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FROM BANACH TO DISTRIBUTIONS AND CURRENTS

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1. Introduction. The Banach spaces as described in the seminal book [B] are still actively studied in many domains, in particular the geometry of Banach spaces (see [JL]).

Moreover, from the beginning of the '30's, Banach spaces with an order relation were introduced and their theory particularly developed in relation with the topology, around 1980 and until now (see [AA]).

They were also used during the '30's where particular distributions were introduced by Sobolev (see Section 3). In the meantime, general distributions were defined and developed in the late '40's (Section 4), then it was necessary to introduce more general function and distribution spaces (Section 5).

We propose to describe the structures of the function and distribution spaces with their relations with Banach spaces and their generalizations to differential forms and currents in global analysis (Sections 5–6). We shall see that the (norm) topology, the completeness, the operators, the dual, the adjoint of an operator will still be used in the generalized context.

No attempt will be done to anymore enter in the intricate history of distributions.

2. Banach spaces.

2.1. *We recall classical definitions* [B], ([AA], Ch. 1). Let E be a K-vector space with $K = \mathbb{C}$ or $\mathbb{R}$. In the following we assume K fixed in every statement. A *norm* on E is a function $\| \,.\, \|: E \to \mathbb{R}_+$ satisfying:

(i) for $x \in E$, $\| x \| = 0$ is equivalent to $x = 0$;

(ii) for every $x \in E$, for every $\alpha \in K$, $\| \alpha.x \| = |\alpha|. \| x \|$;

(iii) for every $(x, y) \in E^2$, $\| x + y \| \leq \| x \| + \| y \|$.

A pair $(E, \| \,.\, \|)$ is called a *normed space* and is generally denoted by E.

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The norm defines a distance d on E by $d(x,y) = \| y-x \|$ such that (E,d) is a metric space. If (E,d) is complete (i.e. every Cauchy sequence is convergent), E is called a *Banach space*.

Let $u : E \to F$ be a linear map (or *operator*) of normed spaces.
Let $\| u \| = \sup_{\|x\| \leq 1} \| ux \| = \sup_{\|x\|=1} \| ux \|$.

If $\| u \| < \infty$, u is called a *bounded operator* and $\| u \|$ its (*operator*) *norm*. Clearly, a bounded operator is exactly a continuous linear map for the topologies defined by the norms.

2.2. *Spaces of operators; topological dual.* The set of all bounded operators : $E \to F$ is denoted $\mathcal{L}(E,F)$, with the addition of functions, their multiplication by the elements of K and the operator norm: it is a normed space. If F is Banach, then $\mathcal{L}(E,F)$ is also Banach.

A *Banach algebra* $\mathcal{A}$ is a Banach space for which the multiplication of $\mathcal{A}$ satisfies, for every $(x,y) \in \mathcal{A}^2$, $\| xy \| \leq \| x \| \| y \|$: the field K is a Banach algebra. We set $E' = \mathcal{L}(E,K)$ and call it the *topological dual* of E; the norm on E' is called the *dual norm*.

2.3. The *adjoint* of a bounded operator $u : E \to F$ is the bounded operator $u' : F' \to E'$ defined, for $y' \in F'$, $x \in E$ by: $(u'y')x = y'(ux)$ translated into:

$$\langle u'y', x \rangle = \langle y', ux \rangle.$$

Then $\| u' \| = \| u \|$.

3. Sobolev spaces [So 1], [AF].

3.1. L^p *spaces.* Let U be an open set of $\mathbb{R}^n$. $L^p(U)$ is the $\mathbb{C}$-vector space of complex-valued functions almost everywhere defined on U, of p-th integrable power ($p \in \mathbb{R} : p \geq 1$), with the norm

$$\| f \|_p = \| f \|_{L^p(U)} = \left[\int_U |f(x)|^p dx \right]^{\frac{1}{p}}.$$

$L^p(U)$ with this norm is a Banach space; $L^\infty(U)$ is the Banach space of essentially bounded measurable functions on U.

Define p' by

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

then, for every $g \in L^{p'}(U)$, the map: $L^p(U) \to \mathbb{C}$ defined by $f \mapsto \int_U f\overline{g} dx$ is a continuous linear form.

F. Riesz representation theorem. For every $p \in [1, \infty[$, the topological dual $L^p(U)'$ of $L^p(U)$ is $L^{p'}(U)$;

$$L^\infty(U)' \supsetneq L^1(U).$$

If measure$(U) < \infty$, then for $1 \leq p \leq q$, $L^q(U) \subset L^p(U) \subset L^1(U)$.

For every $p \geq 1$, $L^p(U) \subset \mathcal{D}'(U)$, the space of distributions on U (see Section 4).

3.2. *Sobolev spaces.* Let

$$m \in \mathbb{Z}_+, \qquad p \in [1, \infty];$$

$$\| u \|_{m,p} = [\sup_{0 \le |\alpha| \le m} \| D^\alpha u \|_p^p]^{\frac{1}{p}}, p < \infty; \qquad \| u \|_{m,\infty} = \sup_{0 \le |\alpha| \le m} \| D^\alpha u \|_\infty .$$

for every function u such that the above expressions have a meaning.

These expressions define norms on the following $\mathbb{C}$-vector spaces:

$$H^{m,p}(U) \quad \text{the completion of } \{u \in C^m(U); \| u \|_{m,p} < \infty\};$$
$$W^{m,p}(U) = \{u \in L^p(U); D^\alpha u \in L^p(U); 0 \le |\alpha| \le m\},$$

where $D^\alpha u$ means the derivative in the distribution sense (see Section 4);

$$W_0^{m,p}(U) = \overline{C_0^\infty(U)} \subset W^{m,p}(U),$$

where $C_0^\infty(U)$ denotes the vector space of C^∞ functions with compact support in U.

With the above norms, these spaces are Banach and are called *Sobolev spaces.*

In 1933, Otton Nikodym [N] introduced[1], following Beppo Levi [L] (1906), a vector subspace of $W^{1,2}(U)$.

3.3. *Properties.* We have the following topological inclusions:

$$\text{for} \quad m \ge 0, \quad W_0^{m,p}(U) \subset W^{m,p}(U) \subset L^p(U);$$
$$\text{for} \quad m \le m', \quad W^{m',p}(U) \subset W^{m,p}(U);$$

$H^m(U) = W^{m,2}(U)$ is a Hilbert space; $H^{m,p}(U)$ is a closed subspace of $W^{m,p}(U)$.

3.4. *Duality.* Let $p \in]1, \infty[$, p and p' related by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Define $W^{-m,p'}(U)$ to be the topological dual of $W_0^{m,p}(U)$; since $\mathcal{D}(U)$ is dense in $W_0^{m,p}(U)$, the elements of $W^{-m,p'}(U)$ are distributions (see Section 5.3). This situation generalizes the duality between $L^p(U)$ and $L^{p'}(U)$.

Remark that the objects under consideration are L^p functions for $m \ge 0$, and test functions are of finite order of differentiability. In the same way, the derivatives in the sense of distributions are always considered L^p functions.

4. Distributions: naive definition [So 2], [So 3], [Sc].

4.1. *Measures.* The *support* of a continuous complex-valued function f on an open set $U \subset \mathbb{R}^n$ is the closure in U of the set of points $x \in U$ where $f(x) \neq 0$; it is denoted $\operatorname{supp} f$.

Let μ be a complex-valued measure on $\mathbb{R}^n$. Let $\mathcal{C}$ be the vector space of complex-valued continuous functions with compact support on $\mathbb{R}^n$. The map: $\mathcal{C} \to \mathbb{C}$ defined

[1] I thank B. Bojarski for giving me references on a more remote origin of these spaces.

by:

$$\varphi \mapsto \mu(\varphi) = \int_{\mathbb{R}^n} \varphi d\mu$$

is a linear form. For any sequence (or net) (φ_j), $\varphi_j \in \mathcal{C}$, supp $\varphi_j \subset K$ a given compact, and if (φ_j) uniformly converges to $\varphi \in \mathcal{C}$, then $\mu(\varphi_j) \to \mu(\varphi)$.

Let $\mathcal{C}_K$ be the subspace of $\mathcal{C}$ of functions with supports contained in K: $\mathcal{C}_K$ is a Banach space for the norm

$$\| \varphi \| = \sup_{x \in \mathbb{R}^n} |\varphi(x)|.$$

Then the continuity condition for μ (said *continuity on* $\mathcal{C}$) means: $\mu|_{\mathcal{C}_K}$ is continuous.

Conversely, from the *F. Riesz representation theorem*, for every continuous linear form L on $\mathcal{C}$ in the above sense, there exists a well-defined measure μ such that $L(\varphi) = \mu(\varphi)$.

Then a measure is a continuous linear form on $\mathcal{C}$; the set of measures on $\mathbb{R}^n$ is a $\mathbb{C}$-vector space $\mathcal{C}'$.

Let dx be the Lebesgue measure on $\mathbb{R}^n$; a continuous function f on $\mathbb{R}^n$ defines the measure $\mu = fdx$; identify f and fdx; then the vector space of continuous functions on $\mathbb{R}^n$ appears as a subspace of $\mathcal{C}'$. The same remark concerns elements of the Sobolev spaces described before.

4.2. *Space of differentiable functions.* For every $r \in \mathbb{Z}_+$, let $\mathcal{C}^{(r)}(U)$ be the $\mathbb{C}$-vector space of C^r maps: $U \to \mathbb{C}$; then:

$$\bigcap_{r \in \mathbb{N}} \mathcal{C}^{(r)}(U) = \mathcal{C}^{\infty}(U) = \mathcal{C}(U)$$

is the space of $\mathcal{C}^\infty$ complex-valued functions on U.

For every compact K of U, for every $s \in \mathbb{N}, f \in \mathcal{C}^{(r)}(U)$, we set, for $s \leq r$,

$$p_{s,K}(f) = \sup_{x \in K; |\nu| \leq s} |D^\nu f(x)|$$

$p_{s,K}$ is a semi-norm on $\mathcal{C}^{(r)}(U)$. A *semi-norm* p satisfies the axioms of a norm, except for (i) which has to be replaced by

(i') $x = 0$ implies $p(x) = 0$.

The family of semi-norms $(p_{s,K})$ defines a (Hausdorff) topology of a topological vector space (t.v.s.) on $\mathcal{C}^{(r)}(U)$; this t.v.s. is denoted $\mathcal{E}^{(r)}(U)$, $r \in \mathbb{N}$, and $\mathcal{E}^{(\infty)}(U)$ usually denoted $\mathcal{E}(U)$ is $\mathcal{C}(U)$ with the family of semi-norms.

4.3. *Locally convex topological vector spaces.* A t.v.s. is called *locally convex* (l.c.) if 0 has a fundamental system of convex neighborhoods.

The l.c.t.v.s. are exactly the vector spaces with a family of semi-norms.

4.4. *Differentiable functions with compact support.* The spaces $\mathcal{E}^{(r)}(U)$, $r \in \mathbb{N} \cup \infty$ are metrizable and complete (i.e. Fréchet spaces, (metrizable and complete l.c.t.v.s.)).

Let K be a compact set contained in U and $\mathcal{E}_0^{(r)}(U)$ be the subspace of $\mathcal{E}^{(r)}(U)$ of functions with compact support. Let $\mathcal{D}^{(r)}(U; K)$ be the subspace of $\mathcal{E}_0^{(r)}(U)$ of functions with support in K: this space is Fréchet, and for $r < \infty$, it is a Banach space.

For the moment, let $\mathcal{D}^{(r)}(U)$ be the vector space, without topology, union of the $\mathcal{D}^{(r)}(U;K)$ and denote by $\mathcal{D}(U)$, respectively $\mathcal{D}(U;K)$, the corresponding spaces for $r=\infty$. We call *test function* every element of $\mathcal{D}(U)$.

4.5. *Distributions.* We call *distribution on* U every $\mathbb{C}$-linear form T on the $\mathbb{C}$-vector space $\mathcal{D}(U)$ such that, for every compact K of U, $T|_{\mathcal{D}(U;K)}$ is continuous.

As the spaces $\mathcal{D}(U;K)$ are metric, this latter condition is equivalent to:

For every compact K of U, with every sequence (φ_j) of C^∞ functions with supports in K converging to 0 *in $\mathcal{E}(U)$, the sequence $(T\varphi_j)$ tends to* 0 *in* $\mathbb{C}$.

The *order* of the distribution T is the smallest r such that $T|_{\mathcal{D}^{(r)}(U;K)}$ is continuous. In particular, the measures are the distributions of order 0.

4.6. *Derivation.* Let f be a function of class C^1. Considering it as a measure and hence as a distribution, and applying, for any $\varphi \in \mathcal{D}(U)$, integration by parts, suggests the following definition:

In $U \subset I\!R^n$ with coordinates $(x_1, \ldots, x_n)$,

$$\frac{\partial T}{\partial x_k}(\varphi) := -T\Big(\frac{\partial \varphi}{\partial x_k}\Big) \quad \text{for } k=1,\ldots,n, \text{ and } \varphi \in \mathcal{D}(U).$$

This process can be iterated indefinitely, for any k and any order.

5. Spaces of functions, distributions and currents [Sc].

5.1. Let E be a $\mathbb{C}$-vector space union of vector subspaces (F_m) such that $F_m \subset F_{m+1}$ is a topological inclusion. Let $j_m : F_m \to E$ be the canonical injection and τ the finest topology of l.c.t.v.s. on E such that, for every $m \in I\!N$, j_m be continuous.

Let E be as above with the topology τ, then for every l.c.t.v.s. G and every linear map $f : E \to G$, the following conditions are equivalent:

(a) *f is continuous:*

(b) *for every $m \in I\!N$, the map $f \circ j_m : F_m \to G$ is continuous.*

E with the topology τ is said to be the *inductive limit of the sequence* (F_m); if the topology of F_{m+1} induces, by restriction to F_m, the topology of F_m, the inductive limit is said to be *strict*.

5.2. Let (K_m) be an exhaustive sequence of compacts of U, then

$$\mathcal{D}(U) = \bigcup_{m \in I\!N} \mathcal{D}(U; K_m)$$

has a well defined topology τ such that it is the strict inductive limit of the sequence $(\mathcal{D}(U;K_m))$. In the following, we assume the topology τ on $\mathcal{D}(U)$. Then, from 5.1,

The distributions on U are the continuous linear forms on U, i.e. the set of the distributions on U is the vector space $\mathcal{D}'(U)$, topological dual of $\mathcal{D}(U)$.

5.3. *Let F be a $\mathbb{C}$-t.v.s. such that:*

(1) *$\mathcal{D}(U)$ is a vector subspace of F;*

(2) *the set $\mathcal{D}(U)$ is dense in the space F;*

(3) *the topology of $\mathcal{D}(U)$ is finer than the topology induced on $\mathcal{D}(U)$ by the topology of F; then the topological dual F' of F is a vector subspace of $\mathcal{D}'(U)$.*

Examples. $\mathcal{E}'(U)$ is the space of distributions with compact supports. The Sobolev spaces are obtained by duality.

5.4. *Adjoint of a continuous linear map.* Let $u : E \to F$ be a continuous linear map of t.v.s.; the $\mathbb{C}$-linear map: $u' : F' \to E'$ defined by: $T \to T \circ u$ is said to be the *adjoint* of u.

We shall use the following notation: for $T \in F'$ and $y \in F$, we set:

$$T(y) =< T, y > .$$

Then, with the above notations, for $x \in E$, $T \in F'$, we have

$$< u'(T), x >=< T, u(x) > .$$

Examples. Assume: $E = F = \mathcal{D}'(U)$.

(a) For every partial derivative D^ν, the map $D^\nu : \mathcal{D}(U) \to \mathcal{D}(U)$ defined by: $\varphi \mapsto D^\nu\varphi$ is continuous linear; the differential operator on distributions (also denoted D^ν) defined by D^ν is $(-1)^{|\nu|}(D^\nu)'$.

(b) For every $\alpha \in \mathcal{E}(U)$, the map $\alpha : \mathcal{D}(U) \to \mathcal{D}(U)$ defined by $\varphi \mapsto \alpha\varphi$ is continuous linear. Then, the multiplication of the distributions by α is the adjoint of the multiplication by α in $\mathcal{D}(U)$.

5.5. *Currents* [R]. This notion has been introduced by G. de Rham, in the '30's in special cases, in particular electrical currents, then, in general, after the L. Schwartz's definition of distributions (see [R]).

A distribution is a local notion since it is defined on an open subset U of $\mathbb{R}^n$. The currents are corresponding global notions and define more general objects, in particular many geometric objects.

Let X be a differential manifold of class $\mathcal{C}^\infty$, countable union of compact sets, supposed to be oriented for simplicity; let n be the dimension of X. Let $\Gamma^{(r)p}(X)$ be the vector space of differential forms of degree p and class C^r on X. On a domain V of coordinates $(x_1, \ldots, x_n)$ of X, every differential form may be expressed as

$$u = \sum_{i_1<\ldots<i_p} a_{i_1,\ldots,i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$

There exists a unique topology of l.c.t.v.s. on $\Gamma^{(r)p}(X)$ such that the following two conditions are equivalent:

(a) a sequence $(u_k)_{k\in\mathbb{N}}, u_k \in \Gamma^{(r)p}(X)$ converges to 0;

(b) for every chart of X of domain V of X, for every compact $K \subset V$, for every multi-index $\nu, |\nu| \leq r$, the sequences of the restrictions to K of the coefficients a_k and of their partial derivatives $D^\nu a_k$ uniformly converge to 0.

Let $\mathcal{E}^{(r)p}(X)$ be the space $\Gamma^{(r)p}(X)$ with the above topology, and $\mathcal{E}^p(X) = \mathcal{E}^{(\infty)p}(X)$.

For every compact K of X, let $\mathcal{D}^p(X;K)$, (resp. $\mathcal{D}^{(r)p}(X;K)$) be the subspaces of $\mathcal{E}^p(X)$, (resp. $\mathcal{E}^{(r)p}(X)$) of differential forms with supports contained in K. Let $\mathcal{D}^p(X)$, (resp. $\mathcal{D}^{(r)p}(X)$) be the union of the $\mathcal{D}^p(X;K)$, resp. ($\mathcal{D}^{(r)p}(X;K)$) with the topology such that they are the strict inductive limits of the above spaces.

$\mathcal{D}^p(X)$ is the space of $\mathcal{C}^\infty$-differential forms, of degree p, with compact support, on X. We denote by $\mathcal{D}^\cdot(X)$ the direct sum, over p, of the $\mathcal{D}^p(X)$, and we call *test form* every element of $\mathcal{D}^\cdot(X)$.

A *p-dimensional current* is a continuous linear form on $\mathcal{D}^p(X)$. So the space of currents of dimension p on X is the topological dual $\mathcal{D}'_p(X) = (\mathcal{D}^p(X))'$ of $\mathcal{D}^p(X)$, and we denote by $\mathcal{D}'_\cdot(X)$ the direct sum of the $\mathcal{D}'_p(X)$.

The *order* of a current is defined as for a distribution.

A q-differential form which is continuous, or more generally L^1_{loc}, over X (i.e. with continuous coefficients or coefficients L^1_{loc} on every coordinate domain), by integration over X, of its exterior product with a test form of degree $p = n - q$, defines a current of dimension p. Hence we define the *degree* of a current T of dimension p as $q = n - p$ and set $T \in \mathcal{D}'^q(X)$.

Let $\mathcal{E}'^\cdot(X)$, $\mathcal{E}'_\cdot(X)$ be the graded vector space, according to degree or dimension, of the currents on X with compact support.

The currents of dimension 0 are called *distributions* by extension of the case $X = U \subset \mathbb{R}^n$. Among them, the currents of order 0 are the *measures.*

5.6. *Operations on currents.* The *right exterior product* by a differential form α: this is the adjoint of the corresponding product in $\mathcal{D}(X)$.

The *boundary* (operator) b is the adjoint of the exterior differential on the space $\mathcal{D}^\cdot(X)$.

The *exterior differential* of a current $T \in \mathcal{D}'^q(X)$ is: $dT = (-1)^{q+1} bT$, generalizing the peculiar case where T is a differential form of degree q.

5.7. Various topologies can be put on the space of currents. The most usual is the weak topology: $T \to 0$, if and only if, for every $\varphi \in \mathcal{D}^\cdot(X)$ implies: $T(\varphi) \to 0$; we also say that T converges to 0 in the sense of currents. Other topologies will be introduced in the next section.

6. Geometrical currents [F]**.** Locally a current is a differential form with coefficients being 0-currents.

We assume that X is an (oriented) Riemannian manifold of dimension n, and we set $n = p + k$.

6.1. *Semi-norms on* $\mathcal{D}'_\cdot(X)$. For every $\varphi \in \mathcal{D}^{(0)p}(X)$, we set:

$$\| \varphi \| (x) = \sup_\gamma \{|\varphi(\gamma)|;\quad \gamma \text{ decomposable } p\text{-(tangent) vector at } x \text{ of length} \leq 1\};$$

then

$$\nu(\varphi) = \sup_{x \in X} \| \varphi \| (x),$$

is called the *comass* of φ.

For every open set $W \subset X$, for every $u \in \mathcal{D}'_\cdot(X)$,

$$M_W(u) = \sup\{|u(\varphi)|;\quad \varphi \in \mathcal{D}^p(X);\quad \text{supp } \varphi \subset W;\quad \nu(\varphi) \leq 1\}$$

is called the *mass* of u on W (it may be $+\infty$).

For fixed u, $W \to M_W(u)$ is a Borel measure on X, denoted by $\| u \|$. For a closed subset $A \subset X$, define $M_A(u) = \| u \| (A)$. The set $\{M_K, K \text{ compact} \subset X\}$ is a set of semi-norms (possibly $+\infty$) inducing on $\mathcal{D}'_{\cdot}(X)$ the *mass topology*; $M_K(u)$ is called the mass of u on K.

6.2. *Measurable currents.* The space $\mathcal{M}_p(X) = \{u \in \mathcal{D}'_p(X);$ for every compact $K \subset X$, $M_K(u) < +\infty\}$ is called the space of *measurable currents* of dimension p. It is a Fréchet space. Moreover $\mathcal{M}_{\cdot}(X) = \bigoplus \mathcal{M}_p(X)$ is the space of currents of order 0, with the topology defined by the semi-norms M_W; it is also said to be the space of *integration currents.*

6.3. *Locally normal currents.* The space of locally normal currents is

$$N^k_{\mathrm{loc}}(X) = \{u \in \mathcal{M}^k(X); du \in \mathcal{M}^{k+1}(X)\}$$

with the semi-norms

$$N_K(u) = M_K(u) + M_K(du).$$

Moreover,

$$N^k(X) = N^k_{loc}(X) \cap \mathcal{E}'^{\cdot}(X)$$

is the space of *normal currents.*

6.4. *Locally flat currents.* The corresponding space is

$$F^k_{\mathrm{loc}}(X) = \mathcal{D}'^k(X) \cap \{\alpha + d\beta; \alpha, \beta \in \mathcal{D}'^{\cdot}(X)$$

with L^1_{loc} coefficients$\}$.

The semi-norms

$$F_W(u) = \sup\{|u(\varphi)|, \varphi \in \mathcal{D}^{\cdot}(X); \quad \mathrm{supp}\ \varphi \subset W; \quad \sup\big(\nu(\varphi), (\nu(d\varphi)\big) \leq 1\}$$

define the *flat topology*; $F^k_{\mathrm{loc}}(X)$ is now equipped with it; $F^k_{\mathrm{loc}}(X)$ is a Fréchet space.

By definition, $F^k(X) = \mathcal{E}'^{\cdot}(X) \cap F^k_{\mathrm{loc}}(X) = \{$flat currents of degree $k\}$.

Properties.

$F^k_{\mathrm{loc}}(X) = \overline{\big(\mathcal{D}^k(X)\big)}_{\mathcal{D}'^k(X)}$ for the flat topology;

$N^k_{\mathrm{loc}}(X) \subset F^k_{\mathrm{loc}}(X) : F^k_{\mathrm{loc}}(X) = \overline{\big(N^k(X)\big)}_{\mathcal{D}'^k(X)}$ for the flat topology;

$F^k_{\mathrm{loc}}(X) \cap \mathcal{M}^k(X) = \overline{\big(N^k_{\mathrm{loc}}(X)\big)}_{\mathcal{D}'^k(X)}$ for the mass topology.

6.5. *Locally rectifiable currents.* Let $\mathcal{S}_p(X)$ be the group of the simplicial C^1 p-chains (subgroup of the additive group of currents on X).

$\overline{\big(\mathcal{S}_p(X)\big)}_{\mathcal{D}'^k(X)}$ for the mass topology is the group of *locally rectifiable* p-currents $\mathcal{R}^k_{\mathrm{loc}}(X)$ having the following property:

$$\mathcal{R}^k_{\mathrm{loc}}(X) \subset F^k_{\mathrm{loc}}(X) \cap \mathcal{M}^k(X).$$

The subgroup

$$\mathcal{R}^k(X) = \mathcal{R}^k_{\mathrm{loc}}(X) \cap \mathcal{E}'^k(X)$$

is the group of *p-rectifiable currents* of X.

$$I^{\mathrm{loc}}_p(X) = \{u \in \mathcal{R}^{\mathrm{loc}}_p(X); du \in \mathcal{R}^{\mathrm{loc}}_{p-1}(X)\}$$

is called the group of *p-locally integral currents.*

By definition, $I^k(X) = \mathcal{E}'^{\cdot}(X) \cap I^k_{\mathrm{loc}}(X) = \{\text{integral currents of degree } k\}$.

6.6. *Examples of applications.*

1. Calculus of variations. Use of the integral currents.
   Ex.: minimal subvarieties: evaluation of the codimension of the singular set [Am 00].
2. Complex geometry. Use of the locally rectifiable currents.
   By definition, a *holomorphic p-chain* is a current $T = \sum n_j[W_j]$ where $n_j \in \mathbb{Z}$ and $[W_j]$ is the integration current on the complex analytic p-dimensional subvariety W_j of a complex analytic manifold.
   (a) Structure theorem of complex analytic subvarieties (or, more generally, holomorphic p-chains) [K 71], [HS 74], [H 77], [Sh 55], [Al 97].
   (b) Characterization of boundaries (in the sense of currents) of complex analytic subvarieties, (or, more generally, holomorphic p-chains) in $\mathbb{C}^n$ [HL 75], $\mathbb{C}P^n \setminus \mathbb{C}P^{n-q}$ [HL 77], [H 77], $\mathbb{C}P^n$, or linearly concave open sets of $\mathbb{C}P^n$ [DH 97], [Di 98].
   (c) Characterization of boundary (in the sense of currents) of a Levi-flat hypersurface in $\mathbb{C}^n$, $n \geq 3$ [DTZ 05].

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