GEOMETRY AND TOPOLOGY OF MANIFOLDS
BANACH CENTER PUBLICATIONS, VOLUME 76
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2007

CONTRIBUTION OF NONCOMMUTATIVE GEOMETRY TO INDEX THEORY ON SINGULAR MANIFOLDS

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Abstract. This survey of the work of the author with several collaborators presents the way groupoids appear and can be used in index theory. We define the general tools, and apply them to the case of manifolds with corners, ending with a topological index theorem.

1. Introduction. Quantum physics, by putting the notion of operator at the very centre of its mathematical development, generated numerous index problems.

In its simplest form, the index of an operator is a number. It is the difference between the dimension of the space of solutions of the equation associated to this operator, and the restrictions it imposes on the image space of this operator:

$$ind(P) = \dim \ker P - \dim \operatorname{coker} P.$$

This has a meaning only if those dimensions are finite, in which case the operator is said to be a Fredholm operator.

Numerous quantities of mathematics, physics and chemistry identify to the index of an operator. Index theory is thus a meeting point of several branches of mathematics, in analysis, topology and geometry. One of the most striking results is the Atiyah-Singer index theorem, which gives a formula for computing the index of numerous operators, and has had a deep impact on many mathematical domains (topology, geometry, PDE), as well as in theoretical physics. This theorem was stated in the context of compact manifolds,

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2000 *Mathematics Subject Classification*: Primary 58B34; Secondary 35S05.

Key words and phrases: noncommutative geometry, index theory, manifolds with corners, groupoids.

This work has begun under the PhD supervision of M. Pimsner, and then G. Skandalis. I thank them for their interest, help and constant support. I am also indebted to my collaborators F. Pierrot, P. Y. Le Gall, R. Lauter, V. Nistor with whom it has been a great pleasure to work and develop this program.

The paper is in final form and no version of it will be published elsewhere.

without boundary. For these manifolds, a pseudodifferential operator is Fredholm if and only if it is elliptic, i.e. its symbol is invertible.

The problem of its extension to "singular" manifolds has been studied by many researchers since the 70's, following the work of Atiyah, Patodi and Singer ([2]) in the case of manifolds with boundary.

This problem is fundamental for applications since it appears in numerous domains through PDEs on non-compact manifolds. One of the technical difficulties lies in the fact that an elliptic operator is no longer Fredholm (in general). It leads to important developments in functional analysis, K-theory, noncommutative geometry.

Among the singular manifolds we examine, manifolds with corners, polyhedral domains play a preeminent role.

The approaches have generally been deeply analytic, focused on the study of pseudodifferential calculus, with authors like Richard Melrose for the case of manifolds with corners, Boutet de Monvel, Schulze for manifolds with conical singularities.

The introduction of noncommutative geometry by Alain Connes, in the 80's, gave new tools to study these problems through a geometrical approach ([7]). By studying "pathological" spaces which appear as models in certain areas of theoretical physics, as the space of leaves of a foliation, A. Connes showed how to define an algebra of pseudodifferential operators adapted to the setting, how to generalize the analytic index as well as the notion of regularizing operators. In the smooth case, a regularizing operator, i.e. an operator whose symbol is null, is compact; in the case of a foliation an operator of null symbol lives in a C^*-algebra caracterized by Connes as the C^*-algebra of the "foliation graph", more precisely the holonomy groupoid of the foliation (see [6]). With Georges Skandalis, he proved an index formula for foliations, based on the structure of groupoid ([8]). This was the first time in global analysis that the tools and objects defined and studied by C. Ehresmann and J. Pradines, the Lie groupoids and algebroids ([31]), were used. This theory was linked to that of operator algebras thanks to the important contribution of Jean Renault, who made explicit the framework of groupoid C^*-algebras ([32, 28] for instance).

In [7], A. Connes also gave a very elegant and deep proof of the index theorem (on manifolds without boundary), by using deformations of groupoids, fully based on noncommutative geometry. He identified the central role that groupoids play in index theory; this new approach opened new perspectives to study the index problems for singular manifolds, and further to extend their application domains.

We studied, in our work, where this new approach can lead, and how to apply it to specific examples.

The first goal was to generalize the constructions of Connes, mainly built in the context of foliations, to more general cases. The problem was to define tools which would be independent of particular cases of applications. While in global analysis each pseudodifferential calculus is defined in the context of a certain type of manifolds, we wanted to define once for all the pseudodifferential calculus on a groupoid so that it is sufficient to define the groupoid adapted to the type of manifold of interest (see section 2.1, and [27, 23, 24, 22, 25]). The first studies have been made in parallel to and independently of those of V. Nistor, A. Weinstein and P. Xu. ([29]). They were limited to the case of Lie groupoids, possibly admitting manifolds with corners.

With R. Lauter and V. Nistor, we tried to find the most general structure of groupoids adapted to singular manifolds; the *continuous family groupoids* (which generalize $\mathcal{C}^{\infty,0}$-foliation groupoids) seem to be good candidates (see section 2.2). This work was done in [13, 14].

We obtained general results like the definition of the analytic index, the generalized Atityah-Singer exact sequence, a necessary and sufficient condition for an operator to be Fredholm (see section 2.4) and also spectral invariance results (see 2.5).

Meanwhile, we constructed in [27, 23, 25] some groupoids adapted to the case of manifolds with corners. These appear naturally in boundary value problems. They can be treated under the Atiyah-Patodi-Singer approach (b-calculus, cusp-calculus...) or that of Boutet de Monvel. We could recover already known pseudodifferential calculi, as the b-calculus (see section 3.2), and we showed in [22, 25] that it was possible to extend these to larger classes of manifolds (section 3.3). In this more general context, the analytic index takes its values in a K-theory group which only depends on the codimension of the manifold; we computed this group in [15] (see section 3.4)

Another challenge was to understand the topological index, and to extend it to the case of manifolds with corners; we did that with V. Nistor in [26] (section 3.5).

In this survey we chose to present first the general tools we built, like the pseudodifferential calculus on a groupoid or the analytic index, then the applications to manifolds with corners. This structure does not follow the chronology.

2. Noncommutative tools of global analysis

2.1. *Pseudodifferential calculus on groupoids.* In order to understand the role played by groupoids in pseudodifferential calculus, let us consider first a simple example.

Let X be a smooth compact manifold. Let $G = X \times X$, and $\kappa \in C_c^\infty(G)$. Then an operator $P : L^2(X) \to L^2(X)$ can be defined by

$$Pf(x) = \int_X \kappa(x, y) f(y) \mathrm{d}y.$$

Here κ is the Schwartz kernel of P. Two such operators can be composed together, through the convolution of their Schwartz kernels. It is possible to extend this by supposing that κ is no longer a function but a distribution satisfying certain conditions, and L. Hörmander showed how to define the pseudodifferential operators on a smooth, compact manifold in this way, by characterizing their Schwartz kernels which are distributions on $X \times X$, smooth outside the diagonal ([11]).

A. Connes showed that in this case the relevant groupoid is indeed $X \times X$. Recall that a groupoid is a small category whose morphisms are invertible, this is thus made of two sets, G and $G^{(0)}$ (the space of units), and of maps called source and range $s, r : G \to G^{(0)}$. The elements $\gamma_1, \gamma_2 \in G$ are composable if and only if $r(\gamma_2) = s(\gamma_1)$.

In our case, we have:

$$G = X \times X, \quad G^{(0)} = X, r(x, y) = x, \quad s(x, y) = y,$$

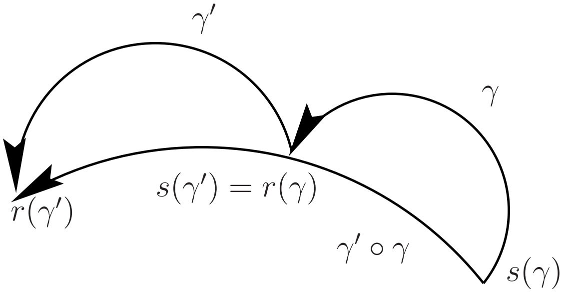


Fig. 1. Composition in a groupoid

$$(x, y) \circ (y, z) = (x, z), \quad (x, y)^{-1} = (y, x).$$

This is thus totally adapted to the convolution of functions.

The notion of Lie groupoid, introduced by C. Ehresmann, adds to the groupoid definition some smoothness conditions on the objects: G and $G^{(0)}$ must be smooth manifolds, the source and range maps, as well as other natural mappings, must be of class C^∞.

One can thus define the algebra of pseudodifferential operators on X through distributions on the groupoid $G = X \times X$, smooth outside the "diagonal" of G, i.e. the space of units $G^{(0)} = X$. In fact, we showed in [22] that this definition can generalize to any Lie groupoid.

Another definition of the algebra of pseudodifferential operators on a Lie groupoid, which is inspired by that of Connes in the setting of foliations, considers families of operators on the fibers of the groupoid. If $x \in G^{(0)}$, let us denote by $G_x = s^{-1}(x)$ the s-fiber of X. Since G is a Lie groupoid, G_x is a smooth manifold, thus one can define a pseudodifferential operator P_x on each fiber. In addition, the groupoid G acts on the fibers, and this induces an operator

$$U_g : C^\infty(G_{s(g)}) \to C^\infty(G_{r(g)}), \quad (U_g f)(g') = f(g'g).$$

Definition 1. A *pseudodifferential operator on* G is a smooth familiy of pseudodifferential operators on the s-fibers, equivariant with respect to the action of G.

This definition was obtained in collaboration with F. Pierrot in [27]. Meanwhile, independently, V. Nistor, A. Weinstein and P. Xu gave an analoguous definition in [29].

The pseudodifferential operators form an algebra (of convolution) by imposing support conditions, for instance if their Schwartz kernels are compactly supported. The algebra of order 0 pseudodifferential operators is a subalgebra of the multiplier algebra of $C^*(G)$, thus it is possible to take its norm closure, to obtain a C^*-algebra denoted by $\overline{\Psi^0}(G)$. The groupoids we consider are always amenable, hence we shall not make any distinction between the max and min C^*-algebras.

2.2. *Groupoids adapted to singular manifolds.* The problem with Lie groupoids is that they do not fit in the setting of singular manifolds, by definition. A possibility, to begin with, is to enlarge this notion to adapt it to the case of certain singular manifolds. This approach was used in [22, 23, 25], considering instead of Lie groupoids some differentiable groupoids whose fibers are smooth, and the space of units is a manifold with corners.

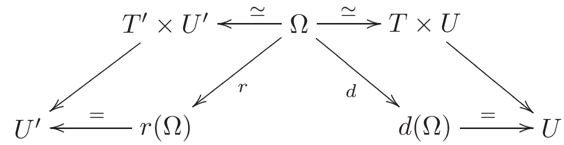
What is needed, indeed, is that the fibers be smooth, in order to define pseudodifferential operators on the fibers.

Nevertheless this notion is not sufficient. For instance, the boundary of such a groupoid does not belong to the same class (the boundary of a square is not a manifold with corners!). Also, it does not apply to other types of singular manifolds. With R. Lauter and V. Nistor ([13]) we built the objects and results in a framework which seems to be the most general for the singular setting: that of *continuous family groupoids*, introduced by A. Paterson ([30]).

A continuous family groupoid is basically a groupoid whose space of units is a topological space, its s-fibers and r-fibers are smooth manifolds, and it is a continuous family of s-fibers or r-fibers. This generalizes the case of the holonomy groupoid of a $\mathcal{C}^{\infty,0}$ foliation introduced by A. Connes.

More precisely, a continuous family groupoid is a locally compact topological groupoid G endowed with a covering by open subsets Ω such that:

- each chart Ω is homeomorphic to two open subspace $\mathbb{R}^k \times G^{(0)}$, $T \times U$ and $T' \times U'$ and the following diagram is commutative:



- each coordinate change is given by $(t, u) \mapsto (\phi(t, u), u)$ where ϕ is of class $\mathcal{C}^{\infty,0}$, i.e. $u \mapsto \phi(., u)$ is a continuous map from U to $\mathcal{C}^\infty(T, T')$.

Moreover, the composition and the inversion must be $\mathcal{C}^{\infty,0}$. We showed in [13] that it is possible to define an algebra of pseudodifferential operators for any continuous family groupoid. In the sequel the groupoids under consideration will always be continuous family groupoids.

2.3. *Atiyah-Singer exact sequence, analytic index and tangent groupoid.* As in the classical case, a pseudodifferential operator has a symbol. More precisely, each operator P_x has a principal symbol in $C(S^*G_x)$. But the union of the tangent spaces of the fibers of G is the Lie algebroid of G, $A(G)$, whose sphere bundle is denoted by $S(G)$. The principal symbol is thus a map

$$\sigma : \overline{\Psi^0}(G) \to C(S^*(G)).$$

As far as the kernel of the symbol is concerned, it is precisely $C^*(G)$. One obtains thus an exact sequence that we shall call *Atiyah-Singer exact sequence* since it generalizes the classical sequence obtained in the case of a smooth manifold:

$$0 \to C^*(G) \to \overline{\Psi^0}(G) \to C(S^*(G)) \to 0.$$

Note that if X is a smooth manifold, and $G = X \times X$, then $C^*(G) = \mathcal{K}$ and $S^*(G) = S^*(X)$.

This Atiyah-Singer exact sequence induces the *analytic index*, $ind_a : K_1(S^*(G)) \to K_0(C^*(G))$. Meanwhile, there is an alternative definition, obtained without using the algebra of pseudodifferential operators, thanks to a generalization of Connes' tangent groupoid.

Let G be a Lie groupoid, its algebroid is the normal bundle of the space of units $G^{(0)}$ in G. The *tangent groupoid* of G, denoted by ${}^TG$, is obtained by gluing $A(G) \times \{0\}$ to $G \times (0, 1]$, through an exponential map. This construction generalizes that of Connes, it relies mainly on previous work of M. Hilsum and G. Skandalis in [10].

The interest of this construction is in the decomposition of $C^*({}^TG)$:

$$0 \to C^*(G \times (0, 1]) \to C^*({}^TG) \overset{e_0}{\to} C_0(A^*(G)).$$

Indeed, $K_*(C^*(G \times (0, 1])) = 0$ so that the K-theory morphism $e_{0,*}$ is an isomorphism, and composing its inverse with $e_{1,*}$ one obtains a morphism which is shown to be equal to the analytic index:

$$ind_a = e_{1,*} \circ e_{0,*}^{-1} : K_0(A(G)) \to K_1(C^*(G)). \tag{1}$$

Remark 1. One can also consider the restriction of ${}^TG$ to $[0, 1)$,

$${}^{ad}G = {}^TG_{G^{(0)} \times [0,1)}.$$

The groupoid is called *adiabatic groupoid*, and also allows to define the analytic index through a boundary map.

2.4. *Total ellipticity.* As seen, the analytic index is not a Fredholm index, it does not take its values in $\mathbb{Z}$. The ellipticity of an operator (the fact that its symbol is invertible) being insufficient to make it Fredholm, one needs to find other conditions.

The fact that the ellipticity is sufficient in the case of a manifold without boundary comes from the fact that $G = X \times X$, and the C^*-algebra of G is the algebra of compact operators. If X is no longer smooth, one cannot take $X \times X$ any longer, since its fibers would not be smooth, preventing from defining families of pseudodifferential operators on the fibers. The relevant groupoids will thus be more complex. Nevertheless, it is legitimate to consider that in a groupoid G on a singular manifold X, the regular part U will be a saturated open subset in G (i.e. stable under the action of G), and the restriction of G to U, G_U will just be the groupoid of couples $U \times U$. In other words, we assume that a pseudodifferential calculus on a singular manifold reduces to the classical calculus on the regular part of the manifold.

Denoting $F = X \backslash U$, there is a restriction morphism $In_F : \overline{\Psi^0}(G) \to \overline{\Psi^0}(G_F)$; the restriction of P to F is defined by $In_F(P)$.

This leads to the following result:

Theorem 1. *Let G be a Lie groupoid, X its space of units, and U a saturated open subset of $G^{(0)}$ such that $G_U = U \times U$. Then an order 0 pseudodifferential operator on G is Fredholm if and only if its symbol is invertible as well as its restriction to the complementary of U.*

This result, which gives a general condition of "total ellipticity", induces some theorems already stated in particular contexts, as for the b-calculus for instance. In this case, it is shown that an operator is Fredholm if and only if it is elliptic and its indicial families are invertible; the latter correspond precisely to the restriction of an operator to the boundary.

The theorem is proven using groupoid techniques. It is in fact a consequence of the commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & C^*(G_U) & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & C^*(G) & \longrightarrow & \overline{\Psi^0}(G) & \xrightarrow{\sigma_0} & C(S^*G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle In_F} & & \downarrow & & \\
0 & \longrightarrow & C^*(G_F) & \longrightarrow & \overline{\Psi^0}(G_F) & \longrightarrow & C_0(S^*G_F) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}
$$

This diagram induces the exact sequence:

$$0 \to C^*(G_U) \simeq \mathcal{K} \to \overline{\Psi^0}(G) \to C_0(S^*G) \times_{C_0(S^*G_F)} \overline{\Psi^0}(G_F) \to 0.$$

Hence we have two types of index problems: the problems of analytic index, with values in $K_*(C^*(G))$ (which will themselves be distinguished in function of the groupoid G, several choices being offered), and the problems of Fredholm index, with values in $\mathbb{Z}$.

2.5. *Spectral invariance.* However, from the point of view of global analysis it is not satisfactory to reason exclusively at the level of C^*-algebras. One of the problems which arise thus is to build algebras of pseudodifferential operators containing operators whose Schwartz kernels have compact support, but which enjoy particular spectral properties.

We indeed expect from such an algebra that it behaves correctly with respect to parametrix. More precisely, if an operator P is Fredholm, i.e. if it is invertible modulo the algebra of compact operators, it possesses a quasi-inverse, or parametrix, Q such as $PQ - Id$ and $QP - Id$ are compact. We wish naturally that the parametrix also belongs to the algebra of pseudodifferential operators. This problem is solved when the algebra is *stable under holomorphic functional calculus*. It has for consequence, in particular, that its K-theory is the same as that of its norm closure, hence K-theory does not make a difference between the C^*-algebra and its subalgebra.

The article [14] is completely dedicated to these questions. We clarify constructions of algebras stable under holomorphic functional calculus, or more precisely we build sub-algebras stable under holomorphic functional calculus $\mathcal{J} \subset C^*(G)$, and we show that $\mathcal{I} := \Psi^0(G) + \mathcal{J}$ is itself stable under holomorphic functional calculus. We give different types of constructions for $\mathcal{J}$. The first one is based on the use of commutators, and induces the construction of an algebra containing $\Psi^0(G)$ as a dense sub-algebra, and such that its elements are pseudodifferential operators on G (and not only limits of pseudodifferential operators).

Another construction relies on a generalization of the Schwartz spaces of functions with rapid decay. It is necessary in that case to have a length function with polynomial growth on G, and we prove that the Schwartz space on G (with respect to this length function) is stable under holomorphic functional calculus in $C^*(G)$.

These techniques are also applied in the case of manifolds with corners, see farther.

3. Application to manifolds with corners. We now have a set of tools available for every continuous family groupoid. From then on, to study the global analysis on a singular variety, it is enough to build a groupoid adapted to this context.

The global analysis on manifolds with corners was studied by R. Melrose and his co-authors ([18, 19, 21, 16, 17]), following the seminal work of Atiyah, Patodi and Singer for the manifolds with boundary ([2, 3, 4, 5]). R. Melrose developed the b-calculus, and studied its structure.

However, his definition forced him to restrict the class of manifolds with corners fitting in his calculus, and in addition the small b-calculus is not stable under holomorphic functional calculus.

We showed in [22, 25] how to build a groupoid for the b-calculus in the case of manifolds with corners, without limitation, and how to build an operator algebra stable under holomorphic functional calculus. We also built groupoids for the *cusp calculi* in [14].

These constructions allow to simplify the analysis of the properties of these operator algebras, by using the general tools of the non-commutative geometry. They also allow to study new index problems.

At the same time, an approach of the pseudo-differential calculus on manifolds with boundary was developed by L. Boutet of Monvel, and studied by E. B. W. Schrohe and. Schulze. In a joint work with J. Aastrup, S. T. Melo and E. Schrohe ([1]) we show how to identify the algebra of regularizing operators of Boutet de Monvel's calculus, the Singular Green Operators, to an ideal of a groupoid algebra.

3.1. *Manifolds with corners.* Let us recall that a manifold with corners is a manifold modelled on $(\mathbb{R}_+)^n$, which will be denoted by $\mathbb{R}^n_+$ in the sequel contrarily to the sometimes used notation for which $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+$ is a half-space. Moreover $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}^*_+ = (0, +\infty)$.

Every point admits a neighborhood diffeomorphic to $(\mathbb{R}_+)^k \times \mathbb{R}^{n-k}$ where k is the *codimension*, or *depth* of x. The set of points of the same depth decomposes into a union of connected components called open faces. The closed faces are the closures of the open faces.

R. Melrose adds a supplementary condition to define the b-calculus, which is that every hyperface (open face of codimension 1) of the manifold be *embedded* in the manifold, or equivalently, that every hyperface H admit a definition function, namely a smooth function $\rho : X \to \mathbb{R}_+$ which is zero over H, and only over H, and the differential of which is non-zero over H. We shall call such a manifold a manifold with embedded corners. The figures below are several examples of manifolds with corners.

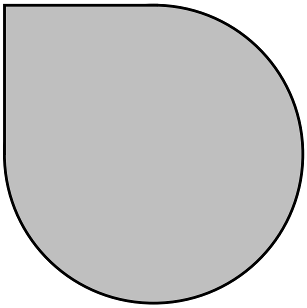


Fig. 2. The drop with one corner

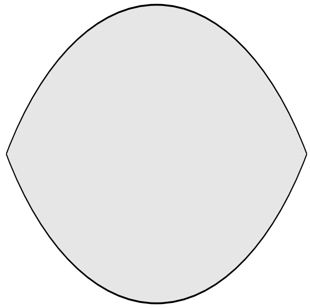


Fig. 3. The drop with two corners

Fig. 4. The square torus with $\pi/4$ twist

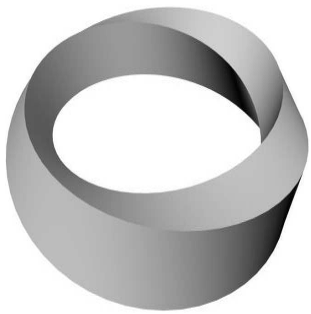


Fig. 5. The square torus with $\pi/2$ twist

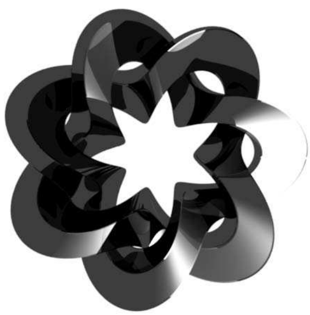


Fig. 6. Another manifold with corners

The drop with one corner (figure 2) is not a manifold with embedded corners, because its only hyperface intersects itself. On the other hand the drop with two corners (figure 3) is indeed a manifold with embedded corners. The square torus with twisting of $\pi/4$ (figure 4) is not either a manifold with embedded corners, it possesses only one hyperface. On the other hand the square torus with twisting of $\pi/2$ (figure 5) is a manifold with embedded corners, which has two hyperfaces. The last figure is a manifold with embedded corners.

3.2. *The groupoid of a manifold with embedded corners.* We showed in [22, 23] that we can define a groupoid the pseudo-differential calculus of which corresponds essentially to the b-calculus. In the case of a manifold with embedded corners X endowed with a family $(\rho_1, \ldots, \rho_N)$ of definition functions of the faces, a non-canonical but simple definition of this groupoid is the following one:

Definition 2. Let X be a manifold with embedded corners. Let

$$\tilde{\Gamma}(X) := \{(x, y, \lambda_1, \ldots, \lambda_N) \in X \times X \times \mathbb{R}_+^{*N},\ \rho_i(x) = \lambda_i \rho_i(y), \text{for all } i\}.$$

Then the *groupoid of the b-calculus*, denoted by $\Gamma(X)$, is the s-connected component of $\tilde{\Gamma}(X)$ (the s-connected component is the largest open subset containing X).

Let us note that it is possible to give a canonical definition of this groupoid: $\Gamma(X)$ is the set of the triples (x, y, α) where x and y are in the same open face F and α is an isomorphism between the normal at y of F in X, N_yF, and N_xF. To make the link with the non-canonical definition, remark that a definition function induces a trivialisation of N_yF and N_xF, which are isomorphic to $\mathbb{R}_+^k$ where k is the codimension of F; the isomorphism α is then reduced to an element of $\mathbb{R}_+^{*k}$. See [23] for the complete definition.

We obtained the following result:

Theorem 2. *The algebra of pseudodifferential operators on* $\Gamma(X)$ *whose Schwartz kernel has compact support is equal to the properly supported b-calculus.*

As already stated, the b-calculus is not stable under holomorphic functional calculus. We studied its regularizing operators, and their decay conditions. It led us to define in [22] a function length on the groupoid $\Gamma(X)$, thus to a Schwartz space a little bigger than the algebra of regularizing operators of the b-calculus, but stable under holomorphic functional calculus ([14]). We were able to characterize the obstruction for the b-calculus to be stable under holomorphic functional calculus by the fact that its regularizing operators is of exponential decay, and not polynomial as those of the Schwartz space.

In the same family of pseudo-differential calculi we find the *cusp calculus*, and more generally the c_n*-calculi*, introduced by R. Melrose and his co-authors (see [16, 20] for example). In [14], we built groupoids for these calculi. To simplify let us consider a manifold with boundary provided with a definition function ρ, and the groupoid

$$\Gamma_n(X) \simeq \{(u, v, \mu) \in X \times X \times \mathbb{R} \,|\, \mu\rho(u)^{n-1}\rho(v)^{n-1} = \rho(u)^{n-1} - \rho(v)^{n-1}\}$$

(where $n \geq 2$). This groupoid induces the c_n-calculus. It is homeomorphic to $\mathcal{G}(X)$, but not diffeomorphic. This implies that the pseudo-differential calculi are different, but nevertheless when we consider the norm closure we obtain isomorphic C^*-algebras.

In this context of c_n-calculi, we defined algebras stable under holomorphic functional calculus, for example by considering a length function leading to a Schwartz space, but we also showed how to build algebras stable under holomorphic functional calculus for cusp-calculi, where kernels are smooth (which is not the case for the Schwartz space of the b-calculus).

3.3. *The groupoid of a manifold with (non-embedded) corners.* In the case of a manifold with corners where the hyperfaces are not embedded, we do not have definition functions of the faces and the "non-canonical" definition of $\Gamma(X)$ is not valid any more. However, the one which uses the isomorphisms can be adapted, and we consider then the groupoid $\mathcal{G}(X)$ as the set of triples (x, y, α) where x and y have the same codimension (they can be in different faces), and $\alpha : N_yF_y \overset{\simeq}{\to} N_xF_x$ is any isomorphism.

In the case of a manifold with embedded corners, the groupoid of the b-calculus, $\Gamma(X)$ is the s-connected component of $\mathcal{G}(X)$. This groupoid possesses interesting properties, although its complexity is bigger (notably, it is not Hausdorff as soon as the manifold is of codimension greater than or equal to 2):

Theorem 3. *Let X and X' be two manifolds of the same maximal codimension. Then their groupoids $\mathcal{G}(X)$ and $\mathcal{G}(Y)$ are equivalent, thus*

$$K_*(C^*(\mathcal{G}(X))) = K_*(C^*(\mathcal{G}(Y))).$$

This result is satisfactory, because $K_*(C^*(\mathcal{G}(X)))$ is the receptacle of the analytic index, it thus says that if we consider the "universal" groupoid $\mathcal{G}$, the analytic index takes its values in a group which depends only on the codimension of the manifold. It is a familiar result in the case of smooth manifolds, where the analytic index, being the Fredholm index, always takes its values in $\mathbb{Z}$.

3.4. *The indicial algebra of a manifold with corners.* An important step in the study of the analytic index is to compute the group in which it takes its values. Two cases appear, as we consider the "universal groupoid" of a manifold with corners, or the groupoid of b-calculus.

We have just seen that in the case of the universal groupoid the receptacle of the index only depends on the codimension. With P.Y. Le Gall we identified these groups in [15]. This work is based on an identification of the role played in $\mathcal{G}(X)$ by the groups $\mathbb{R}^n \rtimes \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symetric group acting on $\mathbb{R}^n$ by permutation of the coordinates.

Indeed, by considering the definition of $\mathcal{G}(X)$, an element is a triple (x, y, α), with $\alpha : N_yF_y \overset{\simeq}{\to} N_xF_x$ an isomorphism. Now if y is of depth k, we have a trivialisation $N_yF_y \simeq \mathbb{R}_+^k$. Let us note that in the case of a manifold with embedded corners, we can obtain a global trivialisation of the normal bundle of a given face.

Remark 2. It is not the case generally, because the local hyperfaces may not be distinguishable globally, as in the case of the Klein bottle where the section is a drop: it is the manifold obtained by gluing both ends of a cylinder of square section, by identifying points through axial symmetry with respect to the diagonal. More precisely, it is $([0,1] \times [0,1] \times \mathbb{R})/\mathbb{Z}$, where the action of $\mathbb{Z}$ is the reflection with respect to a diagonal on $[0,1] \times [0,1]$, and the translation by 1 on $\mathbb{R}$: $1 \cdot (a, b, t) = (b, a, t + 1)$.

The isomorphism α, through these local trivialisations, becomes identified with the product of a diagonal matrix with strictly positive terms, and a matrix of permutation, i.e. an element of ${\mathbb{R}_+^*}^k \rtimes \mathfrak{S}_k$.

We were able to put in evidence a relation of recurrence:

$$0 \to K_*(C^*(\mathcal{G}_n)) \to K_*(C^*({\mathbb{R}_+^*}^n \rtimes \mathfrak{S}_n)) \to K_{1-*}(C^*(\mathcal{G}_{n-1})) \to 0$$

From then on the computation of the groups of the indicial algebra relies on that of the $K_*(C^*({\mathbb{R}_+^*}^n \rtimes \mathfrak{S}_n))$. Now these groups of K-theory have no torsion, which was established by M. Karoubi in his article motivated by our works, [12]. The groups of K-theory thus can be computed thanks to the Chern isomorphism for discrete groups, by reducing to the computation of the homology groups.

This led us to the following result:

THEOREM 4. *$K_0(C^*(\mathcal{G}_n))$ has the form $\mathbb{Z}^{\ell_n}$ and $K_1(C^*(\mathcal{G}_n))$ has the form $\mathbb{Z}^{m_n}$, where ℓ_n and m_n are integers defined by relations of recurrence.*

Let us not deprive any longer the reader, the 100^{th} group is:

$$K_0(C^*(\mathcal{G}_{100})) = \mathbb{Z}^{115826},\ K_1(C^*(\mathcal{G}_{100})) = \mathbb{Z}^{115825}.$$

However we can also consider an analytical index with values in the K-group of the groupoid of b-calculus. As $\Gamma(X)$ is an open subgroupoid of $\mathcal{G}(X)$,we have the following factorization:

$$\begin{array}{ccc} & & K_*(C^*(\mathcal{G}(X))) \\ & \overset{ind_a}{\nearrow} & \uparrow \\ K^*(A^*(G)) & & \\ & \underset{ind_a}{\searrow} & \\ & & K_*(C^*(\Gamma(X))) \end{array}$$

It is thus relevant to know the analytical index with values in $K_*(C^*(\Gamma(X)))$.

3.5. *Topological index theorem for manifolds with embedded corners.* In the case of manifolds with embedded corners, we established a theorem generalizing the theorem of the topological index theorem of Atiyah-Singer. This result allows to compute more easily the indicial algebras of the b-calculus.

Let us recall that the topological index theorem of Atiyah-Singer is based on the fact that we can embed any smooth manifold M in a Euclidian space $\mathbb{R}^n$, and if we denote by ι this embedding there is a mapping $\iota_! : K^0(T^*M) \to K^0(T^*\mathbb{R}^n)$ called the Gysin map. We can then define the topological index as the compound $ind_a^{\mathbb{R}^n} \circ \iota_! : K^0(T^*M) \to \mathbb{Z}$ (where $ind_a^{\mathbb{R}^n}$ is in fact an isomorphism). The topological index theorem gives the equality of the analytical index and the topological index. It can be useful to represent this result through the following diagram:

$$\begin{array}{ccc} \mathbb{Z} & = & \mathbb{Z} \\ \uparrow{\scriptstyle \mathrm{ind}_a^M} & & \uparrow{\scriptstyle \mathrm{ind}_a^{\mathbb{R}^n}} \\ K^0(T^*M) & \xrightarrow{\iota_!} & K^0(T^*\mathbb{R}^n). \end{array}$$

The index theorem indicates that this diagram is commutative.

A natural question is the following one: in the case of a manifold with embedded corners M, how to generalize this result?

More exactly, we wish to prove that if we have an embedding of manifolds with embedded corners $\iota : M \hookrightarrow X$, there is also a commutative diagram as the previous one:

$$
\begin{array}{ccc}
K_*(C^*(\Gamma(M))) & \longrightarrow & K_*(C^*(\Gamma(X))) \\
{\scriptstyle \mathrm{ind}_a^M}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{ind}_a^X} \\
K^0(A^*_M) & \xrightarrow{\iota_!} & K^0(A^*_X)
\end{array}
\qquad (2)
$$

Beyond the commutativity of the diagram, we wish to build a manifold with embedded corners X such that

- M embeds inside,
- the indicial algebras $K_*(C^*(\Gamma(M)))$ and $K_*(C^*(\Gamma(X)))$ are isomorphic,
- the analytic index of X is an isomorphism.

Indeed, it will mean that the analytical index of M relies in fact on the map $\iota_!$ which is topological. We call a manifold with corners X satisfying these properties a *classifying space* of M.

We showed in [26] that the diagram commutes, and how to build such manifolds. We can give sufficient conditions for the manifold X so that it satisfies the conditions above:

- the faces of M are traces on M of faces of X,
- faces of M and X are one-to-one,
- finally open faces of X are Euclidian spaces.

The first two conditions lead to the fact that the groupoids $\Gamma(M)$ and $\Gamma(X)$ are equivalent, which implies that their C^*-algebras are Morita equivalent, and finally their indicial algebras are isomorphic.

Besides, the fact that the open faces are Euclidian spaces lead to the fact that the analytical index of X is an isomorphism.

The commutativity of the diagram is *a priori* non-natural, because the horizontal arrows of the diagram are not of the same nature. To ensure the commutativity, we thus avoided considering these maps separately, and it was made possible thanks to the tangent groupoid, which allows to define the analytic index (see the equality (1) page 226).

More exactly, as in the theorem of Atiyah-Singer, we decompose the embedding ι by considering a tubular neighborhood U of M in X. We then have to consider a diagram related to a fibration $U \to M$, and the second concerning the inclusion $U \to X$. This technique was also used in the case of the etale groupoids by M. Hilsum and G. Skandalis in [10].

The second diagram is naturally commutative, but for the first one it is necessary to use a double deformation, that is a deformation of the tangent groupoid (which is itself a deformation) of U. Here is roughly how to define this double deformation: $\mathcal{G} := \mathcal{G}_1 \sqcup \mathcal{G}_2 \sqcup \mathcal{G}_3$ where

$$
\mathcal{G}_1 := A_U \times \{0\} \times [0,1], \quad \mathcal{G}_2 := A_M \times_M U \times_M U \times (0,1] \times \{0\},
$$
$$
\mathcal{G}_3 := G(U) \times (0,1] \times (0,1].
$$

We can define several evaluation maps, at $s = s_0, t = t_0$, and we prove that the mappings in K-theory $e_{1,0}$ and $e_{0,0}$ are isomorphisms. From then on, it is possible to

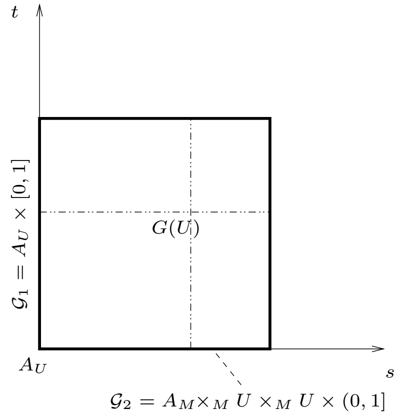


Fig. 7. The groupoid $\mathcal{G}$

consider ind_a^U as the composition $e_{1,1} \circ e_{0,0}^{-1}$ (we can represent it on the diagonal of the square above), and we can also decompose this mapping along the lower and the right sides of the square (at $t = 0$ and $s = 1$). *Modulo* a Morita equivalence, it supplies the Gysin map and the analytical index of M, and the commutativity is direct.

Remark 3. Although it is not handled in [26], let us note that it is possible to generalize the theorem which ensures the commutativity of the diagram in the case of continuous family groupoids such that $G(U) = G(M) \times_{M \times M} (U \times U)$.

As regards the existence of a manifold with embedded corners satisfying the conditions expressed previously, we propose in [26] a construction. It is based on an embedding φ of M in a space $\mathbb{R}^n$ and on the choice of functions of definition of the hyperfaces $\rho_1, \ldots, \rho_p$. The embedding is a refinement of that obtained by considering $\varphi \times \prod_{i=1..n} \rho_i$.

3.6. *Fredholm index for manifolds with embedded corners.* We evoked up to here the analytic index for manifolds with corners, which is not a Fredholm index because the regularizing operators are not compact. As we saw in theorem 1, an order 0 operator is totally elliptic if it is elliptic and its restriction on the boundary, called *indicial operator* within the framework of b-calculus, is invertible.

In that case, it is possible to consider the Fredholm index though a groupoid, as in proposition 3 of [13].

We are going to give a slightly different version, based on the tangent groupoid rather than on the adiabatic groupoid, then show how the topological index theorem is translated in this frame.

Let G be a continuous family groupoid, and ${}^TG = A(G) \cup G \times (0,1]$ be its tangent groupoid. We suppose, as in theorem 1, that there is an open saturated subset U of $G^{(0)}$ such that $G_U = U \times U$; let F be the complement of U. Thus

$${}^TG = A(G) \cup G_F \times (0,1] \cup U \times U \times (0,1].$$

Consider the open sub-groupoid

$$G' = A(G) \cup G_F \times (0,1) \cup U \times U \times (0,1]$$

(we removed $G_F \times \{1\}$). Denote $G_1 = A(G) \cup G_F \times (0,1)$. Then $U \times U \times (0,1]$ is an open sub-groupoid of G', the K-theory of which is trivial, thus there is an isomorphism $\alpha : K_*(C^*(G_1)) \to K_*(C^*(G'))$. If we compose it with the evaluation at 1, we obtain a mapping denoted by ind_f:

$$ind_f = e_1 \circ \alpha : K_*(C^*(G_1)) \to K_*(C^*(U \times U)) = \mathbb{Z}.$$

Let ρ be the evaluation map on $A(G) : \rho : C^*(G_1) \to C^*(A(G))$.

In [13] we showed that we can associate to every Fredholm differential operator P a canonical class $[P] \in K_0(C^*(G_1))$ such that $\rho_*([P]) = [\sigma(P)]$, and $ind_f([P])$ is the Fredholm index of P. The map ind_f is thus crucial for the Fredholm index.

Remark 4. We can improve this result (for manifolds with boundary for the moment) by using the works of C. Debord and J. M. Lescure ([9]). Indeed, the groupoid G_1 is KK-equivalent to their "tangent space" ([9], remark 4). A totally elliptic pseudo-differential operator leads then to an element of the K-homology of the manifold with conical singularities associated. Now the K-duality indicates that this K-homology is isomorphic to the K-theory of $C^*(G_1)$. We thus obtain a class $[P] \in K_0(C^*(G_1))$ as previously, but for any pseudodifferential operator, not only differential operator.

To compute ind_f, we can use a variant of the topological index theorem. Indeed, this one is based on a deformation of the tangent groupoid, which if we restrict it to the sub-groupoid G_1 yields a commutative diagram:

$$\begin{array}{ccc} \mathbb{Z} & = & \mathbb{Z} \\ \uparrow{\scriptstyle \mathrm{ind}_a^M} & & \uparrow{\scriptstyle \mathrm{ind}_a^X} \\ K_0(C^*(G_1(M)))) & \xrightarrow{\beta} & K_0(C^*(G_1(X))). \end{array} \qquad (3)$$

In case X is a classifying space of M, the analytical index of X is an isomorphism, thus the adiabatic groupoid of X, ${}^{ad}G(X) = A(G) \cup G \times (0,1)$ has trivial K-theory. Since

$$G_1 = {}^{ad}G(X) \cup U \times U \times \{1\},$$

the evaluation at 1 yields a K-theory isomorphism. Thus the map ind_f^X is an isomorphism, and we have the theorem:

Theorem 5. *Let M be a manifold with embedded corners, and X be the classifying space of M. Then $ind_f^M = ind_f^X \circ \beta$, where ind_f^X is an isomorphism.*

This result will be developed in a subsequent article.

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