2}^{TQ}(X)$.*

The main, nontrivial Roseman move to check is the pass move R of type $S(c, n + 2, 0)$. Here analysis is very similar to that of Theorem 2.1 and Lemma 4.2.

THEOREM 5.5 (Twisted cocycle invariants). *Consider a knotting $f : M \rightarrow \mathbb{R}^{n+2}$ and fix a quandle $(X; *)$ and $(n + 1)$ -twisted cocycle $\Phi^T : \mathbb{Z}[t^{\pm 1}]X^{n+1} \rightarrow A$ and $(n + 2)$ -cocycle $\tilde{\Phi}^T : \mathbb{Z}[t^{\pm 1}]X^{n+2} \rightarrow A$ where A is a $[t^{\pm 1}]$ -module.*

- (1) *For a fixed coloring $\phi \in \text{Col}_X(D_M)$, the element $\Phi_X(D_M, \phi) = \Phi^T(c_{n+1}^T(D_M, \phi)) \in A$ is preserved by any Roseman move R , that is, $\Phi_X(D_M, \phi) = \Phi_X(RD_M, R_{\#}(\phi))$ in A .*
- (2) *For a fixed shadow coloring $\tilde{\phi} \in \text{Col}_{X,sh}(D_M)$, the element $\tilde{\Phi}_X(D_M, \tilde{\phi}) = \tilde{\Phi}^T(c_{n+2}^T(D_M, \tilde{\phi})) \in A$ is preserved by any Roseman move, that is, $\tilde{\Phi}_X(D_M, \tilde{\phi}) = \tilde{\Phi}_X(RD_M, R_{\#}(\tilde{\phi}))$ in A .*
- (3) *We can sum now over colorings of a finite quandle X , or sum over reduced colorings, or just take a set over colorings, to get twisted cocycle invariants of a knotting.*

REMARK 5.6. If we work with racks and rack (or degenerate) homology we cannot ignore degenerate elements, so the Roseman move of type $S(m, (1, n - 1), 0, p)$ (generalized first Reidemeister move) cannot be performed on the diagrams without possibly changing

(co)homology class of (co)cycles. For other Roseman moves however all our results work well. Thus we have (co)cycle invariants of diagrams of n -knottings up to all Roseman moves except moves of type $S(m, (1, n - 1), 0, p)$, for any rack.

6. General position and Roseman moves in codimension 2. An important tool in our work is given by the work of Roseman on general position of isotopy on moves in co-dimension two and the moves he developed. The next subsections follow [Ros-1, Ros-2, Ros-3]. We have used these notion in the paper; here is more formal development.

Before we can define Roseman moves we need several definitions.

6.1. General position. Let $M = M^n$ be a closed smooth n -dimensional manifold and $f : M \rightarrow \mathbb{R}^{n+2}$ its smooth embedding which is called a smooth knotting. Define $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+1}$ given by $\pi(x_1, \dots, x_{n+1}, x_{n+2}) = (x_1, \dots, x_{n+1})$ to be a projection on the first $n + 1$ coordinates. The projection of the knotting is the set $M^* = \pi f(M)$. Crossing set D^* of the knotting is the closure in M^* of the set of all points $x^* \in M^*$ such that $(\pi f)^{-1}(x^*)$ contains at least two points.

We define the double point set D as $D = (\pi f)^{-1}(D^*)$. The branch set B of f is the set of all points $x \in M$ such that πf is not an immersion at x . In general, if $A \subset M$ then A^* denotes $\pi f(A)$.

DEFINITION 6.1. Let $f : M \rightarrow \mathbb{R}^{n+2}$ be a smooth knotting with branch set B and double point set D . We say that f is in general position with respect to the projection π if the following six conditions hold:

- (1) B is a closed $(n - 2)$ -dimensional submanifold of M .
- (2) D is a union of immersed closed $(n - 1)$ -dimensional submanifolds of M^n with normal crossings. Denote the set of points of D where normal crossings occur as N and call this the self-crossing set of D .
- (3) B is a submanifold of D and for any $b_0 \in B$ there is a small $(n - 1)$ -dimensional open sub-disk V with $b_0 \in V$, $V \subseteq D$ such that $V - B$ has two components V_0 and V_1 , each of which is an $(n - 1)$ -disk which is embedded by the restriction of $\pi \circ f$ but with $V_0^* = V_1^*$ (Fig. 2.1).
- (4) B meets N transversely.
- (5) $(\pi \circ f)|_B$ is an immersion of B with normal crossings.
- (6) The crossing set of B^* is transverse to the crossing set of $(D - B)^*$.

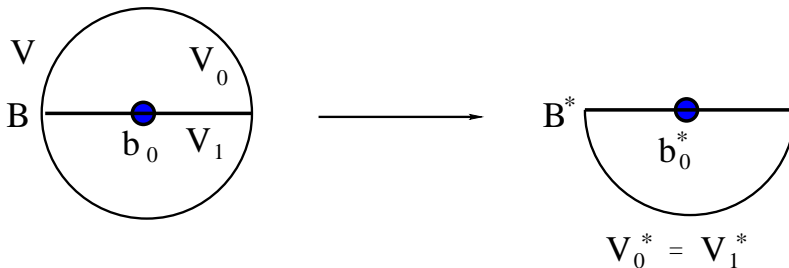


Fig. 6.1. Projecting (folding) of $V = V_0 \cup V_1 \cup (B \cap V)$ onto $(V - B)^* = V_0^* = V_1^*$

THEOREM 6.2. *Given a knotting $f : M^n \rightarrow \mathbb{R}^{n+2}$ we may isotope f to a map which is in general position with respect to the projection π .*

Similarly we define what it means for an isotopy $F : M \times I \rightarrow \mathbb{R}^{n+2} \times I$ to be in general position with respect to the projection $\pi' = \pi \times \text{Id}$. It is just the previous definition for general position of a codimension two knotting except that B and D may have nonempty boundary. In particular, $F_0 = F/(M \times \{0\})$, $F_1 = F/(M \times \{1\}) : M \rightarrow \mathbb{R}^{n+2}$ are smooth knottings in general position.

6.2. Arranging for moves. We put on our isotopy additional conditions called arranging for moves [Ros-3]. Roughly speaking, we filtrate D^* of $F : M \times I \rightarrow \mathbb{R}^{n+2} \times I$ in such a way that the projection $p : \mathbb{R}^{n+1} \times I \rightarrow I$ restricted to any component of each stratum, $Q^{(i)}$, is a Morse style function.

DEFINITION 6.3 (Roseman). Let q be a proper immersion of a manifold Q in $\mathbb{R}^{n+1} \times I$. We say that $q(Q)$ is immersed in Morse style if pq is a Morse function, where $p : \mathbb{R}^{n+1} \times I \rightarrow I$. We assume that a Morse function has critical points on different levels.

For details see [Ros-3]. Here we just mention that $Q^{(0)}$ is the crossing set of B^* , $Q^{(1)} = B^*$, the projection of the branch set of F . $Q^{(2)} = D^*$, generally, $Q^{(k)}$, $k > 1$, is the closure of the subset of D^* such that $F' = \pi'F$ is at least k to 1. Roseman proves:

THEOREM 6.4 (Roseman). Any isotopy $F : M \times I \rightarrow \mathbb{R}^{n+2} \times I$ can be arranged for moves.

6.3. Listing of moves after Roseman. The standard set of moves \mathcal{M}^n is described as follows:

Fix a dimension n and suppose we are given an isotopy $F : M^n \times I \rightarrow \mathbb{R}^{n+2} \times I$ which is arranged for moves. This gives a sequence of elementary singularities. Each singularity will correspond to a standard local knot move in our collection \mathcal{M}^n .

In the notation which follows, we consider three general types of points:

1. *branch type*: critical points of B^* and self-crossing points of B^* for which we use the letter b ,
2. *crossing type*: critical points of D^* and the crossing set of D^* which do *not* belong to B^* for which we use the letter c .
3. *mixed type*: critical points which are in the crossing set of D^* and are in B^* , a “mixed” type, for which we use the letter m .

The first collection of branch type points is denoted by $\{S(b, k, p, q)\}$. If $x^* \in D^*$ is such a singular point, where D^* is the crossing set of an isotopy $F : M^n \times I \rightarrow \mathbb{R}^{n+2} \times I$, let k denote the number of points of $F'^{-1}(x^*)$. In our case the branch point set B^* of F , is codimension 2 in $M^n \times I$, that is, it is of dimension $n - 1$.

If in projection this branch set intersects itself generically, the self-intersection set will have dimension $n - 4$. It follows that $1 \leq k \leq n - 2$. The integer p is the index of the singularity. The integer q has range $0 \leq q \leq k$ and might be called *transverse index* of this critical point. This is defined as follows. If $x \in B$, consider a curve δ in D transverse to B (recall that B has codimension one in D) so that δ^* is, except for the point x , the two-to-one image of δ . In the I direction, the image of this curve has a local maximum or a local minimum at b . Now suppose b^* is a k -fold point of B^* then we have k such curves to consider. The number q is the number of those curves for which we have a local maximum. Of course, it follows that $k - q$ of the curves have a local minimum.

The next collection of crossing type singularities is denoted by $\{S(c, k, p)\}$. If x^* is such a singularity, k denotes the cardinality of $F'^{-1}(x^*)$. Thus k is an integer $2 \leq k \leq n + 2$. Furthermore, on this set of points, where F' is k -to-one, x^* is a critical point in the I direction, of index p . A single point has index 0 by convention.

Finally $S(m, (i, j), p, q)$ denotes mixed singularities. Such a singularity x^* has $F'^{-1}(x^*)$ consisting of $i + j$ points, where exactly i of these points are in B . Again p is the index of the singularity and q is an integer, $0 \leq q \leq i$, which is the number of local maxima we get by looking at those i arcs transverse to B at the points of $F'^{-1}(x^*) \cap B$.

7. A knotting $M^n \xrightarrow{f} F^{n+1} \times [0, 1] \xrightarrow{\pi} F^{n+1}$. The Roseman (local) moves can be used for any n -knotting $f : M^n \rightarrow W^{n+2}$ by the following classical PL-topology result following from Theorem 6.2 in [Hud] (we will use it in a smooth case which can be derived using Whitehead results on triangulation of smooth manifolds).

LEMMA 7.1. *If C is a compact subset of a manifold W and $F : W \times I \rightarrow W$ is the isotopy of W then there is another isotopy $\hat{F} : W \times I \rightarrow W$ such that $F_0 = \hat{F}_0$, $F_1/C = \hat{F}_1/C$ and there exists a number N such that the set $\{x \in W \mid \hat{F}/\{x\} \times (k/N, (k + 1)/N)\}$ is not constant} sits in a ball embedded in W .*

Let $f : M^n \rightarrow F^{n+1} \bar{\times} [0, 1]$ be an n -knotting where F^{n+1} is an $(n + 1)$ -dimensional manifold and $F^{n+1} \bar{\times} [0, 1]$ is an $[0, 1]$ -bundle over F^{n+1} (trivial bundle if F^{n+1} is oriented and the twisted $[0, 1]$ -bundle over F^{n+1} if F^{n+1} is unorientable. In both cases the manifold is oriented). Let $\pi : F^{n+1} \bar{\times} [0, 1] \rightarrow F^{n+1}$. By Lemma 7.1 an embedding f can be assumed to be in general position with respect to π and every ambient isotopy of a knotting can be decomposed into Roseman moves (on D_M). If $\pi_1(F^{n+1}) = 0$ then essentially all results of the paper can be also proven for the knotting (we need $W = F^{n+1} \bar{\times} [0, 1]$ to be simply connected in Lemma 1.12, Remark 3.3, and Theorem 3.4).

REMARK 7.2. If we do not assume that $\pi_1(F^{n+1}) = 0$ in the case of $W^{n+2} = F^{n+1} \times [0, 1]$, we can still develop the theory of (co)cycle invariants by following [FRS-2] where the notion of a reduced fundamental rack is developed (essentially one kills the action of $\pi_1(F^{n+1})$). Then the reduced fundamental rack (or quandle) is, according to Corollary 3.5 of [FRS-2], the same as the fundamental rack (or quandle) obtained by rack (or quandle) abstract coloring of any diagram of the knotting.

8. Speculation on Yang–Baxter homology and invariants of knottings. Yang–Baxter operator can be thought of as a direct generalization of right self-distributivity when we go from the category of sets to the category of k -modules.

We follow here [Leb-1, Leb-2, Prz-1, Prz-2] describing the classical case $n = 1$.

First we note how to get Yang–Baxter operator from a right self-distributive binary operation. Let $(X; *)$ be a shelf and kX be a free module over a commutative ring k with basis X (we can call kX a *linear shelf*). Let $V = kX$, then $V \otimes V = k(V^2)$ and the operation $*$ yields a linear map $Y = Y_{(X;*)} : V \otimes V \rightarrow V \otimes V$ given by $Y(a, b) = (b, a * b)$. Right self-distributivity of $*$ yields the equation of linear maps $V \otimes V \otimes V \rightarrow V \otimes V \otimes V$ as follows:

$$(1) \quad (Y \otimes \text{Id})(\text{Id} \otimes Y)(Y \otimes \text{Id}) = (\text{Id} \otimes Y)(Y \otimes \text{Id})(\text{Id} \otimes Y).$$

In general, the equation of type (1) is called a Yang–Baxter equation and the map Y a Yang–Baxter operator. We usually require that Y is invertible, otherwise the name pre-Yang–Baxter operator is used. For example if Y is given by an invertible $*$, then $Y_{(X;*)}$ is invertible with $Y_{(X;*)}^{-1}(a, b) = (b * a, a)$.

In our case $Y_{(X;*)}$ permutes the base $X \times X$ of $V \otimes V$, so it is called a permutation or a set theoretic Yang–Baxter operator¹⁸. Our distributive homology, in particular our rack homology $((C_n, \partial^Y), \partial^Y = \partial^{(*_0)} - \partial^{(*)})$ can be thought of as the homology of Y . It was generalized from the Yang–Baxter operator coming from a self-distributive $*$ to any set theoretic Yang–Baxter operator (coming from biracks or biquandles), [CES-2]. For a general Yang–Baxter operator, there is no general homology theory (however, compare [Eis-2, Eis-3]). The goal/hope is to define homology for any Yang–Baxter operator and develop the homological invariants of n -knottings (it is done for $n = 1$ and a set theoretic Yang–Baxter equation in [CES-2]). The simple visualization of the distributive face map $d_i^{(*)}$ from Fig. 8.1, observed by I. Dynnikov during Przytycki’s talks in Moscow in May 2012 (and slightly earlier by Victoria Lebed when she was writing her Ph.D. thesis [Leb-1]), easily gives a hint to homology of set theoretic Yang–Baxter homology, and, partially, to general Yang–Baxter homology (this is studied in [Prz-2]). To get the i -th face map of [CES-2] we take the difference of “one term” face maps as illustrated in Fig. 8.2.

The homology invariants of n -knotting should follow, and combining the method of this paper with [Leb-2, Prz-2] looks rather promising.

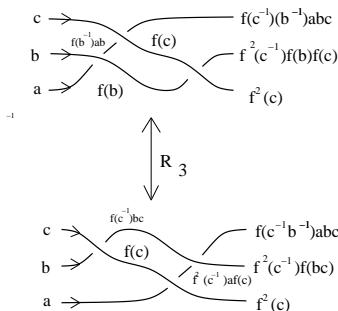
We plan to study (co)cycle invariant for codimension 2 embeddings in the case of set theoretic Yang–Baxter operators (biracks). The main difficulty is that one cannot use

¹⁸The pair $(X; Y)$, where $Y : X \times X \rightarrow X \times X$ leads to pre-Yang–Baxter operator, is called a bishelf, and if Y is invertible — a birack. A classical birack, related to (branched) cyclic covers of links, is defined as follows [Win, Prz-0, P-R, DPT]:

Let G be a group and $f : G \rightarrow G$ a group automorphism, then $Y_{G,f} : G \times G \rightarrow G \times G$ is given by $Y_{G,f}(a, b) = (f(b), f(b^{-1})ab)$. We have $Y_{G,f}^{-1}(c, d) = (cdf^{-1}(c^{-1}), f^{-1}(c))$. In fact, we check that $Y_{G,f}^{-1}Y_{G,f}(a, b) = Y_{G,f}^{-1}(f(b), f(b^{-1})ab) = (f(b)f(b^{-1})abb^{-1}, f^{-1}(f(b))) = (a, b) = Y_{G,f}Y_{G,f}^{-1}(a, b)$. We also check directly the condition of the set theoretic Yang–Baxter equation:

$$\begin{aligned} (Y_{G,f} \times \text{Id})(\text{Id} \times Y_{G,f})(Y_{G,f} \times \text{Id})(a, b, c) &= (f^2(c), f^2(c^{-1})f(b)f(c), f(c^{-1}b^{-1})abc) \\ &= (\text{Id} \times Y_{G,f})(Y_{G,f} \times \text{Id})(\text{Id} \times Y_{G,f})(a, b, c), \end{aligned}$$

as illustrated in the figure below:



In a place of idempotency, we have $Y_{G,f}(f(a), a) = (f(a), a)$.

the general position argument of Fenn, Rourke, and Sanderson, in this case (there is no appropriate, diagram free, topological definition of a fundamental birack for arbitrary n), thus one has to use analysis of all Roseman moves.

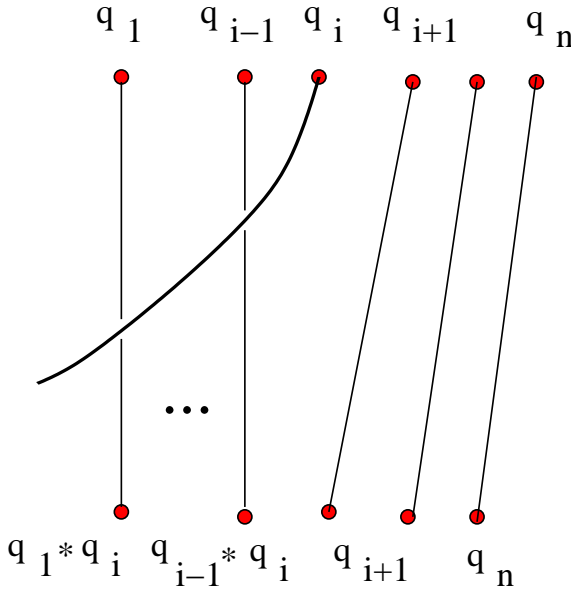


Fig. 8.1. Diagrammatic visualization of a face map gives hint to Yang–Baxter homology.

For a right self-distributive $*$ we have a face map $d_i^{(*)}(q_1, \dots, q_n) = (q_1 * q_i, \dots, q_{i-1} * q_i, q_{i+1}, \dots, q_n)$.

We can also interpret the picture to be applicable to Yang–Baxter theory by using Yang–Baxter operator at each crossing.

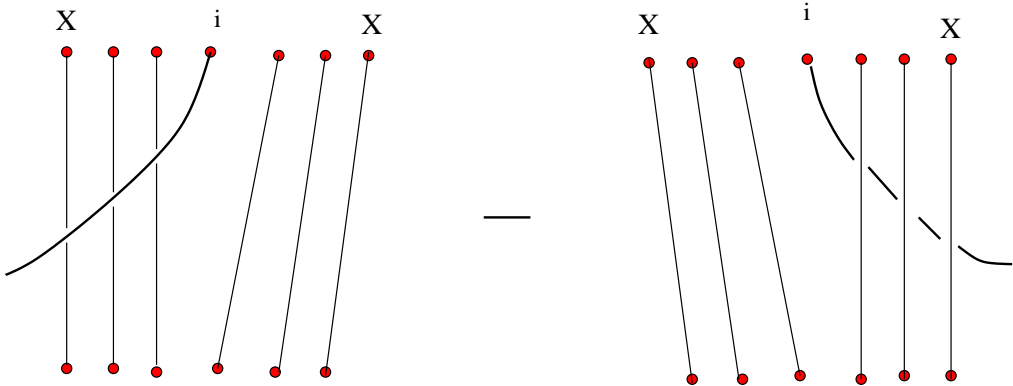


Fig. 8.2. Diagrammatic visualization of an i -th face map of [CES-2]; $Y : X \times X \rightarrow X \times X$ is a set theoretic Yang–Baxter operator

Acknowledgments. The paper has been written as a part of the cooperation agreement between Department of Mathematics, George Washington University, and Institute of Mathematics, University of Gdańsk. J. H. Przytycki was partially supported by the NSA-AMS 091111 grant, and by the GWU REF grant.

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