GEOMETRY AND TOPOLOGY OF MANIFOLDS
BANACH CENTER PUBLICATIONS, VOLUME 76
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2007

SINGULAR POISSON–KÄHLER GEOMETRY OF CERTAIN ADJOINT QUOTIENTS

JOHANNES HUEBSCHMANN

Université des Sciences et Technologies de Lille
UFR de Mathématiques, CNRS-UMR 8524
59655 Villeneuve D'Ascq Cedex, France
E-mail: Johannes.Huebschmann@math.univ-lille1.fr
and
Institute for Theoretical Physics, Universität Leipzig
Augustusplatz 10/11, 04109 Leipzig, Germany

Abstract. The Kähler quotient of a complex reductive Lie group relative to the conjugation action carries a complex algebraic stratified Kähler structure which reflects the geometry of the group. For the group $\mathrm{SL}(n,\mathbb{C})$, we interpret the resulting singular Poisson-Kähler geometry of the quotient in terms of complex discriminant varieties and variants thereof.

1. Adjoint quotients. Let K be a compact Lie group, let $K^{\mathbb{C}}$ be its complexification, and let $\mathfrak{k}$ be the Lie algebra of K. The polar map from $K \times \mathfrak{k}$ to $K^{\mathbb{C}}$ is given by the assignment to $(x, Y) \in K \times \mathfrak{k}$ of $x \exp(iY) \in K^{\mathbb{C}}$, and this map is well known to be a K-bi-invariant diffeomorphism. We endow the Lie algebra $\mathfrak{k}$ with an invariant (positive definite) inner product; by means of this inner product, we identify $\mathfrak{k}$ with its dual $\mathfrak{k}^*$ and, furthermore, the total space $\mathrm{T}K$ of the tangent bundle of K with the total space T^*K of the cotangent bundle of K. The composite

$$\mathrm{T}K \to K \times \mathfrak{k} \to K^{\mathbb{C}}$$

of the inverse $\mathrm{T}K \to K \times \mathfrak{k}$ of left translation with the polar map from $K \times \mathfrak{k}$ to $K^{\mathbb{C}}$ is a

2000 *Mathematics Subject Classification*: Primary 17B63, 17B65, 17B66, 17B81, 53D17, 53D20, 55R80; Secondary 14L24, 16W22, 32S60, 70H45.

Key words and phrases: stratified symplectic space, complex analytic space, complex algebraic space, real semialgebraic space, complex analytic stratified Kähler space, complex algebraic stratified Kähler space, Kähler reduction, constrained system, invariant theory, adjoint quotient, discriminant variety, spherical pendulum.

Support by the Deutsche Forschungsgemeinschaft in the framework of a Mercator-professorship is gratefully acknowledged.

The paper is in final form and no version of it will be published elsewhere.

diffeomorphism, and the resulting complex structure on $\mathrm{T}^*K \cong \mathrm{T}K$ combines with the cotangent bundle symplectic structure to a Kähler structure. The action of $K^{\mathbb{C}}$ on itself by conjugation is holomorphic, and the restriction of the action to K is Hamiltonian, with momentum mapping from $K^{\mathbb{C}}$ to $\mathfrak{k}^*$ which, viewed as a map on $K \times \mathfrak{k} \cong \mathrm{T}^*K$ and with values in $\mathfrak{k}$, amounts to the map

$$\mu\colon K \times \mathfrak{k} \to \mathfrak{k}, \quad \mu(x, Y) = \mathrm{Ad}_x Y - Y.$$

By Proposition 4.2 of [12], the zero momentum reduced space

$$(\mathrm{T}^*K)_0 \cong \mu^{-1}(0)\big/K$$

inherits a *stratified Kähler structure* which is actually complex algebraic in a sense which will be explained below. Here the complex algebraic structure is that of the complex algebraic categorical quotient $K^{\mathbb{C}}//K^{\mathbb{C}}$ relative to conjugation, and this quotient, in turn, may be described as the ordinary orbit space $T^{\mathbb{C}}\big/W$ of the complexification $T^{\mathbb{C}}$ of a maximal torus T of K, relative to the action of the Weyl group W; in the literature, such an orbit space is referred to as an *adjoint quotient*. Certain aspects of the stratified Kähler structure on such an adjoint quotient have been explored in [16]. The purpose of the present paper is to complement the results in [16]: For the special case where $K = \mathrm{SU}(n)$ $(n \geq 2)$ we shall elucidate the complex algebraic stratified Kähler structure in terms of complex discriminant varieties and variants thereof. We will also interpret the resulting stratified Kähler geometry on the adjoint quotient for the special case where $K = \mathrm{SU}(2)$ in terms of the reduced phase space of a spherical pendulum constrained to move with angular momentum zero. In physics, a space of the kind $(\mathrm{T}^*K)_0$ is the *building block* for certain *lattice gauge* theories. For intelligibility we will include an exposition of the notion of stratified Kähler space.

2. Stratified Kähler spaces. To develop Kähler quantization in the presence of singularities, we introduced certain "Kähler spaces with singularities" [12] which we refer to as *stratified Kähler spaces*. In [13] we have shown that ordinary *Kähler quantization* can indeed be extended to a *quantization scheme over stratified Kähler spaces*. A special case of a stratified Kähler space is a complex analytic stratified Kähler space and, for the present paper, this notion suffices; we will now describe it.

Let N be a stratified space, the strata being ordinary smooth manifolds. A *stratified symplectic structure* on N consists of a family of symplectic structures, one on each stratum, together with a Poisson algebra $(C^\infty N, \{\,\cdot\,,\,\cdot\,\})$ of continuous functions on N, and these are required to satisfy the following compatibility requirements:

(1) For each stratum, the restriction map from $C^\infty N$ to the algebra of continuous functions on that stratum goes into the algebra of ordinary smooth functions on the stratum.
(2) For each stratum, the restriction map from $C^\infty N$ to the algebra of smooth functions on that stratum is a morphism of Poisson algebras, where the stratum is endowed with its ordinary smooth symplectic Poisson structure.

A *stratified symplectic space* is defined to be a stratified space together with a stratified symplectic structure. Given a stratified symplectic space $(N, C^\infty N, \{\,\cdot\,,\,\cdot\,\})$, the functions in the structure algebra $C^\infty N$ are not necessarily ordinary smooth functions. A stratified symplectic structure on a space is much more than just a space stratified into symplectic manifolds: The globally defined Poisson algebra encapsulates the *mutual positions* of the symplectic structures on the strata; in other words, it encodes the behaviour of the symplectic structures *across* the strata.

Recall that a *complex analytic space* (in the sense of Grauert) is a topological space X, together with a sheaf of rings $\mathcal{O}_X$, having the following property: The space X can be covered by open sets Y, each of which embeds into the open polydisc

$$U = \{\mathbf{z} = (z_1, \dots, z_n);\ |\mathbf{z}| < 1\}$$

in some $\mathbb{C}^n$ (the dimension n may vary as U varies) as the zero set of a finite system of holomorphic functions $f_1, \dots, f_q$ defined on U, such that the restriction $\mathcal{O}_Y$ of the sheaf $\mathcal{O}_X$ to Y is isomorphic as a sheaf to the quotient sheaf $\mathcal{O}_U/(f_1, \dots, f_q)$; here $\mathcal{O}_U$ is the sheaf of germs of holomorphic functions on U. The sheaf $\mathcal{O}_X$ is then referred to as the *sheaf of holomorphic functions on* X. See [4] for a development of the general theory of complex analytic spaces.

DEFINITION 2.1. A *complex analytic stratified Kähler structure* on the stratified space N consists of

(i) a stratified symplectic structure $(C^\infty N, \{\,\cdot\,,\,\cdot\,\})$ having the given stratification of N as its underlying stratification, together with
(ii) a complex analytic structure on N which is compatible with the stratified symplectic structure.

Here, the complex analytic structure on N being *compatible* with the stratified symplectic structure means that the following requirements are met:

(iii) Each stratum is a complex analytic subspace, and the complex analytic structure, restricted to the stratum, turns that stratum into an ordinary complex manifold; in particular, the stratification of N is a refinement of the complex analytic stratification.
(iv) For each point q of N and each holomorphic function f defined on an open neighborhood U of q, there is an open neighborhood V of q with $V \subset U$ such that, on V, f is the restriction of a function in $C^\infty(N, \mathbb{C}) = C^\infty(N) \otimes \mathbb{C}$.
(v) On each stratum, the symplectic structure combines with the complex analytic structure to a Kähler structure.

A stratified space N, together with a complex analytic stratified Kähler structure, will be said to be a *complex analytic stratified Kähler space*.

A simple example of a complex analytic stratified Kähler space arises as follows: In $\mathbb{R}^3$ with coordinates x, y, r, consider the semicone N given by the equation $x^2 + y^2 = r^2$ and the inequality $r \geq 0$. We refer to this semicone as the *exotic* plane with a single vertex. Consider the algebra $C^\infty N$ of continuous functions on N which are restrictions of ordinary smooth functions on the ambient copy of $\mathbb{R}^3$. Thus the functions in $C^\infty N$ arise from ordinary smooth functions in the variables x, y, r. The Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $C^\infty(\mathbb{R}^3)$ defined by

$$\{x, y\} = 2r,\ \ \{x, r\} = 2y,\ \ \{y, r\} = -2x$$

descends to a Poisson bracket on $C^\infty N$, which we denote by $\{\,\cdot\,,\,\cdot\,\}$ as well. Furthermore, endow N with the complex structure having $z = x + iy$ as holomorphic coordinate. Then the Poisson and complex analytic structures combine to a complex analytic stratified Kähler structure. Here the radius function r is *not* an ordinary smooth function of the variables x and y. Thus the stratified symplectic structure cannot be given in terms of ordinary smooth functions of the variables x and y. The Poisson bracket is *defined at the vertex* as well, away from the vertex the Poisson structure is an ordinary smooth *symplectic* Poisson structure, and the complex structure does *not* "see" the vertex. Thus the vertex is a singular point for the Poisson structure whereas it is *not* a singular point for the complex analytic structure. This semicone N is the classical reduced *phase space* of a single particle moving in ordinary affine space of dimension ≥ 2 with angular momentum zero [12], [17].

This example generalizes to an entire class of examples: The *closure of a holomorphic nilpotent orbit* (in a hermitian Lie algebra) inherits a complex analytic stratified Kähler structure [12]. *Projectivization* of the closure of a holomorphic nilpotent orbit yields what we refer to as an *exotic projective variety*. In physics, spaces of this kind arise as reduced classical phase spaces for systems of harmonic oscillators with zero angular momentum and constant energy. More details may be found in [12]–[14], [17].

Another class of examples arises from moduli spaces of semistable holomorphic vector bundles or, more generally, from moduli spaces of semistable principal bundles on a non-singular complex projective curve. See [9]–[12] and the literature there.

Any ordinary Kähler manifold is plainly a complex analytic stratified Kähler space. More generally, Kähler reduction, applied to an ordinary Kähler manifold, yields a complex analytic stratified Kähler structure on the reduced space. See [12] for details. Thus examples of stratified Kähler spaces abound. In the rest of the paper we will explore a particular class of examples which are actually *algebraic* in a sense which we now explain:

Definition 2.2. Given a stratified space N, a *complex algebraic stratified Kähler structure* on N consists of

(i) a real semialgebraic structure on N such that each stratum inherits the structure of a real algebraic manifold;
(ii) a real algebraic Poisson structure $\{\,\cdot\,,\,\cdot\,\}$ on (the real structure sheaf of) N, together with a real algebraic symplectic structure on each stratum, such that the restriction map from (the sheaf of germs of real algebraic functions on) N to (the sheaf of germs of real algebraic functions on) each stratum is a Poisson map;
(iii) a complex algebraic structure on N which is compatible with the other structure.

Given an affine real semialgebraic space N, we write its real coordinate ring as $\mathbb{R}[N]$. Likewise we write the complex coordinate ring of an affine complex algebraic variety N as $\mathbb{C}[N]$. In this paper, all real semialgebraic structures and all complex algebraic structures will be affine, and we shall not mention sheaves any more.

At the risk of making a mountain out of a molehill we note that, in the definition, when the real structure and the complex structure on N are both affine (beware: this does not mean that $\mathbb{C}[N]$ is the complexification of $\mathbb{R}[N]$), the complex algebraic structure on N being *compatible* with the other structure amounts to the following requirements:

(iv) Each stratum of N is a complex algebraic subspace of N, and the complex algebraic structure, restricted to the stratum, turns that stratum into an ordinary complex algebraic manifold; in particular, the stratification of N is a refinement of the complex algebraic stratification.
(v) For each point q of N and each complex algebraic function f defined on an open neighborhood U of q, there is an open neighborhood V of q with $V \subset U$ such that, on V, f is the restriction of a function in $\mathbb{R}[N]_{\mathbb{C}} = \mathbb{R}[N] \otimes \mathbb{C}$.
(vi) On each stratum, the symplectic structure combines with the complex algebraic structure to a Kähler structure.

A *complex algebraic stratified Kähler space* is, then, a stratified space N together with a complex algebraic stratified Kähler structure; the Poisson algebra will then be referred to as an *algebraic stratified symplectic Poisson algebra* (on N).

3. Complex algebraic stratified Kähler structures on adjoint quotients. Return to the situation at the beginning of the paper. We begin by explaining the Kähler structure on T^*K. Endow K with the bi-invariant Riemannian metric induced by the invariant inner product on the Lie algebra $\mathfrak{k}$. Using this metric, we identify $\mathfrak{k}$ with its dual $\mathfrak{k}^*$ and the total space of the tangent bundle $\mathrm{T}K$ with the total space of the cotangent bundle T^*K. Thus the composite

$$\mathrm{T}^*K \to K \times \mathfrak{k} \to K^{\mathbb{C}}$$

of the inverse of left trivialization with the polar decomposition map, which assigns $x \cdot \exp(iY) \in K^{\mathbb{C}}$ to $(x, Y) \in K \times \mathfrak{k}$, identifies T^*K with $K^{\mathbb{C}}$ in a $(K \times K)$-equivariant fashion. Then the induced complex structure on T^*K combines with the symplectic structure to a (positive) Kähler structure. Indeed, the *real analytic* function

$$\kappa\colon K^{\mathbb{C}} \to \mathbb{R}, \quad \kappa(x \cdot \exp(iY)) = |Y|^2, \quad (x, Y) \in K \times \mathfrak{k},$$

on $K^{\mathbb{C}}$ which is twice the *kinetic energy* associated with the Riemannian metric, is a (globally defined) *Kähler potential*; in other words, the function κ is strictly plurisubharmonic and (the negative of the imaginary part of) its Levi form yields (what corresponds to) the cotangent bundle symplectic structure, that is, the cotangent bundle symplectic structure on T^*K is given by

$$i\partial\overline{\partial}\kappa = -d\vartheta$$

where ϑ is the tautological 1-form on T^*K. An explicit calculation which establishes this fact may be found in [6] (but presumably it is a folk-lore observation). We note that, given $Y_0 \in \mathfrak{k}$, the assignment to $(x, Y) \in K \times \mathfrak{k}$ of $|Y - Y_0|^2$ yields a Kähler potential as well which, in turn, determines the same Kähler structure as κ. There is now a literature on related questions [5], [20], [27].

By Proposition 4.2 in [12], the zero momentum reduced space $(\mathrm{T}^*K)_0$ inherits a complex analytic stratified Kähler structure which is actually a complex algebraic stratified Kähler structure. It is straightforward to describe this structure directly, and we will now do so: The underlying complex algebraic structure is that of the categorical quotient $K^{\mathbb{C}}//K^{\mathbb{C}}$ in the category of complex algebraic varieties, cf. e. g. [25] (§3) for details on the construction of a categorical quotient. Suffice it to mention at this stage that $K^{\mathbb{C}}//K^{\mathbb{C}}$ is the complex affine algebraic variety whose coordinate ring is the algebra $\mathbb{C}[K^{\mathbb{C}}]^{K^{\mathbb{C}}}$ of $K^{\mathbb{C}}$-invariants (relative to the conjugation action) of the complex affine coordinate ring $\mathbb{C}[K^{\mathbb{C}}]$ of $K^{\mathbb{C}}$, where $K^{\mathbb{C}}$ is viewed as a (non-singular) complex affine variety. In view of results of Luna [22], [23], the quotient in the category of algebraic varieties is the categorical quotient in the category of analytic varieties as well; for these matters see also [25] (Theorem 3.6).

Under the present circumstances, the categorical quotient $K^{\mathbb{C}}//K^{\mathbb{C}}$ has a very simple description: Choose a maximal torus T in K and let W be the corresponding Weyl group; then $T^{\mathbb{C}}$ is a maximal torus in $K^{\mathbb{C}}$, and the (algebraic) *adjoint quotient* $\chi\colon K^{\mathbb{C}} \to T^{\mathbb{C}}/W$, cf. [18] (3.4) and [26] (3.2) for this terminology, realizes the categorical quotient. By an abuse of language, we will also refer to the target $T^{\mathbb{C}}/W$ of χ as *adjoint quotient* or as the *adjoint quotient of* $K^{\mathbb{C}}$. In concrete terms, the map χ admits the following description: The closure of the conjugacy class of $x \in K^{\mathbb{C}}$ contains a unique semisimple (equivalently: closed) conjugacy class C_x (say), and semisimple conjugacy classes are parametrized by $T^{\mathbb{C}}/W$; the image of $x \in K^{\mathbb{C}}$ under χ is simply the parameter value in $T^{\mathbb{C}}/W$ of the semisimple conjugacy class C_x. Since W is a finite group, as a complex algebraic space, the quotient $T^{\mathbb{C}}/W$ is simply the space of W-orbits in $T^{\mathbb{C}}$.

The choice of maximal torus T in K also provides a direct description of the algebraic stratified symplectic Poisson algebra on the symplectic quotient $(\mathrm{T}^*K)_0 = \mu^{-1}(0)/K$. Indeed, via the identification (1.1) for $K = T$, the real space which underlies the (complex algebraic) orbit space $T^{\mathbb{C}}/W$ for the action of the Weyl group W on $T^{\mathbb{C}}$ amounts simply to the orbit space T^*T/W, relative to the induced action of the Weyl group W on T^*T, and the orbit space T^*T/W inherits a stratified symplectic structure in an obvious fashion: Strata are the W-orbits, the closures of the strata are affine varieties, in the real category as well as in the complex category, indeed, these closures inherit complex algebraic stratified Kähler structures, and the requisite algebraic stratified symplectic Poisson algebra $(\mathbb{R}[\mathrm{T}^*T/W], \{\,\cdot\,,\,\cdot\,\})$ is simply the algebra $\mathbb{R}[\mathrm{T}^*T]^W$ of W-invariant functions in $\mathbb{R}[\mathrm{T}^*T]$, endowed with the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ coming from the ordinary (algebraic) symplectic Poisson bracket on T^*T. The choice of invariant inner product on $\mathfrak{k}$ determines an injection $\mathrm{T}^*T \to \mathrm{T}^*K$ which induces a homeomorphism from T^*T/W onto $(\mathrm{T}^*K)_0$ compatible with all the structure.

4. The space of normalized degree n polynomials. In the next section we will work out defining equations for the closures of the strata, viewed as complex algebraic varieties, of the adjoint quotient of $\mathrm{SL}(n,\mathbb{C})$. To this end, we need some preparations.

The space $\mathbb{A}^n_{\mathrm{coef}}$ of normalized degree n polynomials

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \tag{4.1}$$

with complex coefficients a_j $(1 \le j \le n)$ is an n-dimensional complex affine space in an obvious way; it may be viewed as the n'th symmetric power $\mathrm{S}^n[\mathbb{C}]$ of a copy $\mathbb{C}$ of the complex numbers. The space $\mathbb{A}^n_{\mathrm{coef}}$ is stratified according to the multiplicities of the roots of the polynomials where strata correspond to partitions of n. We define a *partition* of n of *length* r to be an r-tuple

$$\nu = (n_1, n_2, \dots, n_r) \tag{4.2}$$

of positive natural numbers n_j $(1 \le j \le r)$ such that $\sum n_j = n$, normalized so that

$$n_1 \ge n_2 \ge \ldots \ge n_r.$$

Given the partition ν of n of length r, of the kind (4.2), as a complex manifold, the stratum corresponding to the partition ν is the complex r-dimensional manifold $\mathcal{D}^o_\nu$ of polynomials

$$P(z) = \prod_{j=1}^{r} (z - u_j)^{n_j}$$

with all the roots u_j being pairwise distinct. In particular, the *top* stratum of the space $\mathbb{A}^n_{\mathrm{coef}} \cong \mathrm{S}^n[\mathbb{C}]$ is the space of all polynomials having only single roots and, given a length r partition ν of n of the kind (4.2), as a complex manifold, $\mathcal{D}^o_\nu$ comes down to the top stratum of $\mathrm{S}^r[\mathbb{C}]$, that is, to the space of normalized degree r polynomials having only single roots. For each partition ν of n of the kind (4.2), let $\mathcal{D}_\nu$ be the closure of $\mathcal{D}^o_\nu$ in $\mathbb{A}^n_{\mathrm{coef}}$; this closure is an affine variety. In particular, at the bottom, we have $\mathcal{D}_{(n)} = \mathcal{D}^o_{(n)}$. Given the partition $\nu = (n_1, n_2, \dots, n_r)$ of n, consider the partition of n which arises from ν by the operation of picking a pair (n_j, n_k), taking the sum of n_j and n_k, and reordering the terms (if need be); iterating this operation we obtain a partial ordering among the partitions of n, which we write as $\nu' \preceq \nu$, where ν' arises from ν by a finite sequence of operations of the kind just described, the empty sequence being admitted, so that $\nu \preceq \nu$. When $\nu' \preceq \nu$ and $\nu' \ne \nu$, we write $\nu' \prec \nu$. The stratum $\mathcal{D}^o_{\nu'}$ lies in the closure $\mathcal{D}_\nu$ of $\mathcal{D}^o_\nu$ if and only if $\nu' \prec \nu$; thus, for any partition ν of n,

$$\mathcal{D}_\nu = \bigcup_{\nu' \preceq \nu} \mathcal{D}^o_{\nu'}.$$

It is manifest that $\mathbb{A}^n_{\mathrm{coef}}$ is the disjoint union of the $\mathcal{D}^o_\nu$'s as ν ranges over partitions of n, and this decomposition is a stratification.

The stratification of $\mathbb{A}^n_{\mathrm{coef}}$ can be understood in terms of discriminant varieties. We will now explain this briefly; cf. [19] and the literature there for more details.

Let Σ_1 be the non-singular hypersurface in the $(n+1)$-dimensional complex affine space $\mathbb{C} \times \mathbb{A}^n_{\mathrm{coef}}$ with coordinates $(z, a_1, \dots, a_{n-1}, a_n)$ given by the equation

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0. \tag{4.3}$$

Then a_n is a polynomial function of the other coordinates $z, a_1, \dots, a_{n-1}$ whence the hypersurface Σ_1 admits the parametrization

$$h\colon \mathbb{C}^n \to \mathbb{C} \times \mathbb{A}^n_{\mathrm{coef}}, \quad (z, a_1, \dots, a_{n-1}) \mapsto (z, a_1, \dots, a_{n-1}, a_n), \tag{4.4}$$

and this parametrization is non-singular (smooth).

Given the normalized complex polynomial $P(z)$ of the kind (4.1), for $1 \le k < n$, consider the successive derivatives

$$P^{(k)}(z) = k! z^{n-k} + (k-1)! a_1 z^{n-k-1} + \cdots + a_{n-k};$$

accordingly, let $\Sigma_{k+1} \subseteq \Sigma_1$ be the affine complex variety given by the equations

$$P(z) = 0,\ P'(z) = 0,\ \ldots,\ P^{(k)}(z) = 0; \tag{4.5}$$

the variety Σ_{k+1} is actually non-singular. Let

$$p\colon \mathbb{C} \times \mathbb{A}^n_{\mathrm{coef}} \to \mathbb{A}^n_{\mathrm{coef}}$$

be the projection to the second factor and, for $1 \leq k \leq n$, let $\mathcal{D}_k = p(\Sigma_k) \subseteq \mathbb{A}^n_{\mathrm{coef}}$, so that, in particular, $\mathcal{D}_1 = \mathbb{A}^n_{\mathrm{coef}}$; we refer to $\mathcal{D}_k$ as the k'th *discriminant* variety. We shall see shortly that each $\mathcal{D}_k$ is indeed a complex algebraic variety. For $1 \leq k \leq n-1$, the restriction of the projection p to $\Sigma_k \setminus \Sigma_{k+1}$ is $(n-k+1)$ to 1, the restriction of the projection p from Σ_k to $\mathcal{D}_k$ branches over $\mathcal{D}_{k+1} \subseteq \mathcal{D}_k$, and the restriction of the projection p to Σ_n is injective and identifies Σ_n with $\mathcal{D}_n$. For $1 \leq k \leq n$, $\mathcal{D}_k$ is the space of polynomials having at least one root with multiplicity at least equal to k. In particular, $\mathcal{D}_n$ is a smooth curve, parametrized by

$$w \mapsto \left(-nw, \binom{n}{2} w^2, -\binom{n}{3} w^3, \ldots, (-1)^n w^n\right), \quad w \in \mathbb{C}.$$

By construction, the $\mathcal{D}_k$'s form an ascending sequence

$$\mathcal{D}_n \subseteq \mathcal{D}_{n-1} \subseteq \ldots \subseteq \mathcal{D}_1 = \mathbb{A}^n_{\mathrm{coef}}.$$

Equations for the $\mathcal{D}_k$'s can be obtained by the following classical procedure: Given the polynomial

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \quad a_j \in \mathbb{C} \quad (1 \leq j \leq n), \tag{4.6}$$

with roots $z_1, \ldots, z_n$, the *discriminant*

$$D_n(P) = D_n(a_0, a_1, \ldots, a_n)$$

of this polynomial is defined to be the expression

$$D_n(a_0, a_1, \ldots, a_n) = a_0^{2n-2} \prod_{j<k} (z_j - z_k)^2, \tag{4.7}$$

written as a polynomial in the coefficients $a_0, a_1, \ldots, a_n$.

To reproduce an expression for the discriminant $D_n(a_0, a_1, \ldots, a_n)$ of a polynomial of the kind (4.6), recall from e. g. [28] that, given two polynomials

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n, \tag{4.8}$$

$$h(x) = b_0 x^m + b_1 x^{m-1} + \ldots + b_m, \tag{4.9}$$

the *resultant* $R(f, h)$ of f and h, sometimes referred to as the Sylvester *resultant*, may be computed as the $((m+n) \times (m+n))$-determinant

$$R(f,h) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n & 0 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n \\ b_0 & b_1 & b_2 & \cdots & . & b_m & 0 & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & . & b_{m-1} & b_m & 0 & \cdots & 0 \\ . & . & . & \cdots & . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & b_0 & b_1 & b_2 & \cdots & b_{m-1} & b_m \end{vmatrix}. \tag{4.10}$$

This determinant has m rows of the kind

$$[0,\dots,0,a_0,a_1,a_2,\dots,a_n,0,\dots,0]$$

and n rows of the kind

$$[0,\dots,0,b_0,b_1,b_2,\dots,b_m,0,\dots,0]$$

We quote the following classical facts [28].

Proposition 4.1. *When $a_0 \neq 0 \neq b_0$, the polynomials f and h have a common linear factor if and only if $R(f,h)=0$.*

Proposition 4.2. *The discriminant $D_n(a_0,\dots,a_n)$ of the polynomial*

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$$

satisfies the identity

$$a_0D = \pm R(f,f'). \tag{4.11}$$

Now, by construction, for $1 \le k < n$, the discriminant variety $\mathcal{D}_{k+1} \subseteq \mathcal{D}_1$ is given by the equations

$$\begin{aligned} D_n(1,a_1,\dots,a_n) &= 0, \\ D_{n-1}(n,(n-1)a_1,\dots,a_{n-1}) &= 0, \\ &\dots \\ D_{n-(k-1)}\left(\frac{n!}{(n-k)!},\frac{(n-1)!}{(n-k)!}a_1,\dots,a_{n-(k-1)}\right) &= 0; \end{aligned} \tag{4.12}$$

the dimension of $\mathcal{D}_{k+1}$ equals $n-k$.

For illustration, we give explicit expressions for the discriminants for $2 \le n \le 4$: When $n=2$, the discriminant $D_2(a_0,a_1,a_2)$ of $a_0x^2+a_1x+a_2$ comes down to the familiar expression

$$D_2(a_0,a_1,a_2) = a_1^2 - 4a_0a_2; \tag{4.13}$$

and when $n=3$, the discriminant $D_3(a_0,a_1,a_2,a_3)$ of the cubic polynomial

$$a_0x^3 + a_1x^2 + a_2x + a_3$$

reads

$$D_3(a_0,a_1,a_2,a_3) = a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3. \tag{4.14}$$

Likewise, the discriminant $D_4 = D_4(a_0,a_1,a_2,a_3,a_4)$ of the quartic polynomial

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$$

has the form

$$\begin{aligned} D_4 = {} & (a_1^2a_2^2a_3^2 - 4a_1^3a_3^3 - 4a_0a_2^3a_3^2 + 18a_0a_1a_2a_3^3 - 27a_0^2a_3^4 + 256a_0^3a_4^3) \\ & + (-4a_1^2a_2^3 + 18a_1^3a_2a_3 + 16a_0a_2^4 - 80a_0a_1a_2^2a_3 - 6a_0a_1^2a_3^2 + 144a_0^2a_2a_3^2)a_4 \\ & + (-27a_1^4 + 144a_0a_1^2a_2 - 128a_0^2a_2^2 - 192a_0^2a_1a_3)a_4^2. \end{aligned}$$

For $1 \le k \le n$, when ν is the partition $(k,1,\dots,1)$ of n of length $k-r+1$, $\mathcal{D}_\nu$ equals $\mathcal{D}_k$; moreover, the closure $\mathcal{D}_k$ of $\mathcal{D}_\nu^o$ then contains the $\mathcal{D}_\nu^o$'s for $\nu = (n_1,n_2,\dots,n_s\}$ with $n_1 \ge k$. Given $\mathcal{D}_\nu$ with $\nu = \{k,n_2,\dots,n_r\}$, defining equations for $\mathcal{D}_\nu$ may be obtained by adding suitable equations to those defining $\mathcal{D}_k$.

In low dimensions, these considerations yield the following picture:

$n=2$: In this case, the variety $\mathbb{A}^2_{\mathrm{coef}} = \mathcal{D}_1$ is a 2-dimensional complex affine space, which decomposes into two strata according to the two partitions $(1,1)$ and (2) of the natural number $n=2$. In the coordinates (a_1,a_2) on $\mathbb{A}^2_{\mathrm{coef}}$, the variety $\mathcal{D}_2$ is the non-singular curve of degree 2 given by the equation

$$D_2(1,a_1,a_2) = a_1^2 - 4a_2 = 0. \tag{4.15}$$

$n=3$: Now the variety $\mathbb{A}^3_{\mathrm{coef}} = \mathcal{D}_1$ is a 3-dimensional complex affine space, which decomposes into three strata according to the three partitions $(1,1,1)$, $(2,1)$, (3) of $n=3$. Furthermore, in the coordinates (a_1,a_2,a_3) on $\mathbb{A}^3_{\mathrm{coef}}$, the surface $\mathcal{D}_2 = \mathcal{D}_{(2,1)}$ in $\mathbb{A}^3_{\mathrm{coef}}$ is given by the equation

$$D_3(1,a_1,a_2,a_3) = 0, \tag{4.16}$$

and this surface decomposes as

$$\mathcal{D}_2 = \mathcal{D}^o_{(2,1)} \cup \mathcal{D}_3.$$

Moreover, in the chosen coordinates, $\mathcal{D}_3$ is the curve in $\mathbb{A}^3_{\mathrm{coef}}$ given by the equations (4.16) and

$$D_2(3,2a_1,a_2) = 0, \tag{4.17}$$

and this curve is non-singular. The surface $\mathcal{D}_2 = \mathcal{D}_{(2,1)}$ in $\mathbb{A}^3_{\mathrm{coef}}$ is singular along the curve $\mathcal{D}_3$. Indeed, a calculation shows that the three partial derivatives of the defining equation (4.16) of $\mathcal{D}_2$ vanish identically along $\mathcal{D}_3$. A closer look reveals that $\mathcal{D}_2$ has a "fold" along $\mathcal{D}_3$ but is topologically flat in $\mathbb{A}^3_{\mathrm{coef}}$; cf. Corollary 5.1 of [19] for details.

$n=4$: In this case, by construction, $\mathbb{A}^4_{\mathrm{coef}} = \mathcal{D}_1$ is a 4-dimensional complex affine space; this space decomposes into five strata according to the five partitions $(1,1,1,1)$, $(2,1,1)$, $(2,2)$, $(3,1)$, (4) of $n=4$. Furthermore, in the coordinates (a_1,a_2,a_3,a_4) on $\mathbb{A}^4_{\mathrm{coef}}$, the hypersurface $\mathcal{D}_2 = \mathcal{D}_{(2,1,1)}$ in $\mathcal{D}_1$ is given by the equation

$$D_4(1,a_1,a_2,a_3,a_4) = 0, \tag{4.18}$$

and this hypersurface decomposes as

$$\mathcal{D}_{(2,1,1)} = \mathcal{D}^o_{(2,1,1)} \cup \mathcal{D}^o_{(2,2)} \cup \mathcal{D}^o_{(3,1)} \cup \mathcal{D}_4.$$

Likewise, in the chosen coordinates, the surface $\mathcal{D}_3 = \mathcal{D}_{(3,1)}$ in $\mathbb{A}^4_{\mathrm{coef}}$ is given by the equations (4.18) and

$$D_3(4, 3a_1, 2a_2, a_3) = 0, \tag{4.19}$$

and this surface decomposes as

$$\mathcal{D}_{(3,1)} = \mathcal{D}^o_{(3,1)} \cup \mathcal{D}_4.$$

Moreover, $\mathcal{D}_4$ is a non-singular curve of degree 4 in $\mathbb{A}^4_{\mathrm{coef}}$ which, in the above coordinates, is given by the equations (4.18), (4.19), and

$$D_2(12, 6a_1, 2a_2) = 0. \tag{4.20}$$

The hypersurface $\mathcal{D}_2 = \mathcal{D}_{(2,1,1)}$ in $\mathbb{A}^4_{\mathrm{coef}}$ is the space of normalized complex degree 4 polynomials with at least one multiple root, and the singular locus of this hypersurface $\mathcal{D}_{(2,1,1)}$ is the union

$$\mathcal{D}_3 \cup \mathcal{D}_{(2,2)} = \mathcal{D}^o_{(3,1)} \cup \mathcal{D}^o_{(2,2)} \cup \mathcal{D}_{(4)}.$$

The surface $\mathcal{D}_3 = \mathcal{D}_{(3,1)}$ in $\mathbb{A}^4_{\mathrm{coef}}$ is the space of normalized complex degree 4 polynomials with a root of multiplicity at least 3, and the singular locus of this surface $\mathcal{D}_3$ is the curve $\mathcal{D}_4$. See Example 6.1 in [19] for details. The surface

$$\mathcal{D}_{(2,2)} = \mathcal{D}^o_{(2,2)} \cup \mathcal{D}_{(4)}$$

in $\mathbb{A}^4_{\mathrm{coef}}$ is the space of normalized complex degree 4 polynomials having two roots with multiplicity 2 or a single root with multiplicity 4; this surface is *not* a discriminant variety of the kind $\mathcal{D}_k$. Inspection of the coefficients a_1, a_2, a_3, a_4 of the polynomial

$$P(z) = (z - z_1)^2(z - z_2)^2 = z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$$

shows that, in the coordinates (a_1, a_2, a_3, a_4), the variety $\mathcal{D}_{(2,2)}$ in $\mathbb{A}^4_{\mathrm{coef}}$ is given by the two equations

$$(a_1^2 - 4a_2)a_1 + 4a_3 = 0, \tag{4.21}$$

$$(a_1^2 - 4a_2)^2 - 16a_4 = 0. \tag{4.22}$$

Thus a_3 and a_4 are polynomial functions of a_1 and a_2 whence the obvious parametrization

$$(a_1, a_2) \mapsto (a_1, a_2, a_3, a_4)$$

identifies $\mathcal{D}_{(2,2)}$ with a copy of 2-dimensional complex affine space $\mathbb{A}^2$. Inspection of the coefficients a_1 and a_2 of the polynomial

$$P(z) = (z - z_0)^4 = z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 \tag{4.23}$$

shows that, in terms of the coordinates a_1 and a_2, the variety $\mathcal{D}_4 \subseteq \mathcal{D}_{(2,2)}$ is the curve given by the equation

$$3a_1^2 - 8a_2 = 0. \tag{4.24}$$

We intend to work out elsewhere how equations for varieties of the kind $\mathcal{D}_\nu$ which are not ordinary discriminant varieties (i.e. not of the kind $\mathcal{D}_k$) may be derived.

5. The adjoint quotient of $\mathrm{SL}(n,\mathbb{C})$**.** Let $n \geq 2$. In this section we will explore the stratification of the adjoint quotient of $\mathrm{SL}(n,\mathbb{C})$. For convience, we will consider first the group $\mathrm{GL}(n,\mathbb{C})$. A maximal complex torus in $\mathrm{GL}(n,\mathbb{C})$ is given by the complex diagonal matrices in $\mathrm{GL}(n,\mathbb{C})$. Thus, as a complex Lie group, this torus is isomorphic to the product $(\mathbb{C}^*)^n$ of n copies of the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers. Introduce standard coordinates $z_1,\ldots,z_n$ on $\mathbb{C}^n$. Then the fundamental characters $\sigma_1,\ldots,\sigma_n$ of $\mathrm{GL}(n,\mathbb{C})$, restricted to the maximal torus, come down to the elementary symmetric functions in the variables $z_1,\ldots,z_n$, and the assignment to $\mathbf{z}=(z_1,\ldots,z_n)$ of

$$(a_1,\ldots,a_n) = (-\sigma_1(\mathbf{z}),\sigma_2(\mathbf{z}),\ldots,(-1)^n\sigma_n(\mathbf{z})) \in \mathbb{C}^n$$

yields a holomorphic map

$$(-\sigma_1,\sigma_2\ldots,(-1)^n\sigma_n)\colon (\mathbb{C}^*)^n \to \mathbb{C}^n \tag{5.1}$$

which induces a complex algebraic isomorphism from the adjoint quotient $(\mathbb{C}^*)^n/S_n$ of $\mathrm{GL}(n,\mathbb{C})$ onto the subspace of the space $\mathbb{A}^n_{\mathrm{coef}}$ of complex normalized degree n polynomials which consists of normalized polynomials having non-zero constant coefficient.

Let $K=\mathrm{SU}(n)$, so that $K^{\mathbb{C}}=\mathrm{SL}(n,\mathbb{C})$. A maximal complex torus $T^{\mathbb{C}}$ in $\mathrm{SL}(n,\mathbb{C})$ is given by the complex diagonal matrices in $\mathrm{SL}(n,\mathbb{C})$, that is, by the complex diagonal $(n\times n)$-matrices having determinant 1. Realize the torus $T^{\mathbb{C}}$ as the subspace of the maximal torus of $\mathrm{GL}(n,\mathbb{C})$ in the standard fashion, that is, as the subspace of $(\mathbb{C}^*)^n$ which consists of all $(z_1,\ldots,z_n)$ in $(\mathbb{C}^*)^n$ such that $z_1\ldots z_n=1$. The holomorphic map (5.1) restricts to the holomorphic map

$$(-\sigma_1,\ldots,(-1)^{n-1}\sigma_{n-1})\colon T^{\mathbb{C}} \cong (\mathbb{C}^*)^{n-1} \to \mathbb{C}^{n-1}. \tag{5.2}$$

This map induces a complex algebraic isomorphism from the adjoint quotient $T^{\mathbb{C}}/S_n$ of $\mathrm{SL}(n,\mathbb{C})$ onto the subspace $\mathbb{A}^{n,1}_{\mathrm{coef}}$ of the space $\mathbb{A}^n_{\mathrm{coef}}$ of complex normalized degree n polynomials which consists of polynomials having constant coefficient equal to 1. The space $\mathbb{A}^{n,1}_{\mathrm{coef}}$ is a complex affine space of dimension $n-1$ whence the same is true of the adjoint quotient of $\mathrm{SL}(n,\mathbb{C})$. The discussion in the previous section applies to the adjoint quotient of $\mathrm{SL}(n,\mathbb{C})$ in the following fashion:

For $1\leq k\leq n$, let $\mathcal{D}^{\mathrm{SU}(n)}_k = \mathcal{D}_k \cap \mathbb{A}^{n,1}_{\mathrm{coef}}$ so that, in particular, $\mathcal{D}^{\mathrm{SU}(n)}_1$ is the entire space $\mathbb{A}^{n,1}_{\mathrm{coef}} \cong \mathrm{SL}(n,\mathbb{C})//\mathrm{SL}(n,\mathbb{C})$. Likewise, for each partition ν of n, let

$$\mathcal{D}^{\mathrm{SU}(n)}_\nu = \mathcal{D}_\nu \cap \mathbb{A}^{n,1}_{\mathrm{coef}} \quad \text{and} \quad \mathcal{D}^{(o,\mathrm{SU}(n))}_\nu = \mathcal{D}^o_\nu \cap \mathbb{A}^{n,1}_{\mathrm{coef}}.$$

Then $\mathbb{A}^{n,1}_{\mathrm{coef}} \cong \mathrm{SL}(n,\mathbb{C})//\mathrm{SL}(n,\mathbb{C})$ is the disjoint union of the $\mathcal{D}^{(o,\mathrm{SU}(n))}_\nu$'s as ν ranges over partitions of n, and this decomposition is a stratification. In low dimensions, we are thus led to the following picture:

$n=2$: In this case, the adjoint quotient $\mathbb{A}^{2,1}_{\mathrm{coef}} = \mathcal{D}^{\mathrm{SU}(2)}_1$ is a copy of the complex line, which decomposes into two strata according to the two partitions $(1,1)$ and (2) of the natural number $n=2$. In the coordinate a_1 on $\mathbb{A}^{2,1}_{\mathrm{coef}}$, the stratum $\mathcal{D}^{\mathrm{SU}(2)}_2$ consists of the two points ± 2; these are solutions of the equation

$$D_2(1,a_1,1) = a_1^2 - 4 = 0. \tag{5.3}$$

$n=3$: Now the adjoint quotient $\mathbb{A}^{3,1}_{\mathrm{coef}} = \mathcal{D}^{\mathrm{SU}(3)}_1$ is a 2-dimensional complex affine space, which decomposes into three strata according to the three partitions $(1,1,1)$, $(2,1)$, (3)

of $n = 3$. Furthermore, in the coordinates (a_1, a_2) on $\mathbb{A}^{3,1}_{\mathrm{coef}}$, the variety $\mathcal{D}_2^{\mathrm{SU}(3)} = \mathcal{D}_{(2,1)}^{\mathrm{SU}(3)}$ in $\mathbb{A}^{3,1}_{\mathrm{coef}}$ is the curve given by the equation

$$D_3(1, a_1, a_2, 1) = 0, \tag{5.4}$$

that is, in view of (4.14), by the equation

$$a_1^2 a_2^2 - 4a_2^3 - 4a_1^3 - 27 + 18a_1 a_2 = 0, \tag{5.5}$$

and this curve decomposes as

$$\mathcal{D}_{(2,1)}^{\mathrm{SU}(3)} = \mathcal{D}_{(2,1)}^{(\mathrm{SU}(3),o)} \cup \mathcal{D}_3^{\mathrm{SU}(3)}.$$

Moreover, in the chosen coordinates, the lowest stratum $\mathcal{D}_3^{\mathrm{SU}(3)}$ consists of the three points in $\mathbb{A}^3_{\mathrm{coef}}$ which solve the equations (5.4) and

$$D_2(3, 2a_1, 1) = 0. \tag{5.6}$$

At each of these three points, the curve $\mathcal{D}_2^{\mathrm{SU}(3)}$ in $\mathbb{A}^3_{\mathrm{coef}}$ has a cusp, as a direct examination shows. This fact is also a consequence of the observation spelled out in the previous section that the surface $\mathcal{D}_2$ in $\mathbb{A}^3_{\mathrm{coef}}$ has a "fold" along $\mathcal{D}_3$. Topologically, the curve $\mathcal{D}_2^{\mathrm{SU}(3)}$ in $\mathbb{A}^3_{\mathrm{coef}}$ is flat, i.e. the embedding of $\mathcal{D}_2^{\mathrm{SU}(3)}$ into $\mathbb{A}^3_{\mathrm{coef}}$ is locally homeomorphic to the standard embedding of $\mathbb{R}^2$ into $\mathbb{R}^6$.

$n = 4$: In this case, the adjoint quotient $\mathbb{A}^{4,1}_{\mathrm{coef}} = \mathcal{D}_1^{\mathrm{SU}(4)}$ is a 3-dimensional complex affine space, which decomposes into five strata according to the five partitions $(1,1,1,1)$, $(2,1,1)$, $(2,2)$, $(3,1)$, (4) of $n = 4$. Furthermore, in the coordinates (a_1, a_2, a_3) on $\mathbb{A}^{4,1}_{\mathrm{coef}}$, the surface $\mathcal{D}_2^{\mathrm{SU}(4)} = \mathcal{D}_{(2,1,1)}^{\mathrm{SU}(4)}$ in $\mathbb{A}^{4,1}_{\mathrm{coef}}$ is given by the equation

$$D_4(1, a_1, a_2, a_3, 1) = 0, \tag{5.7}$$

and this surface decomposes as

$$\mathcal{D}_{(2,1,1)}^{\mathrm{SU}(4)} = \mathcal{D}_{(2,1,1)}^{(\mathrm{SU}(4),o)} \cup \mathcal{D}_{(2,2)}^{(\mathrm{SU}(4),o)} \cup \mathcal{D}_{(3,1)}^{(\mathrm{SU}(4),o)} \cup \mathcal{D}_4^{\mathrm{SU}(4)}.$$

Likewise, in the chosen coordinates, the curve $\mathcal{D}_3^{\mathrm{SU}(4)} = \mathcal{D}_{(3,1)}^{\mathrm{SU}(4)}$ in $\mathbb{A}^{4,1}_{\mathrm{coef}}$ is given by the equations (5.7) and

$$D_3(4, 3a_1, 2a_2, a_3) = 0, \tag{5.8}$$

and this curve decomposes as

$$\mathcal{D}_{(3,1)}^{\mathrm{SU}(4)} = \mathcal{D}_{(3,1)}^{(\mathrm{SU}(4),o)} \cup \mathcal{D}_4^{\mathrm{SU}(4)}.$$

Moreover, the lowest stratum $\mathcal{D}_4^{\mathrm{SU}(4)}$ consists of the four points in $\mathbb{A}^{4,1}_{\mathrm{coef}}$ which, in the coordinates (a_1, a_2, a_3), solve the equations (5.7), (5.8), and

$$D_2(12, 6a_1, 2a_2) = 0. \tag{5.9}$$

The singular locus of the surface $\mathcal{D}_2^{\mathrm{SU}(4)} = \mathcal{D}_{(2,1,1)}^{\mathrm{SU}(4)}$ in $\mathbb{A}^{4,1}_{\mathrm{coef}}$ is the union

$$\mathcal{D}_3^{\mathrm{SU}(4)} \cup \mathcal{D}_{(2,2)}^{\mathrm{SU}(4)} = \mathcal{D}_{(3,1)}^{(\mathrm{SU}(4),o)} \cup \mathcal{D}_{(2,2)}^{(\mathrm{SU}(4),o)} \cup \mathcal{D}_4^{\mathrm{SU}(4)}.$$

The singular locus of the curve $\mathcal{D}_3^{\mathrm{SU}(4)}$ in $\mathbb{A}^{4,1}_{\mathrm{coef}}$ consists of the four points in $\mathcal{D}_4^{\mathrm{SU}4}$. This is a consequence of the corresponding observation describing the singular locus of the surface $\mathcal{D}_3$ in $\mathbb{A}^4_{\mathrm{coef}}$, spelled out in the previous section. The curve

$$\mathcal{D}^{\mathrm{SU}(4)}_{(2,2)} = \mathcal{D}^{(\mathrm{SU}(4),o)}_{(2,2)} \cup \mathcal{D}^{\mathrm{SU}(4)}_4$$

in $\mathbb{A}^{4,1}_{\mathrm{coef}}$ is given by the two equations

$$(a_1^2 - 4a_2)a_1 + 4a_3 = 0, \tag{5.10}$$

$$(a_1^2 - 4a_2)^2 - 16 = 0. \tag{5.11}$$

Consequently the embedding

$$(a_1, a_2) \mapsto (a_1, a_2, \frac{1}{4}(4a_2 - a_1^2)a_1)$$

of 2-dimensional complex affine space $\mathbb{A}^2$ into $\mathbb{A}^{4,1}_{\mathrm{coef}}$ identifies the non-singular curve in $\mathbb{A}^2$ given by the equation (5.11) with the curve $\mathcal{D}^{\mathrm{SU}(4)}_{(2,2)}$ in $\mathbb{A}^{4,1}_{\mathrm{coef}}$ which is therefore neccessarily non-singular. The observation related with the polynomial (4.23) in the previous section shows that, in terms of the coordinates a_1 and a_2, the bottom stratum $\mathcal{D}^{\mathrm{SU}(4)}_4$, viewed as a subset of $\mathcal{D}^{\mathrm{SU}(4)}_{(2,2)}$, consists of the four points in $\mathcal{D}^{\mathrm{SU}(4)}_{(2,2)}$ solving the equation (4.24), that is, the four points given by $(a_1, a_2) = (\pm 2\sqrt{2}, \pm 3)$.

The position of the real orbit space $\mathrm{SU}(n)\big/\mathrm{SU}(n)$ relative to conjugation, realized within the adjoint quotient $T^{\mathbb{C}}\big/S_n$, is worth remarking. Thus, consider the real affine space $\mathbb{R}^{n-1}$, embedded into $\mathbb{C}^{n-1}$ as the real affine space of real points of the adjoint quotient $\mathbb{C}^{n-1} \cong T^{\mathbb{C}}\big/S_n$ in the obvious fashion. The space

$$\mathcal{H}_n = \mathcal{D}^{\mathrm{SU}(n)}_2 \cap \mathbb{R}^{n-1} \tag{5.12}$$

of real points of the complex variety $\mathcal{D}^{\mathrm{SU}(n)}_2$ is a real compact hypersurface in $\mathbb{R}^{n-1}$, and the orbit space $\mathrm{SU}(n)\big/\mathrm{SU}(n)$, realized within the adjoint quotient $T^{\mathbb{C}}\big/S_n \cong C^{n-1}$ of $\mathrm{SL}(n,\mathbb{C})$, amounts to the compact region R_n in $\mathbb{R}^{n-1} \subseteq \mathbb{C}^{n-1}$ bounded by this hypersurface. The exponential mapping from the Lie algebra $\mathfrak{t}$ of T to T, restricted to a Weyl alcove, induces a homeomorphism from this alcove onto R_n.

6. The stratified Kähler structure on the adjoint quotient of $\mathrm{SL}(n,\mathbb{C})$**.** Let K be a general compact Lie group, let T be a maximal torus in K, let $K^{\mathbb{C}}$ be the complexification of K and $T^{\mathbb{C}}$ that of T, and let N be the adjoint quotient of $K^{\mathbb{C}}$, that is, N equals the space $T^{\mathbb{C}}\big/W$ of W-orbits relative to the action of the Weyl group W of K on $T^{\mathbb{C}}$. The algebraic stratified symplectic Poisson algebra on the *real* coordinate ring $\mathbb{R}[T^{\mathbb{C}}]$ of the complex torus $T^{\mathbb{C}}$ is obtained in the following fashion: The torus $T^{\mathbb{C}}$ amounts to a product of finitely many copies of the complex 1-dimensional torus $\mathbb{C}^*$. In terms of the standard coordinate $z = x + iy$ on $\mathbb{C}^*$, identified with the multiplicative group of non-zero complex numbers, the corresponding real Poisson structure is given by $\{x, y\} = x^2 + y^2$, and the resulting symplectic structure and the complex structure on $\mathbb{C}^*$ combine to an algebraic Kähler structure which, in turn, is just the structure arising from the standard embedding of $\mathbb{C}^*$ into $\mathrm{SU}(2)^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$ as a maximal complex torus, where $\mathrm{SL}(2,\mathbb{C})$ is endowed with the Kähler structure reproduced in Section 3 above. The symplectic Poisson algebra on the complex torus $T^{\mathbb{C}}$ in the general group $K^{\mathbb{C}}$ is now simply the product structure whence the induced symplectic structure on $T^{\mathbb{C}}$ is plainly real algebraic. The algebraic stratified symplectic Poisson algebra on the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/W] \cong \mathbb{R}[T^{\mathbb{C}}]^W$ of the adjoint quotient is simply obtained by taking invariants. This Poisson algebra is somewhat more easily described in terms of the *complexification* $\mathbb{R}[T^{\mathbb{C}}/W]_{\mathbb{C}}$ of the real coordinate ring of $T^{\mathbb{C}}/W$—this complexification is *not* the complex coordinate ring of $T^{\mathbb{C}}/W$, though. An explicit description of the induced Poisson structure on the complexification of the real coordinate ring of $T^{\mathbb{C}}/W$ has been worked out in [16].

For $K = \mathrm{SU}(n)$, the Weyl group W is the symmetric group S_n on n letters, and the quoted description of the ring $\mathbb{R}[T^{\mathbb{C}}/W]_{\mathbb{C}}$, together with the Poisson structure, takes the following form: As before, consider $\mathrm{SU}(n)$ as a subgroup of $\mathrm{U}(n)$ as usual and, accordingly, identify $T^{\mathbb{C}}$ with the subgroup of $(\mathbb{C}^*)^n$ (taken as a maximal torus of $\mathrm{U}(n)^{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$) given by $z_1 \cdot \ldots \cdot z_n = 1$ where $z_1, \ldots, z_n$ are the obvious coordinates on $(\mathbb{C}^*)^n$. For $r \geq 0$ and $s \geq 0$ such that $1 \leq r + s \leq n$, let

$$\sigma_{(r,s)}(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n) \tag{6.1}$$

be the S_n-orbit sum of the monomial $z_1 \cdot \ldots \cdot z_r \overline{z}_{r+1} \cdot \ldots \cdot \overline{z}_{r+s}$; the function $\sigma_{(r,s)}$ is manifestly S_n-invariant, and we refer to a function of this kind as an *elementary bisymmetric function*. In a degree m, $1 \leq m \leq n$, the construction yields the $m + 1$ bisymmetric functions $\sigma_{(m,0)}$, $\sigma_{(m-1,1)}$, $\ldots$, $\sigma_{(0,m)}$ whence in degrees at most equal to n it yields altogether $\frac{n(n+3)}{2}$ elementary bisymmetric functions. According to a classical result, the ring $\mathbb{Q}[z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n]^{S_n}$ of bisymmetric functions (over the rationals) is generated by the elementary bisymmetric functions. This implies that, as a ring, $\mathbb{R}[T^{\mathbb{C}}/S_n]_{\mathbb{C}}$ is generated by the $\frac{n(n+3)}{2} - 2$ elementary bisymmetric functions $\sigma_{(r,s)}$ for $0 \leq r \leq n$ and $0 \leq s \leq n$ such that $1 \leq r + s \leq n$ and $(r, s) \neq (n, 0)$ and $(r, s) \neq (0, n)$, subject to a certain system of $\frac{n(n-1)}{2}$ relations. See [16] for details. For intelligibility we note that the conditions $(r, s) \neq (n, 0)$ and $(r, s) \neq (0, n)$ mean that, in accordance with the discussion in Section 5 above, in the ring $\mathbb{R}[T^{\mathbb{C}}/S_n]_{\mathbb{C}}$, $\sigma_{(n,0)} = 1$ and $\sigma_{(0,n)} = 1$.

The stratified symplectic Poisson structure is more easily described in terms of another system of multiplicative generators for the complexification of the real coordinate ring of the adjoint quotient; this system is equivalent to the above one and is obtained in the following fashion: For $r \geq 0$ and $s \geq 0$ such that $1 \leq r + s$, let

$$\tau_{(r,s)}(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n) = \sum_{j=1}^{n} z_j^r \overline{z}_j^s; \tag{6.2}$$

such a function $\tau_{(r,s)}$ is manifestly S_n-invariant, and we refer to a function of this kind as a *bisymmetric power sum function*. As a ring, $\mathbb{R}[T^{\mathbb{C}}/S_n]_{\mathbb{C}}$ is generated by the $\frac{n(n+3)}{2}$ bisymmetric power sum functions $\tau_{(r,s)}$ for $r \geq 0$ and $s \geq 0$ such that $1 \leq r + s \leq n$ as well, subject to $\frac{n(n-1)}{2} + 2$ relations; see [16] for details. Suffice it to mention at this stage that rewriting the relation $\sigma_{(n,0)} = 1$ in terms of the power sums $\tau_{(j,0)}$ of the variables z_1, $\ldots$, z_n and the relation $\sigma_{(0,n)} = 1$ in terms of the power sums $\tau_{(0,j)}$ of the variables $\overline{z}_1$, $\ldots$, $\overline{z}_n$ where $1 \leq j \leq n$, we can express the generator $\tau_{(n,0)}$ as a polynomial in the $\tau_{(j,0)}$'s with $1 \leq j < n$ and, likewise, we can express the generator $\tau_{(0,n)}$ as a polynomial in the $\tau_{(0,j)}$'s with $1 \leq j < n$. This procedure reduces the above system to one with $\frac{n(n+3)}{2} - 2$ generators, subject to $\frac{n(n-1)}{2}$ relations.

By a result in [16], the Poisson brackets among the multiplicative generators

$$\tau_{(j,k)},\ 0 \le j \le n,\ 0 \le k \le n,\ 1 \le j+k \le n, \tag{6.3}$$

are given by formulas of the kind

$$\frac{i}{2}\{\tau_{(j_1,k_1)}, \tau_{(j_2,k_2)}\} = (j_1k_2 - j_2k_1)\tau_{(j_1+j_2,k_1+k_2)} + (\ldots) \tag{6.4}$$

where $(\ldots)$ refers to appropriate correction terms (coming from a suitable Dirac bracket defined in terms of the obvious embedding of T into the maximal torus of $\mathrm{U}(n)$). When $j_1 + j_2 + k_1 + k_2 > n$, the right-hand side of (6.4) is here to be rewritten as a polynomial in terms of the multiplicative generators (6.3). Thus we see that, as a *stratified Kähler space*, the adjoint quotient of $\mathrm{SL}(n,\mathbb{C})$ is considerably more complicated than just as a complex algebraic variety; indeed, as a complex algebraic variety, this quotient is just a copy of $(n-1)$-dimensional affine complex space.

We will now spell out explicitly the stratified Kähler structure for $n = 2$ and $n = 3$. Consider first the case where $n = 2$, so that $K = \mathrm{SU}(2)$ and $K^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$. The standard maximal torus $T \cong S^1$ of $\mathrm{SU}(2)$ consists of the diagonal matrices $\mathrm{diag}(\zeta,\zeta^{-1})$ where $|\zeta| = 1$, and the standard maximal torus $T^{\mathbb{C}} \cong \mathbb{C}^*$ of $\mathrm{SL}(2,\mathbb{C})$ consists of the diagonal matrices $\mathrm{diag}(\zeta,\zeta^{-1})$ where $\zeta \ne 0$. In view of the discussion in Section 5, complex algebraically, the categorical quotient $K^{\mathbb{C}}//K^{\mathbb{C}}$ amounts to the space $T^{\mathbb{C}}/S_2 \cong \mathbb{C}$ of orbits relative to the action of the Weyl group S_2 on $\mathbb{C}^* \cong T^{\mathbb{C}}$, and this orbit space is realized as the target of the holomorphic map

$$\chi\colon \mathbb{C}^* \to \mathbb{C},\ \chi(z) = z + z^{-1}. \tag{6.5}$$

Thus $Z = z + z^{-1}$ may be taken as a holomorphic coordinate on the adjoint quotient of $\mathrm{SL}(2,\mathbb{C})$.

In view of the above remarks, the complexification $\mathbb{R}[T^{\mathbb{C}}/S_2]_{\mathbb{C}}$ of the *real* coordinate ring $\mathbb{R}[T^{\mathbb{C}}/S_2]$ of the adjoint quotient $T^{\mathbb{C}}/S_2$ under discussion is generated by the three elementary bisymmetric functions

$$\sigma_1 = \sigma_{(1,0)} = z + z^{-1},\ \overline{\sigma}_1 = \sigma_{(0,1)} = \overline{z} + \overline{z}^{-1},\ \sigma = \sigma_{(1,1)} = \frac{z}{\overline{z}} + \frac{\overline{z}}{z},$$

subject to the single defining relation

$$(\sigma_1^2 - 4)(\overline{\sigma}_1^2 - 4) = (\sigma_1\overline{\sigma}_1 - 2\sigma)^2.$$

See [16] for details. Hence the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}/S_2]$ of the adjoint quotient $T^{\mathbb{C}}/S_2$ under discussion is generated by the three functions

$$X = x + \frac{x}{r^2},\ Y = y - \frac{y}{r^2},\ \tau = \frac{2-\sigma}{4} = \frac{y^2}{r^2},$$

where $z = x + iy$, $Z = X + iY$, and $x^2 + y^2 = r^2$, subject to the relation

$$Y^2 = (X^2 + Y^2 + 4(\tau - 1))\tau. \tag{6.6}$$

The obvious inequality $\tau \ge 0$ brings the semialgebraic nature of the adjoint quotient to the fore. More details concerning the semialgebraic structure may be found in [16]. Moreover, the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $\mathbb{R}[T^{\mathbb{C}}/S_2]$ is given by

$$\{X,Y\} = X^2 + Y^2 + 4(2\tau - 1),\ \{X,\tau\} = 2(1-\tau)Y,\ \{Y,\tau\} = 2\tau X.$$

The resulting complex algebraic stratified Kähler structure is *singular* at the two points -2 and 2. These are solutions of the discriminant equation (5.3), and the Poisson structure vanishes at these points; furthermore, at these two points, the function τ is *not* an ordinary smooth function of the variables X and Y. Indeed, solving (6.6) for τ, we obtain

$$\tau = \frac{1}{2}\sqrt{Y^2 + \frac{(X^2+Y^2-4)^2}{16}} - \frac{X^2+Y^2-4}{8},$$

whence, at $(X,Y) = (\pm 2, 0)$, τ is not smooth as a function of the variables X and Y. Away from these two points, the Poisson structure is symplectic. We refer to the adjoint quotient under discussion as the *exotic plane with two vertices*.

This exotic plane with two vertices admits the following function theoretic interpretation: The composite

$$\chi \circ \exp\colon \mathfrak{t}^{\mathbb{C}} \cong \mathbb{C} \to \mathbb{C} \cong T^{\mathbb{C}}\big/S_2$$

equals the holomorphic function 2cosh. Consequently, under the composite

$$S^1 \times \mathbb{R} \to \mathbb{C}^* \to \mathbb{C}$$

of the polar map with χ, the family of circles $S^1_t = \{(\eta,t); \eta \in S^1\}$ $(t \in \mathbb{R})$ in $S^1 \times \mathbb{R}$ (each circle of the family being parallel to the zero section) goes to the family of curves

$$\eta \mapsto \mathrm{e}^t\eta + \mathrm{e}^{-t}\eta^{-1}, \quad \eta \in S^1,$$

which are ellipses for $t \neq 0$; and the family of lines $L_\eta = \{(\eta,t); t \in \mathbb{R}\}$ $(\eta \in S^1)$ in the tangent directions goes to the family of curves

$$t \mapsto \mathrm{e}^t\eta + \mathrm{e}^{-t}\eta^{-1}, \quad t \in \mathbb{R},$$

which are hyperbolas for $\eta \neq \pm 1$. The image of the circle S^1_0 is a double line segment between -2 and 2, the image of the line L_1 is a double ray, emanating from 2 into the positive real direction, and the image of the line L_{-1} is a double ray, emanating from -2 into the negative real direction. The two families are orthogonal and have the two singular points -2 and 2 as focal points. This is of course well known and entirely classical. In the cotangent bundle picture, the double line segment between -2 and 2 is the adjoint quotient of the base $K = \mathrm{SU}(2)$, indeed this line segment is exactly the fundamental alcove of $\mathrm{SU}(2)$, the family of hyperbolas which meet this line segment between -2 and 2 constitutes a cotangent bundle on this orbit space with the two singular points removed, and the plane, i.e. adjoint quotient of $K^{\mathbb{C}}$, with the two rays emanating from -2 and 2 removed, is the total space of this cotangent bundle; furthermore, the cotangent bundle symplectic structure is precisely that which corresponds to the reduced Poisson structure. However the cotangent bundle structure does not extend to the entire adjoint quotient of $\mathrm{T}^*K \cong K^{\mathbb{C}}$. In particular, this interpretation visualizes the familiar fact that, unless there is a single stratum, the strata arising from cotangent bundle reduction are *not* cotangent bundles on strata of the orbit space of the base space.

Consider now the case where $K = \mathrm{SU}(3)$, with maximal torus T diffeomorphic to the product $S^1 \times S^1$ of two copies of the circle group. Complex algebraically, the map (5.2) for $n = 3$ comes down to the map

$$(-\sigma_1, \sigma_2)\colon T^{\mathbb{C}} \to \mathbb{C}^2, \tag{6.7}$$

and this map induces a complex algebraic isomorphism from the adjoint quotient $T^{\mathbb{C}}\big/S_3$ of $\mathrm{SL}(3,\mathbb{C})$ onto a copy $\mathbb{C}^2$ of 2-dimensional complex affine space; as in Section 5 above, we will take $a_1 = -\sigma_1$ and $a_2 = \sigma_2$ as complex coordinates on the adjoint quotient. The complement of the top stratum $\mathcal{D}_3^{(o,\mathrm{SU}(3))}$ is now the complex affine curve $\mathcal{D}_2^{\mathrm{SU}(3)} = \mathcal{D}_{(2,1)}^{\mathrm{SU}(3)}$ in $\mathbb{A}^{3,1}_{\mathrm{coef}}$ given by the equation (5.4). This curve is plainly parametrized by the restriction

$$\mathbb{C}^* \to \mathbb{C}\times\mathbb{C}, \quad z \mapsto (2z + z^{-2}, z^2 + 2z^{-1}) \tag{6.8}$$

of (6.7) to the diagonal, and this holomorphic curve parametrizes the *closure* $\mathcal{D}_2^{\mathrm{SU}(3)}$ of the stratum $\mathcal{D}_2^{(o,\mathrm{SU}(3))}$. This curve has the three (complex) singularities $(3,3)$, $3(\eta,\eta^2)$, $3(\eta^2,\eta)$, where $\eta^3 = 1, \eta \neq 1$. These points are the images of the (conjugacy classes) of the three central elements under (6.7); as complex curve singularities, these singularities are cuspidal. These three points constitute the stratum $\mathcal{D}_3^{\mathrm{SU}(3)}$.

The real hypersurface written above, for general n, as $\mathcal{H}_n$, cf. (5.12), now comes down to the curve $\mathcal{H}_2$ in $\mathbb{R}^2$; here $\mathbb{R}^2$ is embedded into $\mathbb{C}^2$ as the real affine space of real points of $\mathbb{C}^2$ in the obvious fashion. Since, for a complex number z with $|z| = 1$, $2\overline{z} + \overline{z}^{-2} = z^2 + 2z^{-1}$, the restriction of (6.8) to the *real* torus $T \subseteq T^{\mathbb{C}}$ yields the parametrized real curve

$$S^1 \to \mathbb{R}^2, \quad \mathrm{e}^{i\alpha} \mapsto (u(\alpha), v(\alpha)) \in \mathbb{R}^2, \tag{6.9}$$

where $u(\alpha) + iv(\alpha) = 2\mathrm{e}^{i\alpha} + \mathrm{e}^{-2i\alpha}$, and this parametrizes precisely the curve $\mathcal{H}_2$. This curve is a *hypocycloid*, as noted in [2] (Section 5), and the real orbit space $\mathrm{SU}(3)/\mathrm{SU}(3)$ relative to conjugation, realized within the adjoint quotient $T^{\mathbb{C}}\big/S_3 \cong \mathcal{D}_1^{\mathrm{SU}(3)}$ of $\mathrm{SL}(3,\mathbb{C})$, amounts to the compact region in $\mathbb{R}^2$ enclosed by this hypocycloid.

As a complex algebraic stratified Kähler space, the adjoint quotient looks considerably more complicated. Indeed, cf. [16], the complexification $\mathbb{R}[T^{\mathbb{C}}\big/S_3]_{\mathbb{C}}$ of the real coordinate ring $\mathbb{R}[T^{\mathbb{C}}\big/S_3] \cong \mathbb{R}[T^{\mathbb{C}}]^{S_3}$ of the adjoint quotient $T^{\mathbb{C}}\big/S_3$, viewed as a real semialgebraic space, is generated by the seven functions $\sigma_1 = \sigma_{(1,0)}$, $\overline{\sigma}_1 = \sigma_{(0,1)}$, $\sigma_2 = \sigma_{(2,0)}$, $\overline{\sigma}_2 = \sigma_{(0,2)}$, $\sigma = \sigma_{(1,1)}$, $\rho = \sigma_{(2,1)}$, $\overline{\rho} = \sigma_{(1,2)}$, subject to the following three relations:

$$\begin{aligned}
(\sigma_1^2 - 4\sigma_2)(\overline{\sigma}_1^2 - 4\overline{\sigma}_2) &= (\sigma_1\overline{\sigma}_1 - 2\sigma)^2 + 2\rho\overline{\sigma}_1 + 2\overline{\rho}\sigma_1,\\
D_3(1,-\sigma_1,\sigma_2,-1)\overline{\sigma}_2 &= (9 + \sigma_2^3 - 4\sigma_1\sigma_2)\overline{\sigma}_1^2 + (4\sigma_1^2 - 3\sigma_2 - \sigma_1\sigma_2^2)\overline{\sigma}_1\sigma\\
&\quad + (6\sigma_1 - \sigma_2^2)\overline{\sigma}_1\rho + (\sigma_2^2 - 3\sigma_1)\sigma^2\\
&\quad + (9 - \sigma_1\sigma_2)\sigma\rho + (\sigma_1^2 - 3\sigma_2)\rho^2,\\
D_3(1,-\sigma_1,\sigma_2,-1) &= \overline{\sigma}_1^3 - \sigma_2\overline{\sigma}_1^2\sigma + (\sigma_2^2 - 2\sigma_1)\overline{\sigma}_1^2\rho + \sigma_1\overline{\sigma}_1\sigma^2\\
&\quad - ((\sigma_1^2 - 2\sigma_2)\sigma_1 - \sigma_1^3 + 3\sigma_1\sigma_2 - 3)\overline{\sigma}_1\sigma\rho\\
&\quad - \sigma^3 + (\sigma_1^2 - 2\sigma_2)\overline{\sigma}_1\rho^2 + \sigma_2\sigma^2\rho - \sigma_1\sigma\rho^2 + \rho^3.
\end{aligned}$$

We note that an explicit expression for $D_3(1,-\sigma_1,\sigma_2,-1)$ is given by (4.14). The formula (6.4) yields the stratified symplectic Poisson structure on the complexification of the real coordinate ring of $T^{\mathbb{C}}\big/S_3$ in terms of the nine generators

$$\tau_{(1,0)},\ \tau_{(2,0)},\ \tau_{(3,0)},\ \tau_{(0,1)},\ \tau_{(0,2)},\ \tau_{(0,3)},\ \tau_{(1,1)},\ \tau_{(1,2)},\ \tau_{(2,1)}.$$

This Poisson structure has rank 4 on the top stratum $\mathcal{D}_1^{(o,\mathrm{SU}(3))}$, rank 2 on the stratum $\mathcal{D}_2^{(o,\mathrm{SU}(3))}$, and rank zero at the three points of the stratum $\mathcal{D}_1^{\mathrm{SU}(3)}$, that is, at the three cusps of the complex affine curve $\mathcal{D}_2^{\mathrm{SU}(3)} = \mathcal{D}_{(2,1)}^{\mathrm{SU}(3)}$ in $\mathbb{A}^{3,1}_{\mathrm{coef}}$ given by the equation (5.4). The requisite inequalities which encapsulate the semialgebraic structure may be found in [16].

For $n \geq 4$, the map (5.2) identifies the adjoint quotient $\mathcal{D}_1^{\mathrm{SU}(n)}$ of $\mathrm{SL}(n,\mathbb{C})$ with complex affine $(n-1)$-space $\mathbb{C}^{n-1}$, and the resulting stratified Kähler structure on the adjoint quotient can be described in a way similar to that for the low dimensional cases where $n = 2$ and $n = 3$; the details get more and more involved, though. A function theoretic interpretation extending the interpretation spelled out above for the case where $n = 2$ is available as well. Indeed, let $n \geq 2$, let $T \cong (S^1)^{(n-1)}$ be the standard maximal torus in $\mathrm{SU}(n)$, let $T^{\mathbb{C}} \cong (\mathbb{C}^*)^{n-1}$ be its complexification, and consider the composite

$$T \times \mathfrak{t} \cong T \times \mathbb{R}^{n-1} \to T^{\mathbb{C}} \to \mathbb{C}^{n-1} \tag{6.10}$$

of the polar map with χ. Now the images in the top stratum of the adjoint quotient $\mathcal{D}_1^{\mathrm{SU}(n)} \cong \mathbb{C}^{n-1}$ of the leaves of the horizontal foliation of the total space $T \times \mathbb{R}^{n-1} \cong \mathrm{T}T$ of the tangent bundle of T are smooth submanifolds and generalize the family of ellipses for the case where $n = 2$ and, likewise, the images in the top stratum of the adjoint quotient $\mathbb{C}^{n-1}$ of the leaves of the vertical foliation of the total space $T \times \mathbb{R}^{n-1} \cong \mathrm{T}T$ of the tangent bundle of T are smooth manifolds and generalize the family of hyperbolas for the case where $n = 2$; however, images of such leaves which meet a lower stratum involve singularities, and the geometry of the situation can be described in terms of complex focal points, complex focal lines, etc.: The focal points are the n points of the bottom stratum $\mathcal{D}_n^{\mathrm{SU}(n)}$, a focal curve is given by the next stratum $\mathcal{D}_{n-1}^{\mathrm{SU}(n)}$, etc. The composite of (6.10) with the obvious map from $\mathbb{C}^{n-1}$ to $T \times \mathbb{R}^{n-1}$ generalizes the holomorphic hyperbolic cosine function for the case where $n = 2$, and the function theoretic interpretation extending the interpretation for the case where $n = 2$ is thus immediate.

The above observations involving the singular cotangent bundle projection map for the special case where $n = 2$ extend as follows: The induced projection from the adjoint quotient $\mathcal{D}_1^{\mathrm{SU}(n)} \cong \mathbb{C}^{n-1}$ to the real orbit space $\mathrm{SU}(n)/\mathrm{SU}(n)$ relative to conjugation is a singular cotangent bundle projection map in an obvious manner. Indeed, the map (5.2) induces a map from the *real* torus T to the real affine space $\mathbb{R}^{n-1}$ of real points of the complex adjoint quotient C^{n-1}; just as for the cases where $n = 2$ or $n = 3$, the restriction of this map to the subspace T_1 of the real torus T having non-trivial stabilizer, i.e. points that are non-regular as points of $K = \mathrm{SU}(n)$, has as its image a real closed hypersurface in $\mathbb{R}^{n-1}$, and the real orbit space $\mathrm{SU}(n)/\mathrm{SU}(n)$ relative to conjugation is realized in $\mathbb{R}^{n-1}$ as the semialgebraic space enclosed by this hypersurface. The resulting semialgebraic space is manifestly homeomorphic to an $(n-1)$-simplex or, equivalently, to the standard $(n-1)$-alcove for $\mathrm{SU}(n)$.

Finally we note that this kind of interpretation is available for the closure $\mathcal{D}_\nu^{\mathrm{SU}(n)}$ of each stratum of the adjoint quotient, each such space being a complex algebraic stratified Kähler space; moreover, for $\nu' \preceq \nu$, the injection map from $\mathcal{D}_{\nu'}^{\mathrm{SU}(n)}$ into $\mathcal{D}_\nu^{\mathrm{SU}(n)}$ is compatible with all the structure. For example, when $n = 3$, $\mathcal{D}_2^{\mathrm{SU}(3)}$ is the complex affine curve in the adjoint quotient $\mathcal{D}_1^{\mathrm{SU}(3)} \cong \mathbb{C}^2$ parametrized by (6.8). This parametrization plainly factors through $\mathbb{C}^*/S_2$ and hence induces an embedding

$$\mathcal{D}_1^{\mathrm{SU}(2)} \to \mathcal{D}_1^{\mathrm{SU}(3)}$$

of the adjoint quotient $\mathcal{D}_1^{\mathrm{SU}(2)}$ of $\mathrm{SL}(2,\mathbb{C})$ into the adjoint quotient $\mathcal{D}_1^{\mathrm{SU}(3)}$ of $\mathrm{SL}(3,\mathbb{C})$ which identifies $\mathcal{D}_1^{\mathrm{SU}(2)}$ with $\mathcal{D}_2^{\mathrm{SU}(3)}$ as complex algebraic stratified Kähler spaces. Hence the embedding is one of complex algebraic stratified Kähler spaces, that is, it is compatible with all the structure. In particular, the "focal" geometries correspond.

7. Relationship with the spherical pendulum. In this section we will identify the adjoint quotient of $\mathrm{SL}(2,\mathbb{C})$ with the reduced phase space of a spherical pendulum constrained to move with angular momentum zero about the third axis, so that it amounts to a *planar* pendulum.

The unreduced phase space of the spherical pendulum is the total space T^*S^2 of the cotangent bundle of the ordinary 2-sphere S^2 in 3-space $\mathbb{R}^3$ centered at the origin. By means of the standard inner product, we identify the tangent bundle of $\mathbb{R}^3$ with its cotangent bundle; accordingly, we identify T^*S^2 with the total space $\mathrm{T}S^2$ of the tangent bundle of S^2, and we realize the space $\mathrm{T}S^2$ within $\mathbb{R}^3 \times \mathbb{R}^3 \cong \mathrm{T}\mathbb{R}^3$ in the standard fashion. The map

$$\mathbb{C}^* \to \mathbb{R}^3 \times \mathbb{R}^3,\ \mathrm{e}^t\mathrm{e}^{i\varphi} \mapsto (0, \sin\varphi, \cos\varphi, 0, -t\cos\varphi, t\sin\varphi), \tag{7.1}$$

plainly goes into the subspace $\mathrm{T}S^2$ of $\mathbb{R}^3 \times \mathbb{R}^3$ and induces a homeomorphism from the S_2-orbit space $\mathbb{C}^* \big/ S_2$ onto the angular momentum zero reduced phase space $V_0 = \mu^{-1}(0) \big/ S^1$ of the spherical pendulum. Here the non-trivial element of the cyclic group S_2 with two elements acts on $\mathbb{C}^*$ by inversion, and

$$\mu\colon \mathrm{T}S^2 \to \mathrm{Lie}(S^1)^* \cong \mathbb{R}$$

refers to the momentum mapping for the S^1-action on $\mathrm{T}S^2$ given by rotation about the third axis; actually this is just ordinary angular momentum given by

$$\mu(q_1, q_2, q_3, p_1, p_2, p_3) = q_1 p_2 - q_2 p_1$$

where $(q_1, q_2, q_3, p_1, p_2, p_3)$ are the obvious coordinates on $\mathbb{R}^3 \times \mathbb{R}^3$. Under the identification of $\mathbb{C}^*$ with the complexification $T^{\mathbb{C}}$ of the maximal torus T of $\mathrm{SU}(2)$, the quotient $\mathbb{C}^* \big/ S_2$ gets identified with the adjoint quotient $T^{\mathbb{C}} \big/ S_2$ of $\mathrm{SL}(2,\mathbb{C})$ whence the map (7.1) induces a homeomorphism from the adjoint quotient $T^{\mathbb{C}} \big/ S_2$ of $\mathrm{SL}(2,\mathbb{C})$ onto the reduced phase space V_0 for the spherical pendulum. Moreover, a little thought reveals that the map

$$\Theta\colon \mathbb{C}^* \to \mathbb{R}^3,\ \Theta(\mathrm{e}^t\mathrm{e}^{i\varphi}) = (\cos\varphi, t\sin\varphi, t^2)$$

into $\mathbb{R}^3$ induces a homeomorphism from $\mathbb{C}^* \big/ S_2$ onto the real semialgebraic subspace M_0 of $\mathbb{R}^3$ which, in terms of the coordinates (u, v, w), is given by

$$w(1 - u^2) - v^2 = 0,\ |u| \le 1,\ w - v^2 \ge 0.$$

The space M_0 has received considerable attention in the literature: This space is the "canoe", cf. [1], [3] (pp. 148 ff.) and [21], the two singular points being absolute equilibria for the spherical pendulum. Thus the map Θ identifies the adjoint quotient $T^{\mathbb{C}} \big/ S_2$ of $\mathrm{SL}(2,\mathbb{C})$ and hence the reduced phase space V_0 for the spherical pendulum with the "canoe" M_0.

We have already seen that the holomorphic map χ from $\mathbb{C}^*$ to $\mathbb{C}$ given by $\chi(z) = z + z^{-1}$ induces a complex algebraic isomorphism from the adjoint quotient $T^{\mathbb{C}}\big/S_2$ onto a copy of the complex line $\mathbb{C}$. It is instructive to realize χ on the homeomorphic image V_0 or, equivalently, on the "canoe" M_0. We can do even better: Let α be the real analytic function of the real variable t given by the power series $\alpha(t) = \sum_{j=0}^{\infty} \frac{t^j}{(2j)!}$ and, likewise, let β be the real analytic function of the real variable t given by the power series $\beta(t) = \sum_{j=0}^{\infty} \frac{t^j}{(2j+1)!}$. We note that, by construction,

$$\alpha(t^2) = \cosh(t),\ t\beta(t^2) = \sinh(t).$$

Introduce the real analytic function

$$\Phi\colon \mathbb{R}^3 \to \mathbb{C},\ \Phi(u, v, w) = u\alpha(w) + iv\beta(w), (u, v, w) \in \mathbb{R}^3.$$

Since the composite map $\Phi \circ \Theta \circ \exp$ from $\mathbb{C}$ to $\mathbb{C}$ coincides with 2cosh, the composite $\Phi \circ \Theta$ coincides with χ. In particular, the homeomorphism from the "canoe" M_0 onto the complex line given by the restriction of Φ *flattens out* the "canoe". It is interesting to note that this flattening out is accomplished in the category of real semianalytic spaces, not in that of real semialgebraic spaces, even though the "canoe" and the complex line are real semialgebraic spaces; presumably the two cannot be identified in the category of real semialgebraic spaces.

Since the composite $\Phi \circ \Theta \circ \exp$ equals twice the holomorphic hyperbolic cosine function, the present discussion provides a geometric interpretation of the adjoint quotient of $\mathrm{SU}(2)^{\mathbb{C}} \cong \mathrm{SL}(2, \mathbb{C})$ similar to the function theoretic one given earlier, but in terms of the "canoe". In particular, the two focal points in the earlier interpretation now correspond to the two singular points of the "canoe" which, in turn, correspond to the absolute equilibria of the spherical pendulum.

While the spherical pendulum has been extensively studied, the present complex analytic interpretation seems to be new.

Acknowledgments. This paper was written during a stay at the Institute for Theoretical Physics at the University of Leipzig. This stay was made possible by the Deutsche Forschungsgemeinschaft in the framework of a Mercator-professorship, and I wish to express my gratitude to the Deutsche Forschungsgemeinschaft. It is a pleasure to acknowledge the stimulus of conversations with G. Rudolph and M. Schmidt at Leipzig.

References

[1] J. M. Arms, R. Cushman, and M. J. Gotay, *A universal reduction procedure for Hamiltonian group actions*, in: The Geometry of Hamiltonian Systems, T. Ratiu (ed.), MSRI Publ. 20, Springer Verlag, Berlin, 1991, 33–51.

[2] S. Charzyński, J. Kijowski, G. Rudolph, and M. Schmidt, *On the stratified classical configuration space of lattice QCD*, J. Geom. and Physics 55 (2005), 137–178.

[3] R. H. Cushman and L. M. Bates, *Global Aspects of Classical Integrable Systems*, Birkhäuser Verlag, Boston, 1997.

[4] R. C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, N.J., 1965.

[5] V. W. Guillemin and M. Stenzel, *Grauert tubes and the homogeneous Monge-Ampère equations. I*, J. Diff. Geom. 34 (1992), 561–570.

[6] B. C. Hall, *Geometric quantization and the generalized Segal-Bargmann transform for Lie groups of compact type*, Comm. in Math. Phys. 226 (2002), 233–268.

[7] J. Huebschmann, *Poisson cohomology and quantization*, J. Reine Angew. Math. 408 (1990), 57–113.

[8] J. Huebschmann: *On the quantization of Poisson algebras*, in: Symplectic Geometry and Mathematical Physics, P. Donato *et al.* (eds.), Progress in Mathematics 99, Birkhäuser Verlag, Boston, 1991, 204–233.

[9] J. Huebschmann, *Poisson geometry of flat connections for* SU(2)*-bundles on surfaces*, Math. Z. 221 (1996), 243–259; hep-th/9312113.

[10] J. Huebschmann: *Symplectic and Poisson structures of certain moduli spaces*, Duke Math. J. 80 (1995), 737–756; hep-th/9312112.

[11] J. Huebschmann, *Singularities and Poisson geometry of certain representation spaces*, in: Quantization of Singular Symplectic Quotients (Oberwolfach, 1999), N. P. Landsman *et al.* (eds.), Progress in Mathematics 198, Birkhäuser Verlag, Boston, 2001, 119–135; math.dg/0012184.

[12] J. Huebschmann, *Kähler spaces, nilpotent orbits, and singular reduction*, Memoirs of the AMS 172 (814) (2004); math.DG/0104213.

[13] J. Huebschmann, *Kähler reduction and quantization*, J. Reine Angew. Math. 591 (2006), 75–109; math.SG/0207166.

[14] J. Huebschmann, *Singular Poisson-Kähler geometry of Scorza varieties and their ambient spaces*, Diff. Geometry and its Applications 23 (2005), 79–93; math.DG/0405165.

[15] J. Huebschmann, *Lie-Rinehart algebras, descent, and quantization*, Fields Institute Communications 43 (2004), 295–316; math.SG/0303016.

[16] J. Huebschmann, *Stratified Kähler structures on adjoint quotients,* math.DG/0404141.

[17] J. Huebschmann, *Classical phase space singularities and quantization*, in: Quantum Theory and Symmetries IV, V. K. Dobrev (ed.), Heron Press, Sofia, 2006, 51–65; math-ph/0610047.

[18] J. E. Humphreys, *Conjugacy Classes in Semisimple Algebraic Groups*, Mathematical Surveys and Monographs 43, Amer. Math. Society, Providence, R.I., 1995.

[19] G. Katz, *How tangents solve algebraic equations, or a remarkable geometry of discriminant varieties*, Expo. Math. 21 (2003), 219–261.

[20] L. Lempert and R. Szöke, *Global solutions of the homogeneous complex Monge-Ampère equations and complex structures on the tangent bundle of Riemannian manifolds*, Math. Ann. 290 (1991), 689–712.

[21] E. Lerman, R. Montgomery, and R. Sjamaar, *Examples of singular reduction*, in: Symplectic Geometry (Warwick, 1990), D. A. Salamon (ed.), London Math. Soc. Lecture Note Series 192, Cambridge University Press, Cambridge, 1993, 127–155.

[22] D. Luna, *Sur certaines opérations différentiables des groupes de Lie*, Amer. J. of Math. 97 (1975), 172–181.

[23] D. Luna, *Fonctions différentiables invariantes sous l'opération d'un groupe de Lie réductif*, Ann. Inst. Fourier 26 (1976), 33–49.

[24] M. Oka, *On the stratification of discriminant varieties*, Kodaira Math. J. 12 (1989), 210–227.

[25] G. W. Schwarz, *The topology of algebraic quotients*, in: Topological Methods in Algebraic Transformation Groups, Progress in Mathematics 80, Birkhäuser Verlag, Boston, 1989, 135–152.

[26] P. Slodowy, *Simple Singularities and Simple Algebraic Groups*, Lecture Notes in Mathematics 815, Springer Verlag, Berlin, 1980.

[27] R. Szöke, *Complex structures on tangent bundles of Riemannian manifolds*, Math. Ann. 291 (1991), 409–428.

[28] B. L. van der Waerden, *Algebra*, *Erster Teil*, Springer Verlag, Berlin, 1966.