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HOLONOMY ORBITS OF THE SNAKE CHARMER ALGORITHM

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A *snake* (of length L) is a (continuous) piecewise $\mathcal{C}^1$-curve $\mathcal{S} : [0, L] \to \mathbb{R}^d$, parameterized by arc-length and whose "tail" is at the origin ($\mathcal{S}(0) = 0$). Charming a snake consists in having it move in such a way that its "snout" $\mathcal{S}(L)$ follows a chosen $\mathcal{C}^1$-curve $\gamma(t)$. The *snake charmer algorithm*, initiated in [Ha2] for polygonal snakes and developed in [Ro] in the general case, works as follows. The input is a pair $(\mathcal{S}, \gamma)$, where:

(i) $\mathcal{S} : [0, L] \to \mathbb{R}^d$ is a snake of length L,
(ii) $\gamma : [0, 1] \to \mathbb{R}^d$ is $\mathcal{C}^1$-curve with $\gamma(0) = \mathcal{S}(L)$.

The output will then be a continuous 1-parameter family $\mathcal{S}_t$ of snakes of length L satisfying $\mathcal{S}_0 = \mathcal{S}$ and $\mathcal{S}_t(L) = \gamma(t)$. This algorithm, described in Section 2 below, is Ehresmannian in nature: the output is a horizontal lifting for some connection. A holonomy phenomenon for closed curves γ then occurs: having $\gamma(1) = \gamma(0)$ does not imply that $\mathcal{S}_1 = \mathcal{S}_0$. Given a snake $\mathcal{S}$, one can input in the snake charmer algorithm the pairs $(\mathcal{S}, \gamma)$ for all possible $\mathcal{C}^1$-loops γ at $\mathcal{S}(L)$. The snakes $\mathcal{S}_1$ obtained this way form the *holonomy orbit* of $\mathcal{S}$. The purpose of this note is to study these holonomy orbits, proving that, in good cases, they are compact smooth manifolds diffeomorphic to real Stiefel manifolds.

The paper is organized as follows. After some preliminaries in Section 1, we give in Section 2 a survey of the snake charmer algorithm and of its main properties (more details are to be found in [Ro]). Section 3 presents our new results about holonomy orbits with their proofs. Section 4 contains examples.

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1. Preliminaries

1.1. Let L be a positive real number and let $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = L\}$ be a finite partition of $[0, L]$. If (X, d) is a metric space, a map $z : [0, L] \to X$ is said *piecewise continuous for* $\mathcal{P}$ if, for every $i = 0, \ldots, N-1$, the restriction of z to the semi-open interval $[s_i, s_{i+1})$ extends to a (unique) continuous map z_i defined on the closed interval $[s_i, s_{i+1}]$. In particular, z is continuous on the right and the discontinuities, only possible at points of $\mathcal{P}$, are just jumps. We denote by $\mathcal{C}^0_{\mathcal{P}}([0, L], X)$ the space of maps from $[0, L]$ to X which are piecewise continuous for $\mathcal{P}$; this space is endowed with the uniform convergence distance

$$d(z_1, z_2) = \sup_{s \in [0,L]} \{z_1(s), z_2(s)\}.$$

When $\mathcal{P}$ is empty, the map z is just continuous and $\mathcal{C}^0_{\emptyset}([0, L], X) = \mathcal{C}^0([0, L], X)$. The map $z \mapsto (z_1, \ldots, z_N)$ provides a homeomorphism

$$\mathcal{C}^0_{\mathcal{P}}([0, L], X) \approx \prod_{i=0}^{N-1} \mathcal{C}^0([s_i, s_{i+1}], X). \tag{1}$$

If X is a Riemannian manifold, the space $\mathcal{C}^0([s_i, s_{i+1}], X)$ naturally inherits a Banach manifold structure (see [Ee]), so $\mathcal{C}^0_{\mathcal{P}}([0, L], X)$ is a Banach manifold using (1). The tangent space $T_z\mathcal{C}^0_{\mathcal{P}}([0, L], X)$ to $\mathcal{C}^0_{\mathcal{P}}([0, L], X)$ at z is the space of those $v \in \mathcal{C}^0_{\mathcal{P}}([0, L], TX)$ satisfying $p \circ v = z$, where $p : TX \to X$ is the natural projection.

1.2. The unit sphere in $\mathbb{R}^d$ centered at the origin is denoted by $\mathbb{S}^{d-1}$. Let $\text{Möb}(d-1)$ be the group of Möbius transformations of $\mathbb{S}^{d-1}$. It is a Lie group of dimension $d(d+1)/2$, with $SO(d)$ as a compact maximal subgroup. Its Lie algebra is denoted by $\mathfrak{möb}(d-1)$. For $0 \neq v \in \mathbb{R}^d$, we define a 1-parameter subgroup Γ^v_t of $\text{Möb}(d-1)$ by

$$\Gamma^v_t = \varphi_v^{-1} \circ \rho^v_t \circ \varphi_v$$

where

- $\varphi_v : \mathbb{S}^{d-1} \to \widehat{\mathbb{R}}^d$ is the stereographic projection sending $v/|v|$ to ∞ and $-v/|v|$ to 0;
- $\rho^v_t : \widehat{\mathbb{R}}^d \to \widehat{\mathbb{R}}^d$ is the homothety $\rho^v_t(x) = e^{t|v|}x$.

Thus, Γ^v_t is a purely hyperbolic flow with stable fixed point $v/|v|$ and unstable fixed point $-v/|v|$. We agree that $\Gamma^0_t = \text{id}$. Let $C_v \in \mathfrak{möb}(d-1)$ such that $\Gamma^v_t = \exp(tC_v)$ (with $C_0 = 0$). The correspondence $v \mapsto C_v$ gives an injective linear map $\chi : \mathbb{R}^d \to \mathfrak{möb}(d-1)$; its image is a d-dimensional vector subspace $\mathcal{H}$ of $\mathfrak{möb}(d-1)$, complementary to $\mathfrak{so}(d)$, and $\mathcal{H}$ generates $\mathfrak{möb}(d-1)$ as a Lie algebra. Let $\Delta^{\mathcal{H}}$ be the right invariant distribution on $\text{Möb}(d-1)$ which is equal to $\mathcal{H}$ at the unit element.

The 1-parameter subgroups Γ^v_t may be used to build up a diffeomorphism $\Psi : \mathbb{R}^d \times SO(d-1) \xrightarrow{\approx} \text{Möb}(d-1)$ defined by $\Psi(v, \rho) = \Gamma^v_1 \cdot \rho$.

2. The algorithm.

In this section, we give a survey of the snake charmer algorithm and some of its properties. For details, see [Ro].

2.1. Fix a positive real number L and a finite set $\mathcal{P} \subset [0, L]$ as in 1.1. Let $\text{Conf} = \mathcal{C}^0_{\mathcal{P}}([0, L], \mathbb{S}^{d-1})$, with its Banach manifold structure coming from the standard Riemannian structure on $\mathbb{S}^{d-1}$. The inclusion of $\mathbb{S}^{d-1}$ into $\mathbb{R}^d$ makes Conf a submanifold of the Banach space $\prod_{i=0}^{N-1} \mathcal{C}^0([s_i, s_{i+1}], \mathbb{R}^d)$.

The space Conf is the *space of configurations* for the snakes of length L which are continuous and "piecewise $\mathcal{C}^1$ for $\mathcal{P}$". The snake $\mathcal{S}_z$ associated to $z \in$ Conf is the map $\mathcal{S}_z(s) = \int_0^s z(\tau)d\tau$. Taking its "snout" $S_z(L)$ provides a map $f : \text{Conf} \to \mathbb{R}^d$, defined by

$$f(z) = \int_0^L z(s)ds$$

which is proven to be smooth. The image of f is the closed ball of radius L centered at the origin.

The snakes corresponding to piecewise constant configurations, $z(s) = z_i$ for $s \in [s_{i-1}, s_i)$, are called *polygonal snakes*. In this case, we see f as a map from $(\mathbb{S}^{d-1})^N$ to $\mathbb{R}^d$ sending $z = (z_1, \dots, z_N)$ to $\sum_{i=1}^N (s_i - s_{i-1})z_i$. For more details on this particular case see [Ha2]. If all the $(s_i - s_{i-1})$ are equal, the snake is called an *isosceles polygonal snake*.

The critical points of f are the *lined* configurations, where $\{z(s) \mid s \in [0, L]\} \subset \{\pm p\}$ for a point $p \in \mathbb{S}^{d-1}$ (for polygonal snakes, this is [Ha1, Theorem 3.1]). The snake associated to such a configuration is then contained in the line through p. The set of critical values is thus a finite collection of $(d-1)$-spheres centered at the origin, which depends on $\mathcal{P}$.

Charming snakes will now be a path-lifting ability for the map f: given an initial configuration $z \in$ Conf and a $\mathcal{C}^1$-curve $\gamma : [0,1] \to \mathbb{R}^d$ such that $\gamma(0) = f(z)$, we are looking for a curve $t \mapsto z_t \in$ Conf such that $z_0 = z$ and $f(z_t) = \gamma(t)$. In Ehresmann's spirit, we are looking for a connection for the map f. The tangent space T_z Conf to Conf at z is the vector space of those maps $v \in \mathcal{C}^0_{\mathcal{P}}([0, L], \mathbb{R}^d)$ such that $\langle v(s), z(s)\rangle = 0$ for all $s \in [0, L]$, where $\langle , \rangle$ denotes the usual scalar product in $\mathbb{R}^d$. We endow T_z Conf with the scalar product $\langle v, w\rangle = \int_0^L \langle v(s), w(s)\rangle ds$. For each smooth map $\varphi : \mathbb{R}^d \to \mathbb{R}$, one gets a vector field Grad$\,(\varphi\circ f)$ on Conf defined by

$$\text{Grad}_{\,z}(\varphi\circ f)(s) = \text{grad}_{\,f(z)}\varphi - \langle z(s), \text{grad}_{\,f(z)}\varphi\rangle z(s)\ .$$

This vector field plays the role of the gradient of $\varphi\circ f$, that is

$$\langle \text{Grad}_{\,z}(\varphi\circ f), v\rangle = T_z(\varphi\circ f)(v)$$

for each $v \in T_z$ Conf (as the metric induced by our scalar product is not complete, gradients do not exist in general). For $z \in$ Conf, the set of all Grad$_{\,z}(\varphi\circ f)$ for $\varphi \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ is a vector subspace Δ_z of T_z Conf, of dimension $d-1$ if z is a lined configuration and d otherwise. The correspondence $z \mapsto \Delta_z$ is a distribution Δ (of non-constant dimension). For a pair $(z, \gamma) \in \text{Conf} \times \mathcal{C}^1([0,1], \mathbb{R}^d)$ such that $f(z) = \gamma(0)$, the snake charmer algorithm takes for z_t the horizontal lifting of γ for the connection Δ.

As the map f is not proper and the dimension of Δ_z is not constant, the existence of horizontal liftings has to be established. We use the $\mathcal{C}^\infty$-action of the Möbius group Möb$(d-1)$ on Conf by post-composition: $g\cdot z = g\circ z$. For $z_0 \in$ Conf, let $\mathcal{A}(z_0)$ be the subspace of those $z \in$ Conf which can be joined to z_0 by a succession of Δ-horizontal curves. One of the main results ([Ro, Theorem 2.19], proven in [Ha2] for isosceles polygonal snakes) says that $\mathcal{A}(z_0)$ coincides with the orbit of z_0 under the action of Möb$(d-1)$:

THEOREM 2.2. *For all* $z_0 \in \mathrm{Conf}$, *one has* $\mathcal{A}(z_0) = \mathrm{Möb}(d-1) \cdot z_0$.

The proof of Theorem 2.2 uses the following fact about the action of the 1-parameter subgroup Γ_t^v of $\mathrm{Möb}(d-1)$ introduced in 1.2:

LEMMA 2.3. *The flow* $\phi_t(z) = \Gamma_t^v \cdot z$ *on* Conf *is the gradient flow of the map* $z \mapsto \langle f(z), v\rangle$.

The flow ϕ_t on Conf is therefore Δ-horizontal. If $z \in \mathrm{Conf}$, denote by $\beta_z : \mathrm{Möb}(d-1) \to \mathrm{Conf}$ the smooth map $\beta_z(g) = g \cdot z$. Lemma 2.3 implies that the map β_z sends the distribution $\Delta^{\mathcal{H}}$ onto the distribution Δ. If z is not a lined configuration, $T_g\beta_z : \Delta_g^{\mathcal{H}} \to \Delta_{g\cdot z}$ is an isomorphism; if z is lined, there is a 1-dimensional kernel (for instance, if $z(s) \in \{\pm p\}$, then $\Gamma_t^p \cdot z = z$ for all t).

2.4. *The differential equation.* As a consequence of Theorem 2.2, a Δ-horizontal curve z_t starting at z_0 may be written as $g(t) \cdot z_0$ where $g(t) \in \mathrm{Möb}(d-1)$ and $g(0) = \mathrm{id}$. For a given $\mathcal{C}^1$-curve $\gamma(t) \in \mathbb{R}^d$, the Δ-horizontal lifting $g(t) \cdot z_0$ of $\gamma(t)$ is obtained by taking for $g(t)$ the solution of an ordinary differential equation in the Möbius group $\mathrm{Möb}(d-1)$ that we describe below (more details can be found in [Ro, Chapter 3]).

For $z \in \mathrm{Conf}$, we define the $(d \times d)$-matrix $M(z)$ by

$$M(z) = \begin{pmatrix} L & & 0 \\ & \ddots & \\ 0 & & L \end{pmatrix} - \begin{pmatrix} \int_0^L z_1(s)z_1(s)ds & \cdots & \int_0^L z_1(s)z_d(s)ds \\ \vdots & & \vdots \\ \int_0^L z_1(s)z_d(s)ds & \cdots & \int_0^L z_d(s)z_d(s)ds \end{pmatrix}.$$

(Observe that the second term is the Gram matrix of the vectors $(z_1, \ldots, z_d)$ for the scalar product). It can be proved that $M(z)$ is invertible if and only if z is not lined. Let $F = F_{z_0} : \mathrm{Möb}(d-1) \to \mathbb{R}^d$ be the composition $F = f \circ \beta_{z_0}$. Let $g \in \mathrm{Möb}(d-1)$. Using the linear injective map χ of 1.2, with image $\mathcal{H}$, and the right translation diffeomorphism R_g in $\mathrm{Möb}(d-1)$, we get a linear map

$$\mathbb{R}^d \xrightarrow[\approx]{\chi} \mathcal{H} \xrightarrow[\approx]{T_eR_g} \Delta_g^{\mathcal{H}} \xrightarrow{T_gF} T_{F(g)}\mathbb{R}^d \approx \mathbb{R}^d. \tag{2}$$

It turns out that the matrix of the linear map (2) is $M(g \cdot z_0)$. If $g \cdot z_0$ is not a lined configuration, T_gF is bijective and $M(g \cdot z_0)$ is invertible. One way to insure that $g \cdot z_0$ is not lined for all $g \in \mathrm{Möb}(d-1)$ is to assume that z_0 takes at least 3 distinct values. The following result is proven in [Ro, Prop. 3.10]:

PROPOSITION 2.5. *Let* $z_0 \in \mathrm{Conf}$ *be a configuration that takes at least* 3 *distinct values. Let* $\gamma : [0,1] \to \mathbb{R}^d$ *be a* $\mathcal{C}^1$*-curve with* $\gamma(0) = f(z)$. *Suppose that the* $\mathcal{C}^1$*-curve* $g : [0,1] \to \mathrm{Möb}(d-1)$ *satisfies the following differential equation:*

$$\dot g(t) = T_eR_{g(t)} \circ \chi(M^{-1}(g(t) \cdot z_0)\,\dot\gamma(t)), \quad g(0) = \mathrm{id}. \tag{3}$$

Then, the curve $g(t) \cdot z_0$ *is the unique* Δ*-horizontal lifting of* γ *starting at* z_0.

As the Möbius group is not compact and the map F is not proper, differential equation (3) may not have a solution for all $t \in [0,1]$. The notion of "sedentariness", described in 2.7, is the main tool to study the global existence of Δ-horizontal liftings.

2.6. *The cases $d = 2$ or $d = 3$.* For planar snakes ($d = 2$), one can use that Möb(1) is isomorphic to $PSU(1,1) = SU(1,1)/\{\pm\mathrm{id}\}$, where $SU(1,1) = \{\left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a}\end{smallmatrix}\right) \mid a, b \in \mathbb{C} \text{ and } |a|^2 - |b|^2 = 1\}$. The isomorphism between Möb(1) and $PSU(1,1)$ comes from the $SU(1,1)$-action on $\mathbb{C}$ by homographic transformations, which preserves the unit circle. The group $SU(1,1)$ being thus a 2-fold covering of Möb(1), the curve $g(t) \in$ Möb(1) of Proposition 2.5 admits a unique lifting in $\tilde{g}(t) \in SU(1,1)$ with $\tilde{g}(0) = \mathrm{id}$. The output of the snake charmer algorithm will then be of the form $z_t = \tilde{g}(t) \cdot z_0$, using the action of $SU(1,1)$ on Conf(1). Working in the matrix group $SU(1,1)$ is of course favorable for numerical computations.

The Lie algebra $\mathfrak{psu}(1,1) = \mathfrak{su}(1,1)$ consists of matrices of the form $\left(\begin{smallmatrix} iu & b \\ \bar{b} & -iu\end{smallmatrix}\right)$ with $u \in \mathbb{R}$ and $b \in \mathbb{C}$. Under the isomorphism $\mathfrak{su}(1,1) \approx \mathfrak{m\ddot{o}b}(1)$, the element $\chi(v) \in \mathcal{H} \subset \mathfrak{m\ddot{o}b}(1)$ corresponds to the matrix $\left(\begin{smallmatrix} 0 & v \\ \bar{v} & 0\end{smallmatrix}\right) \in \mathfrak{su}(1,1)$. These matrices form a 2-dimensional vector space $\tilde{\mathcal{H}} \subset \mathfrak{su}(1,1)$ giving rise to a right invariant distribution $\Delta^{\tilde{\mathcal{H}}}$ on $SU(1,1)$. By Proposition 2.5, the curve $\tilde{g}(t) \cdot z_0 \in \mathrm{Conf}(1)$ is horizontal if $\tilde{g}(t)$ is $\Delta^{\tilde{\mathcal{H}}}$-horizontal. In this language, Equation (3) becomes

$$\dot{\tilde{g}}(t) = \begin{pmatrix} 0 & v/2 \\ \bar{v}/2 & 0 \end{pmatrix} \tilde{g}(t), \qquad g(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

where $v = v_1(g,t) + iv_2(g,t)$ is obtained by

$$\begin{pmatrix} v_1(g(t),t) \\ v_2(g(t),t) \end{pmatrix} = M(\tilde{g}(t) \cdot z_0) \begin{pmatrix} \dot{\gamma}_1(t) \\ \dot{\gamma}_2(t) \end{pmatrix}.$$

For more details, see [Ro, Chapter 5]. An analogous (and formally similar) matrix approach is available for spatial snakes ($d = 3$), using the isomorphism $PSL(2,\mathbb{C}) \approx$ Möb(2); see [Ro, Prop. 5.6].

2.7. *Sedentariness.* Denote by μ the Lebesgue measure on $\mathbb{R}$. Let $z \in \mathrm{Conf}$. The *sedentariness* $\mathrm{sed}(z) \in [0, L]$ of z is defined by $\mathrm{sed}(z) = \max_{p \in \mathbb{S}^{d-1}} \mu(z^{-1}(p))$. This maximum exists since the set $\{p \in \mathbb{S}^{d-1} \mid \mu(z^{-1}(p)) > r\}$ is finite for all $r > 0$. By Theorem 2.2, $\mathrm{sed}(z)$ is an invariant of $\mathcal{A}(z)$. Observe that, if $\mathrm{sed}(z) < L/2$, then z takes at least 3 distinct values. The sedentariness of z is used in [Ro, Section 3.3] to get global existence results for horizontal liftings of a path starting at $f(z)$. The one we need is the following

PROPOSITION 2.8. *Let $z \in \mathrm{Conf}$ and let $\gamma : [0,1] \to \mathbb{R}^d$ be a $\mathcal{C}^1$-path with $\gamma(0) = f(z)$. Suppose that $\gamma([0,1])$ is contained in the open ball centered at 0 of radius $L - 2\,\mathrm{sed}(z)$. Then there exists a (unique) Δ-horizontal lifting $\tilde{\gamma} : [0,1] \to \mathrm{Conf}$ for γ with $\tilde{\gamma}(0) = z$.*

As the sedentariness of a configuration is preserved along horizontal curves, Proposition 2.8 is also true for continuous piecewise $\mathcal{C}^1$-paths. A configuration z is called *nomadic* if $\mathrm{sed}(z) = 0$. Proposition 2.8 guarantees that, if z is nomadic, any $\mathcal{C}^1$-path starting at $f(z)$ admits a horizontal lifting, provided its image stays in the open ball of radius L. More general results may be found in [Ro, Section 3.3].

2.9. *Continuity of the algorithm.* We now describe how the snake charmer algorithm behaves as we vary the initial configuration z_0 and the $\mathcal{C}^1$-curve γ. Our goal is not to present the most general statements but only those needed in Section 3.

Let $z_0 \in \mathrm{Conf}$, $b = f(z_0)$ and let $\mathrm{Conf}^b = f^{-1}(b)$. Set $\sigma = \mathrm{sed}(z_0)$ and suppose that $|f(z_0)| < L - 2\sigma$. Denote by $B_{L-2\sigma}(0)$ the open ball in $\mathbb{R}^d$ of radius $L - 2\sigma$ centered at the origin. Consider a $\mathcal{C}^1$-curve $\gamma : [0,1] \to B_{L-2\sigma}(0)$ with $\gamma(0) = b$. Define $E_{\sigma,b} = \{z \in \mathrm{Conf}^b \mid \mathrm{sed}(z) \leq \sigma\}$; it is a metric space with the induced metric from Conf. Consider the map

$$X : \mathrm{M\ddot{o}b}(d-1) \times E_{\sigma,b} \times [0,1] \mapsto T\mathrm{M\ddot{o}b}(d-1),$$

defined by

$$X(g,z,t) = R_{g(t)} \circ \chi(M^{-1}(g(t) \cdot z_0)\, \dot\gamma(t)).$$

This is a vector field on $\mathrm{M\ddot{o}b}(d-1)$ depending on the time t and on the parameter $z \in E_{\sigma,b}$. Differential equation (3) becomes

$$\dot g(t) = X(g(t), z, t)\ , \quad g(0) = \mathrm{id}. \tag{5}$$

For any given $z \in E_{\sigma,b}$, the solution g_z of equation (5) exists for all $t \in [0,1]$ by Proposition 2.8. Since X is continuous and its derivative in g is a continuous map in the variables z and t, classical results on the dependence of parameters for the solution (see for example [Sc, 4.3.11]) imply that the map $(z,t) \mapsto g_z(t)$ is continuous. This yields:

Proposition 2.10 (Continuity in z). *Let $\gamma : [0,1] \to B_{L-2\sigma}(0)$ be a $\mathcal{C}^1$-curve such that $\gamma(0) = b$. For any $z \in E_{\sigma,b}$, denote by $\tilde\gamma_z(t) = g_z(t) \cdot z$ the Δ-horizontal lift of γ starting at z. The map $E_{\sigma,b} \times [0,1] \to \mathrm{Conf}$ that sends (z,t) to $\tilde\gamma_z(t)$ is continuous.*

Let $M \subset E_{\sigma,b}$ be a smooth submanifold of Conf and suppose that γ is of class $\mathcal{C}^2$. The map X restricted to M is therefore $\mathcal{C}^1$ (in all variables) and using once again [Sc, 4.3.11] we get:

Proposition 2.11 (Differentiability in z). *Let $\gamma : [0,1] \to B_{L-2\sigma}(0)$ be a $\mathcal{C}^2$-curve such that $\gamma(0) = b$. For any $z \in M$, denote by $\tilde\gamma_z(t) = g_z(t) \cdot z$ the Δ-horizontal lift of γ starting at z. The map $M \times [0,1] \to \mathrm{Conf}$ that sends (z,t) to $\tilde\gamma_z(t)$ is of class $\mathcal{C}^1$.*

We have an analogous result when we fix the configuration z_0 and vary γ. Consider the set $C_{\sigma,b} = \{\gamma \in \mathcal{C}^1([0,1], \mathbb{R}^d) \mid \gamma(0) = b,\ \gamma([0,1]) \subset B_{L-2\sigma}(0)\}$ that is an open subset of the affine space of $\mathcal{C}^1$-paths starting at b.

Proposition 2.12 (Continuity in γ). *Let $z_0 \in \mathrm{Conf}$, $b = f(z_0)$ and $\sigma = \mathrm{sed}(z_0)$. Suppose that $|f(z_0)| < L - 2\sigma$. For any $\gamma \in C_{\sigma,b}$, denote by $\tilde\gamma_z(t)$ the Δ-horizontal lift of γ starting at z_0. The map $C_{\sigma,b} \times [0,1] \to \mathrm{Conf}$ that sends (γ,t) to $\tilde\gamma_z(t)$ is continuous.*

2.13. *Bivalued configurations.* Let z_0 be a bivalued configuration, that is $z_0([0,L]) = \{p_0, q_0\}$, with $p_0 \neq q_0$. Let L_p and L_q be the Lebesgue measures of $z_0^{-1}(p_0)$ and $z_0^{-1}(q_0)$ (these preimages are finite unions of intervals). By Theorem 2.2, a horizontal curve z_t starting at z_0 will stay bivalued: $z_t([0,L]) = \{p(t), q(t)\}$, with $p(0) = p_0$, $q(0) = q_0$ and $p(t) \neq q(t)$. Also, one has $z_t^{-1}(p(t)) = z_0^{-1}(p_0)$, $z_t^{-1}(q(t)) = z_0^{-1}(q_0)$ and therefore z_t is determined by the pair $(p(t), q(t))$. Hence, $\mathcal{A}(z_0)$ is contained in a compact submanifold W of Conf naturally parameterized by $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$. The restriction of f to W gives a $\mathcal{C}^\infty$-map $\hat f : W \to \mathbb{R}^d$ which takes the explicit form $\hat f(p,q) = L_p p + L_q q$. The definition of the distribution Δ implies that $\Delta_{(p,q)} \subset T_{(p,q)}W$ when $(p,q) \in W$.

Consider the open sets $W^0 \subset W$ and $U^0 \subset \mathbb{R}^d$ defined by

$$W^0 := \{(p,q) \in W \mid p \neq \pm q\}\ , \quad U^0 := \{x \in \mathbb{R}^d \mid |x| < L \text{ and } |x| \neq L_p - L_q\}$$

As W^0 contains no lined configurations, the map $\hat{f}$ restricts to a submersion $\hat{f}^0 = f^0 : W^0 \to U^0$ for which Δ is an ordinary connection. As f^0 extends to $\hat{f} : W \to \mathbb{R}^d$ and as W is compact, the map f^0 is proper. Ehresmann's c original onstruction of horizontal liftings [Eh] then applies: if $z_0 \in W^0$ and $\gamma : [0,1] \to U^0$ is a $\mathcal{C}^1$-curve $\gamma(0) = z_0(L)$, then γ admits a Δ-horizontal lifting $\tilde{\gamma} : [0,1] \to W^0$ of f^0 with $\tilde{\gamma}(0) = z_0$.

Let us specialize to planar snakes ($d = 2$). Each fiber of f^0 then consists of two points; as f^0 is proper, it is thus a 2-fold cover of U^0. Therefore, $\tilde{\gamma}(t)$ is determined by $f(\tilde{\gamma}(t)) = \gamma(t)$. We deduce that *the unique lifting of γ into* Conf *is horizontal.*

If, for $t = t_0$, $\gamma(t)$ crosses the sphere of radius $L_p - L_q$, it may happen that $\tilde{\gamma}(t)$ tends to a lined configuration when $t \to t_0$ (see Example 4.2 below). To understand when the unique lifting $\tilde{\gamma}(t)$ is horizontal at t_0, we must study horizontal liftings around a lined configuration, which, after changing notations, we call again z_0, corresponding to $(p_0, q_0) \in W$ with $p_0 = -q_0$. The vector space Δ_{z_0} is then of dimension 1 and it turns out that $T_{z_0} f(\Delta_{z_0})$ is the line orthogonal to p_0, see [Ro, Remark 1.17]. Let $\gamma : [0,1] \to \mathbb{R}^d$ be a $\mathcal{C}^1$-curve with $\gamma(0) = z_0(L)$ and $\dot{\gamma}(0) \neq 0$. A necessary condition for γ to admit a horizontal lifting is then $\langle \dot{\gamma}(0), p_0 \rangle = 0$. If we orient the plane with the basis $(p_0, \dot{\gamma}(0))$, the curve γ has a signed curvature $\kappa(0)$ at $t = 0$. In [Ro, Prop. 3.18], it is proven that, around $t = 0$, the unique lifting of γ into Conf is horizontal if and only if $\dot{\gamma}(0)$ is orthogonal to p_0 and

$$\kappa(0) = \frac{L_{p_0} - L_{q_0}}{L^2}. \tag{6}$$

Condition (6) has been detected by the numerician Ernst Hairer. Such a second-order condition for the existence of a horizontal lifting for a non-constant rank distribution is worth being studied. As far as we know, no such phenomenon is mentioned in the literature.

2.14. *Miscellanies.* We finish this section by listing a few more properties of the snake charmer algorithm. Let $(\mathcal{S}, \gamma)$ be an input for the algorithm, with $\mathcal{S}$ a snake of length L. Let $\mathcal{S}_t : [0,L] \to \mathbb{R}^d$ be the output. The following two results follow directly from Theorem 2.2.

Proposition 2.15 (Regularity). *Suppose that, on some open subset $U \subset [0,L]$, the snake $\mathcal{S}$ is of class $\mathcal{C}^k$, $k \in \{1, 2, \dots, \infty\}$. Then the same holds true for the snake $\mathcal{S}_t$ for all $t \in [0,1]$.*

Proposition 2.16 (Periodicity). *Suppose that there exists $T \in \mathbb{R}$ such that $z(s) = z(s+T)$ for all s such that s and $s+T$ belong to $[0,L]$. Then, $z_t(s) = z_t(s+T)$ for all $t \in [0,1]$.*

The following proposition follows either from Theorem 2.2 or simply from the fact that $\mathcal{S}_t$ is a horizontal lifting.

Proposition 2.17 (Reparameterization). *Let $\varphi : [0,1] \to [0,1]$ be an orientation preserving $\mathcal{C}^1$-diffeomorphism. Then, the deformation of $\mathcal{S}$ following the curve $\gamma(\varphi(u))$ is $\mathcal{S}_{\varphi(u)}$.*

Finally, by construction, the distribution Δ is orthogonal to fibers of $f : \mathrm{Conf} \to \mathbb{R}^d$. This can be rephrased in the following

PROPOSITION 2.18. *z_t is the unique lifting of γ which, for all t, minimizes the infinitesimal kinetic energy of the hodograph, that is $\frac{1}{2}\int_0^L |\frac{d}{dt} z_t(s)|^2\, ds$.*

Again, the existence of such a minimizer $\mathcal{S}_t$ is only guaranteed by an analysis like in 2.7.

3. Holonomy orbits. Let $b \in \mathbb{R}^d$. Define $\mathrm{Conf}^b = f^{-1}(b)$, the space of all configurations with associated snake ending at b. Define the *holonomy orbit* $\mathrm{horb}(z_0) \subset \mathrm{Conf}^b$ of $z_0 \in \mathrm{Conf}^b$ by $\mathrm{horb}(z_0) = \mathcal{A}(z_0) \cap \mathrm{Conf}^b$; it is thus the subspace of those $z \in \mathrm{Conf}$ which are the result of the holonomy of the snake charmer algorithm for a pair (z_0, γ) with γ a piecewise $\mathcal{C}^1$-loop at b. Even if z_0 is not lined, the point b might not be a regular value of f, so $f^{-1}(b)$ might not be a submanifold of Conf. But Lemma 3.1 below tells us that this is the case for $\mathrm{horb}(z_0)$. For $z \in \mathrm{Conf}$, define the *spherical dimension* $\mathrm{spdim}(z)$ of z to be the minimal dimension of a sub-sphere of $\mathbb{S}^{d-1}$ containing the set $z([0, L])$. By Theorem 2.2, $\mathrm{spdim}(z)$ is an invariant of $\mathcal{A}(z)$. Notice that $\mathrm{spdim}(z) = 0$ if and only if the configuration z takes one or two values.

LEMMA 3.1. *Let $z_0 \in \mathrm{Conf}$. Let $k = \mathrm{spdim}(z_0)$ and suppose that $k > 0$. Then $\mathrm{horb}(z_0)$ is a smooth submanifold of Conf of dimension $\sum_{i=1}^{k+1}(d-i)$.*

Generically, $\mathrm{spdim}(z_0) = d-1$, so $\mathrm{horb}(z_0)$ is of dimension $d(d-1)/2$. The case where $\mathrm{spdim}(z_0) = 0$ is not covered by Lemma 3.1. It contains the monovalued case (z being constant) where $\mathrm{horb}(z_0) = f^{-1}(f(z_0)) = \{z_0\}$. The other case, formed by the bivalued configurations, is interesting and is treated in Example 4.2 and Proposition 4.4.

Proof of Lemma 3.1. Let $\beta : \mathrm{M\ddot{o}b}(d-1) \to \mathrm{Conf}$ be the map $\beta(g) = g \cdot z_0$ and let $F : \mathrm{M\ddot{o}b}(d-1) \to \mathbb{R}^d$ be the composition $F = f \circ \beta$. Let $K = F^{-1}(f(z_0))$. By Theorem 2.2, one has $\mathrm{horb}(z_0) = \beta(K)$. The map F is smooth.

Since $\mathrm{spdim}(z_0) > 0$, the configuration z_0 takes at least three values in $\mathbb{S}^{d-1}$ and so does $g \cdot z_0$ for all $g \in \mathrm{M\ddot{o}b}(d-1)$. Therefore, $\mathcal{A}(z_0)$ contains no lined configurations and the tangent map $T_g F : T_g\mathrm{M\ddot{o}b}(d-1) \to T_{F(g)}\mathbb{R}^d \approx \mathbb{R}^d$ is surjective for all $g \in \mathrm{M\ddot{o}b}(d-1)$ (restricted to $\Delta_g^{\mathcal{H}}$, it is an isomorphism). Hence, F is a submersion. Thus, K is a smooth submanifold of $\mathrm{M\ddot{o}b}(d-1)$ of dimension $\dim \mathrm{M\ddot{o}b}(d-1) - d = d(d+1)/2 - d = d(d-1)/2$.

The manifold K contains the stabilizer A of z_0 and $K \cdot A = K$. Let V^k be the smallest sub-sphere of $\mathbb{S}^{d-1}$ containing the image of z_0. The group A is then the stabilizer of the points of V^k; since $k > 0$, A is conjugate in $\mathrm{M\ddot{o}b}(d-1)$ to $SO(d-k-1)$. Therefore, the quotient space K/A is of dimension

$$\frac{d(d-1)}{2} - \frac{(d-k-1)(d-k-2)}{2} = \sum_{i=1}^{d-1} i - \sum_{j=1}^{d-k-2} j = \sum_{i=1}^{k+1}(d-i). \tag{7}$$

By [Ro, Proposition 2.33], the map β induces an embedding of $\mathrm{M\ddot{o}b}(d-1)/A$ into Conf, hence an embedding of K/A into Conf with image $\mathrm{horb}(z_0)$. This proves Lemma 3.1. ∎

In general, $\operatorname{horb}(z)$ is not closed; see examples in [Ha2, pp. 112–113][1]. But this is the case if $f(z)$ is near the origin:

THEOREM 3.2. *Let $z_0 \in \operatorname{Conf}$ such that $|f(z_0)| < L - 2\operatorname{sed}(z_0)$. Let $k = \operatorname{spdim}(z_0)$ and suppose that $k > 0$. Then, each connected component of $\operatorname{horb}(z_0)$ is a closed smooth submanifold of Conf which is diffeomorphic to the homogeneous space $SO(d)/SO(d-k-1)$.*

Observe that $SO(d)/SO(d-k-1)$ is the Stiefel manifold of orthonormal $(k+1)$-frames in $\mathbb{R}^d$.

Proof. Suppose first that $f(z_0) = 0$. Let $z \in \operatorname{horb}(z_0)$. For each $g \in SO(d)$, one has $f(g\cdot z) = g\cdot f(z) = 0$, hence $SO(d)\cdot z \subset \operatorname{horb}(z_0) = \operatorname{horb}(z)$. As in the proof of Lemma 3.1, let $\beta_z : \text{Möb}(d-1) \to \operatorname{Conf}$ be the map $\beta_z(g) = g \cdot z$. As $\operatorname{spdim}(z) = \operatorname{spdim}(z_0) = k > 0$, the stabilizer of z is conjugate in $SO(d)$ to $SO(d-k-1)$. Therefore, $\beta_z(SO(d))$ is a connected, compact submanifold of Conf diffeomorphic to $SO(d)/SO(d-k-1)$. By Lemma 3.1, $\beta_z(SO(d))$ must be the connected component of z in $\operatorname{horb}(z_0)$. This proves Theorem 3.2 when $f(z_0) = 0$.

In the general case, let $b = f(z_0)$ and let $\gamma : [0,1] \to \mathbb{R}^d$ be the linear parameterization $\gamma(t) = tb$ of the segment joining 0 to b. If $\delta : [0,1] \to X$ is a map, one denotes by $\delta^- : [0,1] \to X$ the map $\delta^-(t) = \delta(1-t)$. Let $z \in \operatorname{Conf}^b$ such that $\operatorname{sed}(z) \le \operatorname{sed}(z_0)$. By Proposition 2.8, the path γ admits a Δ-horizontal lifting $\tilde\gamma_z$ in Conf, starting at z. The Δ-parallel transport $\tau(z) = \tilde\gamma_z(1)$ is then defined. Of course, γ^- also admits a horizontal lifting $\tilde\gamma^-_{\tau(z)}$ which is equal to $(\gamma_z)^-$. By Proposition 2.10, the parallel transport gives rise to a homeomorphism

$$\tau : \{z \in \operatorname{Conf}^b \mid \operatorname{sed}(z) \le \operatorname{sed}(z_0)\} \xrightarrow{\approx} \{z \in \operatorname{Conf}^0 \mid \operatorname{sed}(z) \le \operatorname{sed}(z_0)\}.$$

One has $\tau(\operatorname{horb}(z_0)) = \operatorname{horb}(\tau(z_0))$. As $\operatorname{horb}(z_0)$ is a smooth submanifold of Conf, $\tau_{|\operatorname{horb}(z_0)}$ and $\tau^{-1}_{|\tau(\operatorname{horb}(z_0))}$ are smooth by Proposition 2.11. Therefore τ induces a diffeomorphism from $\operatorname{horb}(z_0)$ onto $\operatorname{horb}(\tau(z_0))$. This finishes the proof of Theorem 3.2. ■

In the particular case of planar snakes ($d = 2$), we get:

COROLLARY 3.3. *Let $z_0 \in \operatorname{Conf}(1)$ such that $|f(z_0)| < L - 2\operatorname{sed}(z_0)$ and $\operatorname{spdim}(z_0) = 1$. Then $\operatorname{horb}(z_0)$ is a disjoint union of circles.*

We do not know, under the hypotheses of Theorem 3.2 or Corollary 3.3, whether the manifold $\operatorname{horb}(z_0)$ is connected. This is not true if $|f(z_0)| > L-2\operatorname{sed}(z_0)$, see Example 4.2. It is however the case if $\operatorname{sed}(z_0) = 0$:

PROPOSITION 3.4. *Let $z_0 \in \operatorname{Conf}$ be a nomadic configuration. Let $k = \operatorname{spdim}(z_0)$. Then, $\operatorname{horb}(z_0)$ is a closed smooth submanifold of Conf which is diffeomorphic to the homogeneous space $SO(d)/SO(d-k-1)$.*

To prove Proposition 3.4, we need the following lemma whose proof is postponed till the end of this section.

[1] In [Ha2], one studies the set $\mathcal{H}_b(U) \cdot z_0$: if $b = f(z_0)$ and U is an open neighborhood of b, $\mathcal{H}_b(U)$ denotes the subgroup of diffeomorphisms of Conf^b obtained as the holonomy of a piecewise $\mathcal{C}^1$-loop at b that stays in U. Notice that $\mathcal{H}_b(U) \cdot z_0 \subset \operatorname{horb}(z_0)$ and that $\mathcal{H}_b(U) \cdot z_0$ contains the connected component of z_0 of $\operatorname{horb}(z_0)$.

LEMMA 3.5. *Let $z_0 \in \mathrm{Conf}$ and let $z \in \mathrm{horb}(z_0)$. Then, there exists a loop $\gamma : [0,1] \to \mathbb{R}^d$ at $f(z_0)$ which is of class $\mathcal{C}^\infty$ and which admits a horizontal lifting joining z_0 to z.*

Proof of Proposition 3.4. A nomadic configuration is not lined. Therefore, $k > 0$ and the condition $|f(z_0)| < L - 2\,\mathrm{sed}(z_0) = L$ is automatic. By Theorem 3.2, it is then enough to prove that $\mathrm{horb}(z_0)$ is connected.

Let $z \in \mathrm{horb}(z_0)$. By Lemma 3.5, there exists a loop $\gamma : [0,1] \to \mathbb{R}^d$ at $f(z_0)$, of class $\mathcal{C}^\infty$, admitting a horizontal lifting joining z_0 to z. For $s \in [0,1]$, define $\gamma_s(t) = (1-s)\gamma(0) + s\gamma(t)$. The map $s \mapsto \gamma_s$ is a homotopy of $\mathcal{C}^\infty$-loops at $f(z_0)$, from γ to the constant loop. By Proposition 2.8, each loop γ_s admits a horizontal lifting $\tilde{\gamma}_s$ starting at z_0. By Proposition 2.12, the curve $s \mapsto \tilde{\gamma}_s(1)$ is continuous, producing a path in $\mathrm{horb}(z_0)$ from z to z_0. This proves that $\mathrm{horb}(z_0)$ is connected, provided we prove Lemma 3.5. ■

Proof of Lemma 3.5. By Theorem 2.2, there exists $g \in \mathrm{M\ddot{o}b}(d-1)$ with $z = g \cdot z_0$. The Möbius group $\mathrm{M\ddot{o}b}(d-1)$ is arcwise connected using only curves that are piecewise trajectories of flows of the type Γ_t^v (see [Ha2, p. 102]). Therefore, there exists $(v_i, \lambda_i) \in \mathbb{R}^d \times \mathbb{R}$, $i = 1, \dots, r$, such that $g = \Gamma_{\lambda_r}^{v_r} \cdots \Gamma_{\lambda_1}^{v_1}$. For $i = 1, \dots, r$, let $\phi_i : [0,1] \to [0,1]$ be a $\mathcal{C}^\infty$-function such that $\phi_i([0, \frac{i-1}{r}]) = \{0\}$ and $\phi_i([\frac{i}{r}, 1]) = \{1\}$. The map $g(t) : [0,1] \to \mathrm{M\ddot{o}b}(d-1)$ given by

$$g(t) = \Gamma_{\phi_r(t)\lambda_r}^{v_r} \cdots \Gamma_{\phi_1(t)\lambda_1}^{v_1}$$

is a $\mathcal{C}^\infty$-curve in $\mathrm{M\ddot{o}b}(d-1)$, joining the unit element to g. By Lemma 2.3, the curve $g(t)$ is $\Delta^{\mathcal{H}}$-horizontal. Then $g(t) \cdot z_0$ is the horizontal lifting of the $\mathcal{C}^\infty$-path $\gamma(t) = f(g(t) \cdot z_0)$, which is a loop of class $\mathcal{C}^\infty$ at $f(z_0)$. ■

4. Examples. All our examples are planar snakes ($d = 2$). We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Configurations $z \in \mathrm{Conf}(1)$ are in the form $s \mapsto e^{i\theta(s)}$ with $\theta(s) \in \mathbb{R}$.

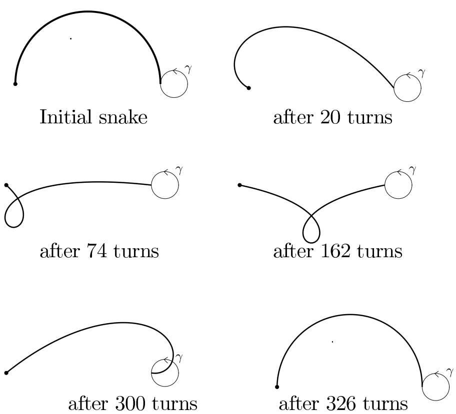


Fig. A

4.1. In our first example, the snake is a semicircle with configuration $z : [0, \pi] \to \mathbb{S}^1$ given by $z(s) = ie^{-is}$, thus $\mathcal{S} = \mathcal{S}_z$ is given by $\mathcal{S}(s) = 1 - e^{-is}$. We have $f(z) = \mathcal{S}(\pi) = 2$. The curve γ is a small circle, centered at $(2.1875, 0)$ and of radius 0.1875, followed in the trigonometric direction. The snake charmer algorithm has been solved using the method of 2.6. After one turn, the snake slightly leans to the left. Figure A below shows the snake after various numbers of turns of γ. One sees that after 326 turns, the snake seems being back in its initial position.

The curve $t \mapsto g(t) \in \text{Möb}(1)$ obtained from equation (3) (such that $z_t = g(t) \cdot z_0$), may be visualized using the diffeomorphism $\mathbb{R}^2 \times \mathbb{S}^1 \xrightarrow{\approx} \text{Möb}(1)$ described at the end of 1.2. The two fold covering $SU(1,1) \to \mathbb{R}^2 \times \mathbb{S}^1$ thus obtained is given in formula by

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mapsto (v, e^{i\theta})$$

with $\theta = 2\arg(a)$ and $v = 2\,\text{arccosh}(|a|)e^{i\arg(ab)}$. Figure B below illustrates the subset $\{(v(n \cdot 2\pi), \theta(n \cdot 2\pi)) \mid n = 1, \dots, 326\} \subset \mathbb{R}^2 \times \mathbb{S}^1$. The picture on the left-hand side shows the 326 points of the set $\{v(n \cdot 2\pi)\}$ $(v = (v_1, v_2))$ and the one on the right-hand side is the graph of the function $n \mapsto \theta(n \cdot 2\pi)$ (also formed by 326 points, but hardly distinguishable). That the points look more concentrated between 80 and 250 seems to be related to the fact, seen in Figure A, that the snake's shape changes less drastically in this range. As in Figure A, we observe that $\theta(t)$ and $v(t)$ seem to have returned to their original position after 326 turns.

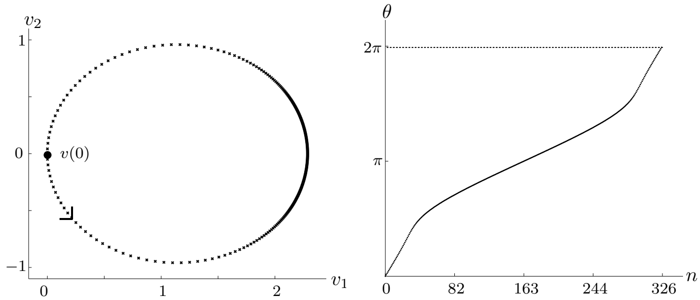


Fig. B

Figure C below shows the entire curve $(\theta(t), v(t))$ for $t \in [0, 326 \cdot 2\pi]$.

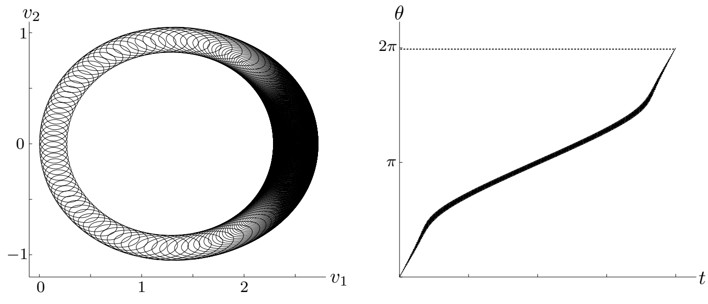


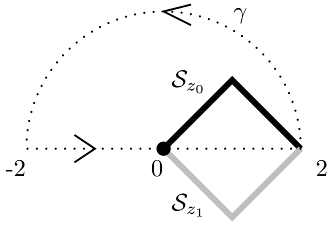
Fig. C

4.2. *A non-connected holonomy orbit.* Let $z_0 : [0, 2\sqrt{2}] \to \mathbb{S}^1$ be the 2-valued configuration

$$z_0(s) = \begin{cases} e^{i\pi/4} & \text{if } s < \sqrt{2}, \\ e^{-i\pi/4} & \text{if } s \geq \sqrt{2}. \end{cases}$$

Thus, $f(z) = 2$. Let $\gamma : [0,1] \to \mathbb{C}$ be the piecewise $\mathcal{C}^\infty$-loop at 2 defined by

$$\gamma(t) = \begin{cases} 2\,e^{2i\pi t} & \text{if } t \leq 1/2, \\ 8(t - 3/4) & \text{if } t \geq 1/2. \end{cases}$$



By 2.13, the unique possible lifting z_t, starting at z_0 and determined by $f(z_t) = \gamma(t)$ is horizontal. It follows that $z_1 = \bar{z}_0$ (the complex conjugate of z_0). Hence $\mathrm{horb}(z_0) = \{z_0, \bar{z}_0\}$ is not connected.

Using an arbitrarily small bump around 0, one can perturb γ so that it avoids the origin. But z_1 would then be equal to z_0. So z_t does not depend continuously on γ, in contrast to what happens in Proposition 2.12. Here, two hypotheses of Proposition 2.12 are not satisfied: $\mathrm{spdim}(z_0) = 0$ and, what seems more serious, the condition $|f(z_0)| < L - 2\,\mathrm{sed}(z_0)$ is impossible since $\mathrm{sed}(z_0) = \sqrt{2} = L/2$.

4.3. Example 4.2 is an illustration of the following computation of $\mathrm{horb}(z_0)$ when z_0 is bivalued.

PROPOSITION 4.4. *Let z_0 be a bivalued configuration and let $b = f(z_0)$. Then*

(i) $\mathrm{horb}(z_0) \approx \mathbb{S}^{d-2}$ *if z_0 is not lined;*
(ii) $\mathrm{horb}(z_0)$ *is one point if z_0 is lined and $f(z_0) \neq 0$;*
(iii) $\mathrm{horb}(z_0) \approx \mathbb{S}^{d-1}$ *if z_0 is lined and $f(z_0) = 0$.*

Proof. Let p_0, q_0, L_p, L_q, $\hat{f} : W \to \mathbb{R}^d$ and W^0 as in 2.13. One has $\mathrm{horb}(z_0) \subset \hat{f}^{-1}(b)$. If $w \in \mathbb{R}^d$, denote by B_w be the stabilizer of w in $SO(d)$. We divide the proof into four cases.

Case (i) with $d = 2$. Let γ be a loop at b, with $|\gamma(t)| < L$, such that γ hits the sphere of radius $|L_p - L_q|$ in a single point, tangentially and respecting the curvature formula (6). By 2.13, γ has a Δ-horizontal lifting z_t in W and z_t intersects the submanifold W^0 of lined configurations in one point and transversally. This implies that $z_1 \neq z_0$, which proves Case (i) when $d = 2$. Example 4.2 illustrates this argument with $L_p = L_q$.

Case (i) with $d > 2$. The vectors p, q and b are co-planar and $p \neq b \neq 0$, so $B_b \cdot z_0 \approx B_b/(B_b \cap B_p) \approx SO(d-1)/SO(d-2) \approx \mathbb{S}^{d-2}$. One has $B_b \cdot z_0 \subset \mathrm{horb}(z_0) \subset \hat{f}^{-1}(b)$. Case (i) with $d > 2$ then follows from the fact that $\hat{f}^{-1}(b) \approx \mathbb{S}^{d-2}$. The latter is proven in [Ha1, Prop. 4.1] but, for the convenience of the reader, we repeat the argument here in our language. Let $W_b = \hat{f}^{-1}(\mathbb{R}_{>0}\, b)$, which is a submanifold of W of dimension $d-1$ (see [Ha1, (1.3)]). Let $h : W_b \to \mathbb{R}$ defined by $h(z) = |\hat{f}(z)|$, which is a Morse function whose critical point are the lined configurations of W_b (see [Ha1, Theorem 3.2]). If $z \in W_b$ is a critical point of h with $h(z) \geq b$, then $z = (b/|b|, b/|b|)$, a non-degenerate maximum, and $h(z) = L > |b|$. Therefore, $\hat{f}^{-1}(b) = h^{-1}(|b|)$ is diffeomorphic to $\mathbb{S}^{d-2}$ by the Morse Lemma.

Case (ii). This case is trivial since $\hat{f}^{-1}(b)$ consists of the single point $(p_0, -p_0)$.

Case (iii). The map $\mathbb{S}^{d-1} \to W$ given by $p \mapsto (p, -p)$ is an embedding with image $\hat{f}^{-1}(0)$, as well as the inclusion $B_0 \cdot z_0 \subset \mathrm{horb}(z_0) \subset \hat{f}^{-1}(0)$. ∎

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