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A FEW REMARKS ON THE GEOMETRY OF THE SPACE OF LEAF CLOSURES OF A RIEMANNIAN FOLIATION

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Abstract. The space of the closures of leaves of a Riemannian foliation is a nice topological space, a stratified singular space which can be topologically embedded in $\mathbb{R}^k$ for k sufficiently large. In the case of Orbit Like Foliations (OLF) the smooth structure induced by the embedding and the smooth structure defined by basic functions is the same. We study geometric structures adapted to the foliation and present conditions which assure that the given structure descends to the leaf closure space. In Section 5 we introduce the notion of an Ehresmann connection on a stratified foliated space and study the properties of the strata which depend on the existence of such a connection. We also give conditions which ensure that a connection understood as a differential operator defines an Ehresmann connection as above. In the last section we present some curvature estimates for metric structures on the leaf closure space.

In recent years physicists and mathematicians working on mathematical models of physical phenomena have realized that modelling based on geometric structures on now classical smooth manifolds is insufficient, more complicated topological spaces appear naturally. One of the well-known examples is the orbit space of a smooth action of a compact Lie group. Such a space is a stratified pseudomanifold of Goresky-MacPherson, cf. [14, 25, 9, 32]. This fact has been used to describe the topology and structure of the reduced space of the momentum map in the singular case, cf. [35].

The study of the Riemannian geometry of the orbit space of a smooth action of a compact Lie group has been initiated in [1]. One should also mention K. Richardson's paper, cf. [34], in which the author demonstrates that any space of orbits of such an action is homeomorphic to the space of the closures of leaves of a regular Riemannian foliation, and, obviously, cf. [28], vice versa. Therefore this foliated Riemannian manifold can be

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regarded as a desingularization of the orbit space. Likewise for stratified pseudomanifolds which are homeomorphic to the leaf closure space of an SRF, the foliated Riemannian manifold can be seen as a desingularisation of this pseudomanifold. Therefore we think that it is important to study relations between the geometry of a foliated Riemannian manifold and the geometry of its leaf closure space.

In this paper we initiate the study of the geometry of the space of leaf closures $M/\bar{\mathcal{F}}$ of a Riemannian foliation, either regular or singular. The space $M/\bar{\mathcal{F}}$ is naturally stratified. In Section 2 we construct a natural embedding of $M/\bar{\mathcal{F}}$ into R^k for some k sufficiently large. Then we show that the smooth structure defined by this embedding is the same as the smooth structure defined by basic functions. It means that, in the regular case, when the space $M/\bar{\mathcal{F}}$ is algebraic, we have at our disposal the stratified Morse theory of Goresky-Macpherson, cf. [15]. The singular case is more complicated and requires more serious studies. In Section 3 we study transversally hermitian and Kähler foliations. We investigate the conditions under which the transverse symplectic and complex structures descend to the leaf closure space. In the case of singular foliations, first we define foliated complex and Kähler structures. and then study the projected geometric structures on the leaf closure space. We prove a theorem asserting that under some natural conditions the leaf closure space is a Kähler singular space, cf. [17]. Section 5 presents a new definition of an Ehresmann connection on a stratified manifold. The last section presents some curvature estimations and corollaries of the Lovrić, Min-Oo, Ruh theorem for the Ricci flow of bundle-like metrics.

One of the reasons for the study of such foliations is the fact that recently there have been a renewed interest in non-integrable geometries associated to Riemannian structures—in fact these geometries correspond to the choice of an additional geometric structure compatible with the Riemannian metric, cf. [13]. Almost Hermitian structures are one of the best known examples of such structures. Foliated (almost) Hermitian manifolds or foliated Kähler manifolds are of interest as they combine three foliated structures: a Riemannian, an almost complex and a symplectic ones. These structures appear quite naturally in geometry as Sasakian manifolds form a special class manifolds foliated by transversally Kähler 1-dimensional Riemannian foliations, cf. [46] and $\mathcal{K}$-manifolds give another example of such foliated manifolds, cf. [10].

1. Riemannian foliations. In this section we recall some basic facts about Riemannian foliations. The notion of a singular Riemannian foliation was introduced by Pierre Molino in [27], see also [28, 29].

A Riemannian foliation $\mathcal{F}$ on a Riemannian manifold (M, g) is a foliation for which any geodesic of (M, g) orthogonal to $\mathcal{F}$ at one point remains orthogonal to $\mathcal{F}$ at any point of its domain. In the regular case it is equivalent to the fact that g is bundle-like.

Let $\mathcal{F}$ be a regular foliation on a manifold M. The foliation $\mathcal{F}$ is given by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}$ modelled on a manifold N_0, i.e.

i) $\{U_i\}$ is an open covering of M,
ii) $f_i\colon U_i \longrightarrow N_0$ are submersions with connected fibres defining $\mathcal{F}$,
iii) g_{ij} are local diffeomorphisms of N_0 and $g_{ij} \circ f_j = f_i$ on $U_i \cap U_j$.

The manifold $N = \coprod f_i(U_i)$ we call the transverse manifold of $\mathcal{F}$ associated to the cocycle $\mathcal{U}$ and the pseudogroup H generated by g_{ij} the holonomy pseudogroup (representative) on the transverse manifold N.

The foliated geometric structures, i.e., those which in local coordinates can be expressed in the transverse coordinates only, correspond bijectively to holonomy invariant ones on the transverse manifold. In particular, the foliation is Riemannian if the transverse manifold is a Riemannian manifold and the holonomy pseudogroup consists of local isometries, likewise the transverse manifold of a transversally Kähler foliation is a Kähler manifold whose Kähler structure is H–invariant.

1.1. *The structure of the leaf closure space.* First let us look at the regular case.

Let $\mathcal{F}$ be a regular Riemannian foliation of codimension q on a compact manifold M of dimension n. In fact, it is sufficient to assume that there is a complete bundle-like metric g for $\mathcal{F}$ or even less that the geodesics of the bundle-like metric g orthogonal to the leaves of $\mathcal{F}$ are globally defined, i.e., the metric g is transversally complete. The bundle of transverse orthonormal frames $B(M, O(q); \mathcal{F})$ of the foliated Riemannian manifold $(M, g; \mathcal{F})$ admits a foliation $\mathcal{F}_1$ whose leaves are covering spaces of leaves of $\mathcal{F}$ and which is transversally parallelisable and invariant for the natural action of the group $O(q)$. The closures of leaves of $\mathcal{F}_1$ define a global submersion π onto a manifold W whose fibres are just these closures. Since $\mathcal{F}_1$ is invariant for the natural action of the group $O(q)$, so is the foliation by the closures of leaves. The manifold W inherits an action of $O(q)$ making the submersion π $O(q)$-equivariant. As the fibre of $B(M, O(q); \mathcal{F})$ is compact the closures of leaves of $\mathcal{F}_1$ project onto the closures of leaves of $\mathcal{F}$. To be precise, let L be a leaf of $\mathcal{F}$ and L_1 be a leaf of $\mathcal{F}_1$ which covers L. Then $\bar{L} = p(\bar{L}_1)$ where p is the natural projection $p\colon B \to M$. Therefore the closures are submanifolds of M and minimal subsets of $(M, \mathcal{F})$, In fact, the closure $\bar{L}$ corresponds to the $O(q)$ orbit of the point $\pi(\bar{L}_1) \in W$, cf. [28]. The above correspondence defines a homeomorphism between the leaf closure space $M/\overline{\mathcal{F}}$ and the orbit space $W/O(q)$.

It is well-known that the closures of leaves of a Riemannian foliations are the orbits of the commuting sheaf of this foliation, cf. [28], which is defined using the bundle of transverse orthonormal frames. The local vector fields of the commuting sheaf are local Killing vector fields of the induced Riemannian metric. For transversally Hermitian or Kähler foliations we can refine the definition, cf. [41, 40]. The compatible foliated Riemannian and symplectic structures define a foliated $U(q)$-reduction $B(M, U(q); \mathcal{F})$ of the bundle $L(M, \mathcal{F})$ of transverse frames, i.e. the frames of the normal bundle $N(M, \mathcal{F})$. The group $U(q)$ is of type 1, so the foliation $\mathcal{F}_1$ of the total space of $B(M, U(q); \mathcal{F})$ is transversally parallelisable (TP) and the closures of leaves form a regular foliation with compact leaves. The projections of these leaves onto M are the closures of leaves of $\mathcal{F}$. From the general theory of TP foliations, cf. [28], we know that these closures are the orbits, in the foliated sense, of local vector fields commuting with global foliated vector fields, in particular with vector fields of the transverse parallelism. These vector fields are the lifts to $B(M, U(q); \mathcal{F})$ of local foliated vector fields on $(M, \mathcal{F})$, which are infinitesimal automorphisms of the transverse $U(q)$-structure, so they preserve both the transverse Riemannian metric, almost complex structure and the associated 2-form.

1.2. *Stratification.* The manifold M with an SRF $\mathcal{F}$ is stratified by the dimension of leaves of $\mathcal{F}$, i.e., let for any $x \in M$ denote by L_x the leaf of $\mathcal{F}$ passing through x. Then for $r = 0, \dots, n = \dim M$, let $M_r = \{x \in M : \dim L_x = r\}$. Obviously, there exist $r_{\min}$ and $r_{\max}$ such that $M_r = \emptyset$ for $r < i_{\min}$ or $r > r_{\max}$; $r_{\min}$ is the smallest dimension of leaves of the foliation $\mathcal{F}$ and $r_{\max}$ is the greatest dimension of leaves of this foliation. The set $M^* = M_{r_{\max}}$ is open and dense in M. The set $\Sigma = M - M^*$ is a closed subset of M of measure 0.

If the manifold M is foliated by an RRF $\mathcal{F}$ then $\bar{\mathcal{F}}$ is an SRF and M admits the following stratification, cf. [28].

Let k be any number between 0 and n. Define

$$\Sigma_k = \{x \in M : x \in L \in \mathcal{F}, dim\overline{L} = k\}.$$

The leaf closures of $\mathcal{F}$ define a singular Riemannian foliation which in any Σ_k defines a regular Riemannian foliation. P. Molino demonstrated that connected components of these subsets are submanifolds of M and that $\overline{\Sigma}_k \subset \bigcup_{i \leq k} \Sigma_i$. For some i the sets Σ_i may be empty. Let k_0 be the greatest dimension of leaves of $\bar{\mathcal{F}}$. Then the set Σ_{k_0} is open and dense in M. It is called the principal stratum.

The holonomy group of each $\overline{L}$ is finite, and there are a finite number of types of the holonomy groups. Therefore each set Σ_i decomposes into a finite number of submanifolds

$$\Sigma_{p\alpha} = \{x \in L \in \mathcal{F} \colon dim\overline{L} = p,\ h(\overline{L}, x) \in \alpha\}$$

where $h(\overline{L}, x)$ is the holonomy group of $\overline{L}$ in Σ_p and α is the conjugacy class of finite subgroups of $O(q_p)$ where q_p is the codimension of $\overline{\mathcal{F}}$ in Σ_p. In this way, we have obtained a stratification $\mathcal{S} = \{\Sigma_\gamma\}$ of $(M, \mathcal{F})$ into submanifolds on which $\overline{\mathcal{F}}$ define regular Riemannian foliations and the foliation $\overline{\mathcal{F}}$ is without holonomy. On each stratum S_γ of $\mathcal{S}$, the foliation $\overline{\mathcal{F}}$ defines a compact RF without holonomy, so the space of leaves $S_\gamma/\overline{\mathcal{F}}$ is a smooth manifold denoted by $\bar{S_\gamma}$ and the natural projection $p_\gamma \colon S_\gamma \to \bar{S_\gamma}$ is a locally trivial fibre bundle. The manifolds $\bar{S_\gamma}$ define a stratification of the leaf closure space $M/\bar{\mathcal{F}}$ which we denote $\overline{\mathcal{S}}$.

Let us return to the singular case. We have the stratification $\mathcal{S} = \{\Sigma_i\}$ where

$$\Sigma_k = \{x \in M : x \in L \in \mathcal{F}, dimL = k\}.$$

From the very definition the foliation, $\mathcal{F}$ defines an RRF in each stratum. Therefore this stratification can be refined in the following way:

Let k be any number between 0 and n. Define

$$\Sigma_{kl} = \{x \in \Sigma_k : x \in L \in \mathcal{F}, dim\overline{L} = l\}.$$

In each stratum Σ_{kl} the foliations $\mathcal{F}$ and $\bar{\mathcal{F}}$ are regular. Therefore the holonomy group of each $\overline{L}$ in the corresponding stratum Σ_{kl} is finite, and there is a finite number of types of the holonomy groups. Hence each set Σ_{kl} decomposes itself into a finite number of submanifolds

$$\Sigma_{kl\alpha} = \{x \in L \in \mathcal{F} \colon dimL = k,\ \ \overline{L} = l,\ h(\overline{L}, x) \in \alpha\}$$

where $h(\overline{L}, x)$ is the holonomy group of $\overline{L}$ in Σ_p, α is the conjugacy class of finite subgroups of $O(q_p)$, and q_p is the codimension of $\overline{\mathcal{F}}$ in Σ_{kl}. In this way, we have obtained a stratification $\mathcal{S} = \{\Sigma_\gamma\}$ of $(M, \mathcal{F})$ consisting of submanifolds on which $\mathcal{F}$ and $\overline{\mathcal{F}}$ define regular Riemannian foliations. On each stratum S_γ of $\mathcal{S}$ the foliation $\overline{\mathcal{F}}$ defines a compact RF without holonomy, so the space of leaves $S_\gamma/\overline{\mathcal{F}}$ is a smooth manifold denoted by $\bar{S_\gamma}$ and the natural projection $p_\gamma\colon S_\gamma \to \bar{S_\gamma}$ is a locally trivial fibre bundle. The manifolds $\bar{S_\gamma}$ define a stratification of the leaf closure space $M/\bar{\mathcal{F}}$ which we denote $\overline{\mathcal{S}}$.

In the subsequent sections we will use the following natural splittings of the tangent bundle TM along each stratum. Let us choose a stratum $\Sigma_\gamma \in \mathcal{S}$. The foliation $\overline{\mathcal{F}}$ is regular on Σ_γ, so we have the following orthogonal splitting of the bundle $TM|\Sigma_\gamma$:

$$(*) \qquad TM = T\mathcal{F} \oplus Q_1 \oplus Q_2 \oplus Q_3$$

where $T\mathcal{F} \oplus Q_1 = T\overline{\mathcal{F}}$ and $T\mathcal{F} \oplus Q_1 \oplus Q_2 = T\Sigma_\gamma$. Therefore the normal bundle of $\mathcal{F}$ on the stratum Σ_γ can be identified with $Q_1 \oplus Q_2 \oplus Q_3$.

2. Smooth structure on leaf closure space. First, let us concentrate our attention on the regular case. The leaf closure space $M/\bar{\mathcal{F}}$ can be directly embedded in some R^k. In fact, any foliated open covering of $(M, \mathcal{F})$ is also foliated for $(M, \bar{\mathcal{F}})$ and admits a subordinated partition of unity by basic functions. Let S be a leaf closure and U a foliated tubular neighbourhood of S. Then U is an R^s bundle over S and there exists a compact Lie group $G \subset O(s)$ such that the traces of leaf closures on R^s are the orbits of G. The basic functions on U correspond to G-invariant functions on R^s. Using the classical theory of G invariant functions (polynomials) we can embed $R^s/G = U/\bar{\mathcal{F}}$ into some R^r. The manifold M admits a finite covering by such foliated open sets, and then using a subordinated partition of unity we can embed the leaf closure space $M/\bar{\mathcal{F}}$ in R^k for some k sufficiently large. One can follow the proof of the classical compact manifold embedding theorem, cf. [16], using instead of charts local embeddings of $M/\bar{\mathcal{F}}$ and basic functions and basic partitions of unity.

The smooth structure on $M/\overline{\mathcal{F}}$ can be defined by basic functions on open foliated subsets of $(M, \mathcal{F})$ and the smooth structure on $W/O(q)$ by $O(q)$-invariant functions. The above considerations assure that the homeomorphism is a diffeomorphism for these smooth structures. The natural embedding of $W/O(q)$ in R^k, cf. [4], makes this space an algebraic set, and the induced smooth structure from R^k coincides with the structure introduced via $O(q)$-invariant functions, as locally the leaf closure space is an orbit space Therefore the results of the singular Morse theory developed by M. Goresky and R. Macpherson in [15] can be translated to the foliated Morse theory of basic functions of regular foliated Riemannian manifolds.

The case of SRF is more complicated. We have a description of a tubular neighbourhood of a leaf closure due to H. Boualem and P. Molino, cf. [8], but it is not fine enough to obtain a local embedding. Only in the case of orbit-like foliations (OLF), cf. [29], any leaf closure admits an open foliated tubular neighbourhood U such that the leaf space $U/\bar{\mathcal{F}}$ is identified with the orbit space R^r/G for some compact Lie group G. Therefore for OLF on compact manifolds we have the leaf closure embedding, the proof being the same as for regular Riemannian foliations. Note that regular Riemannian foliations are OLF.

THEOREM 1. *Let $\mathcal{F}$ be an OLF on a compact manifold M. Then the leaf closure space $M/\bar{\mathcal{F}}$ admits a topological embedding into R^k for some sufficiently large k. The smooth structure induced by this embedding is the same as the smooth structure defined by basic functions of $(M, \mathcal{F})$.*

More information on embeddings of stratified spaces and induced smooth structures can be found in [32, 31].

3. Foliated structures. In this section we will look at the way geometric structures descend to the leaf and leaf closure spaces with their natural stratifications. We will pay particular attention to symplectic and complex structures as transversally Hermitian and Kähler foliations are of great interest in view of applications.

3.1. *Symplectic structure.* A foliation $\mathcal{F}$ is transversally symplectic if it admits a basic closed 2-form ω of maximal rank. Therefore the codimension of $\mathcal{F}$ is even, say $2q$. Such manifolds are also called presymplectic and play and important role in physics, cf. [36, 23]. We have already seen that the smooth structure of the leaf closure space is given by smooth basic functions. A symplectic form ω on the foliated manifold $(M, \mathcal{F})$ defines a Poisson structure on the algebra $C^\infty(M)$ of smooth functions on M, however the basic functions $C^\infty(M, \mathcal{F})$ rarely form its Poisson subalgebra, cf. [21, 22] and Proposition 9.7 of [23]. However, for transversally symplectic foliations we have the following proposition:

PROPOSITION 1. *If $\mathcal{F}$ is transversally symplectic, then $C^\infty(M, \mathcal{F}) = C^\infty(M, \overline{\mathcal{F}}) = C^\infty(M/\overline{\mathcal{F}})$ admits the structure of a Poisson algebra.*

Proof. The form ω projects on the holonomy invariant form $\overline{\omega}$, which is a symplectic form of N. The symplectic form $\overline{\omega}$ defines a Poisson structure $\{,\}_N$ on N which assigns to two H-invariant functions an H-invariant function. Therefore this Poisson structure lifts to a Poisson structure $\{,\}_B$ on the set of basic functions $C^\infty(M, \mathcal{F})$. The basic functions are the same for both foliations $\mathcal{F}$ and $\overline{\mathcal{F}}$, therefore $\{,\}_B$ is a Poisson structure on the set $C^\infty(M, \overline{\mathcal{F}})$. ∎

The above proposition is the first step in the proof of the fact that our stratified pseudomanifold $M/\overline{\mathcal{F}}$ is a symplectic stratified space, cf. [35], also [32, 31].

To proceed any further we have to ensure that the induced structure on the strata is symplectic. The condition formulated below seems to be the most natural one. Assume that $\mathcal{F}$ is transversally Kähler for the Riemannian metric g, the complex structure J and the symplectic form ω on the normal bundle $N(M, \mathcal{F})$, and that J satisfies the following condition:

$$(**) \quad J(T\overline{\mathcal{F}}/T\mathcal{F}) \subset T\overline{\mathcal{F}}/T\mathcal{F}$$

We shall look at the transverse symplectic and holomorphic structures and check whether some of their components project onto the leaf space $M/\bar{\mathcal{F}}$ and verify what structures they induce.

First consider the principal stratum Σ_0. The splitting of TM reduces itself to $T\mathcal{F} \oplus Q_1 \oplus Q_2$ as Σ_0 is an open and dense subset M. The normal bundle of $\mathcal{F}$ can be identified with $Q_1 \oplus Q_2$ and $T\overline{\mathcal{F}}/T\mathcal{F}$ with Q_1. So J acts on $Q_1 \oplus Q_2$. Since $J(Q_1) \subset Q_1$, then

$J(Q_2) \subset Q_2$. Therefore as the splitting is g-orthogonal, it is also ω-orthogonal and the transverse symplectic form ω can be written as $\omega = \omega^{2,0} + \omega^{0,2}$, where $\omega^{2,0}$ and $\omega^{0,2}$ are homogeneous components with respect to the splitting. Locally, the subbundle Q_1 is spanned by foliated Killing vector fields X such that $L_X\omega = 0$. Their flows preserve the splitting so

$$L_X(\omega^{2,0} + \omega^{0,2}) = L_X\omega^{2,0} + L_X\omega^{0,2}$$

and $L_X\omega^{2,0} = 0$ and $L_X\omega^{0,2} = 0$. The restrictions of both forms to Q_1 and Q_2, respectively, are of maximal rank. Therefore to prove that $\omega^{0,2}$ is a transverse symplectic form for the foliation $\bar{\mathcal{F}}$ on Σ_0, it is sufficient to demonstrate that $d\omega^{0,2} = 0$.

In fact, $0 = d\omega = d\omega^{2,0} + d\omega^{0,2} = d_1\omega^{2,0} + d_2\omega^{2,0} + \partial\omega^{2,0} + d_1\omega^{0,2} + d_2\omega^{0,2}$.

Thus $d_1\omega^{2,0} = 0, \quad d_2\omega^{2,0} = 0, \quad \partial\omega^{2,0} + d_1\omega^{0,2} = 0, \quad d_2\omega^{0,2} = 0$. Therefore it remains to prove that $d_1\omega^{0,2} = 0$. Now, for any vector field X of the commuting sheaf

$$L_X\omega^{0,2} = 0 = i_X d\omega^{0,2} + di_X\omega^{0,2} = i_X d\omega^{0,2} = i_X d_1\omega^{0,2} + i_X d_2\omega^{0,2} = i_X d_1\omega^{0,2}.$$

As these vector fields span the subbundle Q_1 we obtain $d_1\omega^{0,2} = 0$, and hence $d\omega^{0,2} = 0$. Therefore our foliation $\overline{\mathcal{F}}$ is transversally symplectic for the 2-form $\omega^{0,2} = \bar{\omega}$. Thus the projection $\pi_0 \colon \Sigma_0 \to \overline{\Sigma}_0$ projects the 2-form $\bar{\omega}$ to a symplectic form on $\overline{\Sigma}_0$, which we denote by the same letter.

Let Σ_α be any stratum of $(M, \mathcal{F})$. $\overline{\mathcal{F}}$ induces a regular foliation without holonomy. In [43] we have proved that global i.a. of $\mathcal{F}$ are tangent to the strata and that the module $X(M, \mathcal{F})$ of these global vector fields is transverse to $\overline{\mathcal{F}}$ in each stratum. If X is an i.a., so is JX. Therefore each stratum is J-invariant, i.e. $J(Q_1 \oplus Q_2) \subset Q_1 \oplus Q_2$, hence the splitting (*) over any stratum is also ω-orthogonal. In this case the standard reasoning ensures that the 2-form $\omega_\alpha = i_\alpha^*\omega$, where i_α is the inclusion of the stratum Σ_α into M, is a transverse symplectic form of the foliated manifold $(\Sigma_\alpha, \mathcal{F})$. Now the same considerations as for the principal stratum demonstrate that each stratum of the stratification $\overline{\mathcal{S}}$ is a symplectic manifold.

Let $\pi_\alpha \colon \Sigma_\alpha \to \overline{\Sigma}_\alpha$ be the local trivial fibre bundle defining the foliation $\overline{\mathcal{F}}$ on Σ_α. Clearly, the mapping π_α is a morphism of the Poisson algebras $C^\infty(\Sigma_\alpha, \overline{\mathcal{F}}), \{,\}_\alpha$ and $C^\infty(\overline{\Sigma}_\alpha), \overline{\{,\}}_\alpha$, where the Poisson brackets $\{,\}_\alpha$ and $\overline{\{,\}}_\alpha$ are defined by ω_α and $\bar{\omega}_\alpha$, respectively.

To complete the proof that $M/\overline{\mathcal{F}}$ is a singular symplectic space we have to show that for any $\overline{\Sigma}_\alpha \in \overline{\mathcal{S}}$ the inclusion $i_\alpha \colon \overline{\Sigma}_\alpha \to M/\overline{\mathcal{F}}$ is a Poisson morphism, i.e.

$$\forall f, g \in C^\infty(M/\overline{\mathcal{F}}) \quad \{f, g\}_B|\overline{\Sigma}_\alpha = \overline{\{f|\overline{\Sigma}_\alpha, g|\overline{\Sigma}_\alpha\}}_\alpha.$$

In fact,

$$\{f, g\}_B|\overline{\Sigma}_\alpha\pi_\alpha = \omega(X_f, X_g)|\Sigma_\alpha = \omega_\alpha(X_f|\Sigma_\alpha, X_g|\Sigma_\alpha) = \overline{\{f|\overline{\Sigma}_\alpha, g|\overline{\Sigma}_\alpha\}}_\alpha\pi_\alpha$$

as vector fields X_f, X_g are tangent to strata.

Therefore we have proved the following lemma:

LEMMA 1. *Let $\mathcal{F}$ be transversally almost Kähler for the Riemannian metric g, the almost complex structure J and the symplectic form ω on the normal bundle $N(M, \mathcal{F})$. If $J(T\overline{\mathcal{F}}/T\mathcal{F}) \subset T\overline{\mathcal{F}}/T\mathcal{F}$, then $M/\overline{\mathcal{F}}$ is a singular symplectic space.*

3.2. *Complex structure.* At the beginning of the section we introduce the notion of an almost complex structure adapted to a singular foliation.

Definition 1. A (1,1)-tensor field $J \in Hom(TM, TM)$ is a foliated almost complex structure iff

i) $J(T\mathcal{F}) \subset T\mathcal{F}$;

ii) for any i.a X of $\mathcal{F}$ the vector field JX is also an i.a. of $\mathcal{F}$;

iii) for any i.a X of $\mathcal{F}$ $J^2X = -X \ \ mod\mathcal{F}$, i.e., they differ by a vector field tangent to the leaves of $\mathcal{F}$.

Properties. 1) In the regular case any transverse almost complex structure can be extended to a foliated almost complex structure, however such an extension is not unique.

2) If the foliation $\mathcal{F}$ is Riemannian and $J(T\overline{\mathcal{F}}) \subset T\overline{\mathcal{F}}$, then the strata of $(M, \mathcal{F})$ are J-invariant as global i.a. are transverse to leaf closures in the strata, cf. [33, 43].

3) On any stratum Σ_α the foliations $\mathcal{F}$ and $\overline{\mathcal{F}}$ are regular. Then the tensor field $J_\alpha = J|T\Sigma_\alpha$ is well defined and induces transverse almost complex structures for $\mathcal{F}|\Sigma_\alpha$.

A foliated Riemannian metric g and a foliated almost complex structure J are said to be compatible if $g(JX, JY) = g(X, Y)$ for any vectors $X, Y \in TM$. Then on any stratum Σ_α the induced Riemannian metric g_α and the induced foliated almost complex structure J_α are compatible. Then the 2-form $\omega_\alpha(X, Y) = g_\alpha(J_\alpha X, Y)$ for $X, Y \in T\Sigma_\alpha$ is basic. Moreover, J_α induces a transverse almost complex structure for $\overline{\mathcal{F}}|\Sigma_\alpha$.

The above considerations suggest the following definition.

Definition 2. A singular foliation $\mathcal{F}$ on M is said to be transversally almost Kähler if it admits a foliated Riemannian metric g and a foliated almost complex structure, which are compatible, and such that on any stratum Σ_α of the associated stratification of M the 2-forms ω_α are closed. Such a structure is called transversally Kähler if the induced almost complex structures are transversally integrable for $\mathcal{F}_\alpha$.

Before formulating the theorem we prove the following simple property.

Proposition 2. *Let $\mathcal{F}$ be an RRF of a compact manifold M. If J is a foliated almost complex structure for $\mathcal{F}$ and the closures of leaves are J-invariant then J is also foliated for the leaf closure foliation $\bar{\mathcal{F}}$.*

Remark. The assumption about the J-invariance of the closures of leaves is essential as indicated by the example presented in [11].

Proof. Our assumption means that the condition (i) of Definition 1 is satisfied. We have to verify the conditions (ii) and (iii) for any i.a. X of $\bar{\mathcal{F}}$. Let X be an i.a. of $\mathcal{F}$ and Σ_α be a stratum. Then $X|\Sigma_\alpha = X_\alpha^\top + X_\alpha^\perp$, where $X_\alpha^\top$ is the part tangent to the leaf closures and $X_\alpha^\perp$ is the part orthogonal to the leaf closures. There exist global i.a.s of $\mathcal{F}$ $X_1^\alpha, ..., X_s^\alpha$ such that $X_\alpha^\perp = \Sigma_1^s f_j X_j^\alpha$. Therefore $JX|\Sigma_\alpha = J(X_\alpha^\top + X_\alpha^\perp) = J(X_\alpha^\top) + J(X_\alpha^\perp) = J(X_\alpha^\top) + J(\Sigma_1^s f_j X_j^\alpha)$. Hence JX is an i.a. of $\bar{\mathcal{F}}$. The above calculations also show that the third condition is satisfied. ■

Taking into account the previous considerations and Definition 2 the following theorem is true:

THEOREM 2. *Let $\mathcal{F}$ be a transversally almost Kähler singular foliation on a compact manifold M. Then the space of leaf closures $M/\overline{\mathcal{F}}$ is an almost Kähler singular space.*

4. Connections. Connections have played a very important role in the theory of regular foliations. There are two classes of connections "adapted" to a regular foliation, "Bott" connections and foliated or transversally projectable ones, cf. [7, 19, 28, 42]. In the singular case it is more difficult to give a satisfactory definition of an adapted connection. Let us propose a definition of a connection "adapted" to a singular foliation based on the classical definition of a connection as a differential operator. We will study conditions which will permit us to project the connection to the leaf space, a singular stratified space.

Let $\mathcal{F}$ be a singular foliation on a smooth, compact, connected manifold M. Let $\mathcal{X}(\mathcal{F})$ denote the sheaf of local vector fields which are tangent to leaves of $\mathcal{F}$. The sheaf $\mathcal{X}(\mathcal{F})$ is transitive on these leaves. Like-wise the Lie algebra $X(\mathcal{F}) = \Gamma(M, \mathcal{X}(\mathcal{F}))$ of global sections of $\mathcal{X}(\mathcal{F})$, i.e. of global vector fields tangent to leaves of $\mathcal{F}$, is transitive on leaves of $\mathcal{F}$. Let $\mathcal{X}(M,\mathcal{F})$ denote the sheaf of local infinitesimal automorphisms of $\mathcal{F}$, i.e. $Y \in \mathcal{X}(M,\mathcal{F})$ iff for any $X \in \mathcal{X}(\mathcal{F})$ $[Y,X] \in \mathcal{X}(\mathcal{F})$. Obviously $\mathcal{X}(\mathcal{F}) \subset \mathcal{X}(M,\mathcal{F})$ and $X(\mathcal{F}) \subset X(M,\mathcal{F})$. $\mathcal{X}(\mathcal{F})$ and $X(\mathcal{F})$ are double ideals of $\mathcal{X}(M,\mathcal{F})$ and $X(M,\mathcal{F})$, respectively. On compact manifolds, for some regular foliations and all singular ones the Lie algebra $X(M,\mathcal{F})$ is not transitive on M.

Let ∇ be a connection on the manifold M. We say that ∇ is *adapted* to the foliation $\mathcal{F}$ or ∇ is a *foliated connection* on $(M,\mathcal{F})$ if

1) for any $X, Y \in \mathcal{X}(M,\mathcal{F})$ $\nabla_X Y \in \mathcal{X}(M,\mathcal{F})$;
2) for any $X \in \mathcal{X}(\mathcal{F})$ and any $Y \in \mathcal{X}(M,\mathcal{F})$ $\nabla_X Y \in \mathcal{X}(\mathcal{F})$ and $\nabla_Y X \in \mathcal{X}(\mathcal{F})$.

Now we shall discuss these two conditions (1) and (2). Let's check how restrictive they are in the regular case.

A) Let $\mathcal{F}$ be a Riemannian foliation and ∇^{LC} be the Levi-Civita connection. Using the local form of a bundle-like metric it is easy to check that the Levi-Civita connection of such a metric satisfies (1). Condition (2) implies that the foliation is totally geodesic and that the orthogonal complement subbundle $\mathcal{F}^{\perp}$ is integrable.

However, in the Riemannian case, we can define another connection. Consider the splitting $TM = T\mathcal{F} \oplus T\mathcal{F}^{\perp}$ given by the bundle-like metric and let $p_2 \colon TM \to T\mathcal{F}^{\perp}$ be the orthogonal projection. We will define a connection ∇ as follows: let ∇^1 be any metric connection in $T\mathcal{F}$, the connection ∇^2 in $T\mathcal{F}^{\perp}$ is defined as follows: for any $Y, Z \in \Gamma(T\mathcal{F}^{\perp})$ and $X \in \mathcal{X}(\mathcal{F})$, let $\nabla^2_X Y = p_2[X,Y]$ and $\nabla^2_Z Y = \overline{\nabla}_Z Y$ where $\overline{\nabla}$ is the transverse Levi-Civita connection. The connection ∇ in TM is defined as $\nabla^1 \oplus \nabla^2$. It satisfies both conditions (1) and (2), however it has torsion. It is a metric connection preserving the natural splitting given by the bundle-like metric. Therefore it is a connection in the $O(p) \times O(q)$ reduction $B(M, O(p) \times O(q))$ of the bundle of orthogonal frames $B(M, O(n))$ given by the bundle-like metric. The existence of the torsion-free connection is linked to the vanishing of the structure tensor of $B(M, O(p) \times O(q))$, cf. [37, 42].

In [39], I. Vaisman proposed the following conditions for a linear connection ∇ on a manifold M foliated by a regular foliation $\mathcal{F}$:

i) $\mathcal{F}$ is parallel with respect to ∇ (i.e., $\nabla(\mathcal{X}(\mathcal{F})) \subseteq \mathcal{X}(\mathcal{F})$),

ii) the torsion T_∇ takes values in $T\mathcal{F}$,

iii) the curvature R_∇ satisfies the condition

$$R_\nabla(Z,X)Y \in T\mathcal{F} \quad \forall Z \in T\mathcal{F}, \ \ X,Y \in T\mathcal{F}^\perp.$$

In the regular case, these conditions are "almost" equivalent (1) and (2). In fact, (i) and (ii) imply easily (2), and (2) assures (i) and (ii) in the case when one of the vectors is tangent to the foliation. The condition (iii) implies (1), cf. [39], vice versa: $R_\nabla(Z,X)Y = \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y$, therefore for any $Z \in \mathcal{X}(\mathcal{F}), X, Y \in \mathcal{X}(M,\mathcal{F})$ the conditions (1) and (2) imply (iii).

In the case of singular foliations studied in this note we can replace the condition (2) by

(2') for any $Z \in \mathcal{X}(\mathcal{F})$, and $X, Y \in \mathcal{X}(M,\mathcal{F})$

$$R_\nabla(Z,X)Y \in \mathcal{X}(\mathcal{F}).$$

Therefore we have the following theorem:

THEOREM 3. *Let $\mathcal{F}$ be a compact SRF on a compact manifold M. Let ∇ be a connection satisfying the conditions:*

(a) for any $X \in \mathcal{X}(\mathcal{F})$ and $Y \in \mathcal{X}(M,\mathcal{F})$ $\nabla_X Y \in \mathcal{X}(\mathcal{F})$ and $\nabla_Y X \in \mathcal{X}(\mathcal{F})$;

(b) $R_\nabla(Z,X)Y \in T\mathcal{F}$ $\forall Z \in T\mathcal{F}$, $X,Y \in T\mathcal{F}^\perp$,

then any stratum Σ_α of the natural stratification $\mathcal{S}$ of $(M,\mathcal{F})$ is totally geodesic and the restricted connection ∇^α to the stratum Σ_α is transversally projectable and induces a connection $\bar{\nabla}^\alpha$ on the corresponding stratum $\bar{\Sigma}_\alpha$ of the leaf space $M/\mathcal{F}$.

Let $\mathcal{F}$ be transversally Kähler foliation of a compact manifold M and $\mathcal{S} = \{\Sigma_\alpha\}$ be the associated stratification of M. The restriction $\mathcal{F}_\alpha$ to any stratum Σ_α is transversally Kähler and therefore transversally symplectic for the associated transverse symplectic form ω_α, which defines a presymplectic structure. A torsion-free connection ∇ is called a presymplectic connection if any stratum Σ_α is ∇-totally geodesic (parallel) and $\nabla\omega_\alpha(X,Y) = 0$ for any $X, Y \in T\Sigma_\alpha$, cf. [39] for the definition in the regular case. Symplectic and presymplectic connections have been a very hot research topic, cf., e.g., [38, 39, 2, 3], particular attention being paid to their behaviour with respect to the Marsden-Weinstein reduction, cf. [30].

Theorem 3 and Proposition 3.2 of [39] yield the following:

THEOREM 4. *Let $\mathcal{F}$ be transversally Kähler foliation of a compact manifold M and $\mathcal{S} = \{\Sigma_\alpha\}$ be the associated stratification of M. Then any presymplectic connection ∇ induces a family of symplectic connections on the strata of the singular Kähler space $M/\overline{\mathcal{F}}$, the leaf closure space of the foliated manifold $(M,\mathcal{F})$.*

5. Ehresmann connections. Ehresmann connections for regular foliations have been introduced by [12] and studied by many authors, cf., e.g., [5, 6, 44, 45]. The prime example is the orthogonal complement of a totally geodesic foliation of a compact Riemannian manifold. However, the singular case is much more complicated.

In [26], V. Miquel Molina and the second author have proved that on a compact manifold an SRF which is minimal for a foliated Riemannian metric must be regular, thus an SRF cannot be totally geodesic for a foliated Riemannian metric. Therefore we cannot model our definition in the singular case on this classical example. There are several possible approaches. In [47, 48], N. Zhukova proposes to consider a distribution which is complementary to the tangent distribution to the leaves of the foliation, no stratification is needed but this distribution is not differentiable. In [46], the second author proposes a slightly different definition using an adapted stratification and obtains different results.

As our main interest are singular foliations which admit an adapted stratification, we present a definition that is more suitable and according to which an Ehresmann connection is a differentiable distribution. The distribution is not complementary to the foliation on the manifold M. It is impossible in the case of a singular foliation, but this condition is satisfied in each stratum. According to [43], in the case of an SRF with all leaves compact on a compact manifold, the orthogonal complement in each stratum defines a global smooth distribution, spanned by global foliated vector fields. It will be our model example of an Ehresmann connection. But there is no chance that local diffeomorphisms of leaves defined by the orthogonal geodesics are local isometries, cf. [18]. In the language of Ehresmann connections, the induced Ehresmann connections in each stratum are not ∇^{LC}-preserving, where ∇^{LC} is the Levi-Civita connection of the foliated Riemannian metric, cf. [6]. But the results obtained by the second author and his collaborators suggest that a change of a Riemannian metric in each stratum can do the trick.

Let $(M, \mathcal{F})$ be a compact manifold M with singular foliation $\mathcal{F}$ and let $\mathcal{S}$ be the associated stratification of M. We assume that the sheaf $\mathcal{X}(M, \mathcal{F})$ is transitive on the strata. A smooth distribution Q on M is called a complementary distribution on $(M, \mathcal{F})$ if for any $\Sigma_\alpha \in \mathcal{S}$ the distribution $Q\mid_{\Sigma_\alpha}$ is tangent to Σ_α and $T\Sigma_\alpha = T\mathcal{F}\mid_{\Sigma_\alpha} \oplus Q\mid_{\Sigma_\alpha}$. A smooth curve tangent to Q is called a Q-curve.

A complementary distribution on $(M, \mathcal{F})$ is called a *singular Ehresmann connection* of $\mathcal{F}$ if for any stratum Σ_α, $\gamma\colon [a, b] \to \Sigma_\alpha$ leaf curve and $\delta\colon [c, d] \to \Sigma_\alpha$ a Q-curve (i.e. tangent to Q) such that $\gamma(c) = \alpha(a)$, there exists a mapping $\sigma\colon [a, b] \times [c, d] \to M$, of the following properties:

a) $\sigma_s\colon [a, b] \to M$, $\sigma|_{[a,b]\times\{s\}}$ is a leaf curve for any $s \in [c, d]$ and $\sigma_c = \gamma$;
b) for any $t \in [a, b]$ $\sigma^t\colon [c, d] \to M$, $\sigma|_{\{t\}\times[c,d]}$ is a Q-curve and $\sigma^a = \delta$.

Let $\mathcal{G}$ be a singular foliation on M whose tangent bundle Q is a complementary distribution of $\mathcal{F}$, and let ∇ be a torsion free connection on M such that the leaves of $\mathcal{F}$ and the strata of $\mathcal{S}$ are ∇-totally geodesic. Following [6], we say that

(i) the distribution Q is $\mathcal{F}$-invariant if $\nabla_Y X \in Q \ \ \forall Y \in T\mathcal{F}, X \in Q$;

(ii) Q preserves ∇ if in each stratum the elements of holonomy along Q-curves are affine transformations.

In the singular case we have the following proposition, cf. Proposition 5.3 of [6].

PROPOSITION 3. *If Q is $\mathcal{F}$-invariant and $R(X,Y)Z \in Q$ for any $X \in Q$, $Y,Z \in T\mathcal{F}$, then Q preserves ∇.*

It is a simple extension of the above mentioned proposition as each stratum is ∇ totally geodesic and Q is tangent to strata. Likewise the singular version of Proposition 5.4 is true.

PROPOSITION 4. *Let Q be $\mathcal{F}$-invariant. Then if Q preserves ∇, then $R(X,Y)Z \in Q$ for any $X \in Q$, $Y,Z \in T\mathcal{F}$.*

The assumption induces the same conditions in any stratum. Therefore the proposition is true in any stratum, according to Proposition 5.4 of [6]. So our proposition follows.

PROPOSITION 5. *If the connection ∇ is complete, Q is $\mathcal{F}$-invariant and $R(X,Y)Z \in Q$ for any $X \in Q$, $Y,Z \in T\mathcal{F}$, then Q is a singular Ehresmann connection.*

We have to prove that for any stratum Σ_α the restriction bundle $Q|_{\Sigma_\alpha}$ is an Ehresmann connection for the regular foliation $\mathcal{F}|_{\Sigma_\alpha}$ of Σ_α. The strata are foliated so we can apply the same method as that used in [6] to prove Lemma 5.7.

THEOREM 5. *Let $(M,\mathcal{F})$ be a compact manifold M with singular foliation $\mathcal{F}$ and let $\mathcal{S}$ be the associated stratification of M. Let $\mathcal{G}$ be a singular foliation on M whose tangent bundle Q is a complementary distribution of $\mathcal{F}$. Let ∇ be a complete torsion free connection on M such that the leaves of $\mathcal{F}$ and the strata of $\mathcal{S}$ are ∇-totally geodesic. If Q is $\mathcal{F}$-invariant and $R(X,Y)Z \in Q$ for any $X \in Q$, $Y,Z \in T\mathcal{F}$, then*

1) the universal covering $\tilde{\Sigma}_\alpha$ of any stratum Σ_α is topologically a product $\tilde{L}_\alpha \times \tilde{G}_\alpha$ where $\tilde{L}_\alpha, \tilde{G}_\alpha$ are the universal covers of leaves of the foliations $\mathcal{F}_\alpha$ and Q_α, respectively, in Σ_α;

2) the lifts of $\mathcal{F}_\alpha$ and Q_α, to $\tilde{\Sigma}_\alpha$ is the foliation by $\tilde{L}_\alpha \times \{q\}$ and $\{p\} \times \tilde{G}_\alpha$, respectively, where $p \in \tilde{L}_\alpha, q \in \tilde{G}_\alpha$;

3) the projection $\tilde{\Sigma}_\alpha \to \tilde{L}_\alpha$ is an affine transformation if we restrict the lifted connection $\tilde{\nabla}_\alpha$ to leaves of the lifted foliation $\tilde{\mathcal{F}}_\alpha$.

6. Curvature properties. Let $(M,g,\mathcal{F})$ be a regular Riemannian foliation on a compact manifold, $\bar{\mathcal{F}}$ the SRF by leaf closures and let $\mathcal{S} = \{\Sigma_\alpha\}_{\alpha \in A}$ be the corresponding stratification of M. The strata of $\mathcal{S}$ have the following nice property:

any geodesic orthogonal to $\mathcal{F}$ and tangent to a stratum at one point remains tangent to the stratum in a neighbourhood of this point.

The standard argument, cf. [20], vol. 2, shows that if ∇ is the Levi-Civita connection of (M,g) $\nabla_X Y$ is tangent to Σ_α for any vector fields X,Y orthogonal to $\mathcal{F}$ and tangent to Σ_α. Therefore on Σ_α, if $p_\alpha \colon T\Sigma_\alpha \to T\mathcal{F}_\alpha^\perp$ denotes the orthogonal projection, $\bar{\nabla}$ the Levi-Civita connection of the induced Riemannian metric g_α on Σ_α we have the following:

$$p_\alpha(\nabla_X Y) = p_\alpha(\bar{\nabla}_X Y)$$

for any vector fields X, Y tangent to Σ_α and orthogonal to $\bar{\mathcal{F}}$. This equality permits us to establish some relations between the curvatures of the manifolds (M, g), $(\Sigma_\alpha, g_\alpha)$ and the transverse curvature of the foliation $\mathcal{F}$.

Let $Sec_{\mathcal{F}}$ denote the transverse sectional curvature of $(M, g, \mathcal{F})$, Sec the sectional curvature of (M, g) and $Sec^{\alpha}_{\mathcal{F}}$ the transverse sectional curvature of $(\Sigma_\alpha, g_\alpha, \mathcal{F}_\alpha)$. For any 2-subspace σ orthogonal to the foliation $\mathcal{F}$

$$Sec(\sigma) \leq Sec_{\mathcal{F}}(\sigma).$$

Moreover, for any $\sigma \subset T\Sigma_\alpha \cap T\mathcal{F}_\alpha^{\perp}$

$$Sec_{\mathcal{F}}(\sigma) = Sec^{\alpha}_{\mathcal{F}}(\sigma).$$

On each stratum Σ_α, the foliation $\bar{\mathcal{F}}$ by leaf closures is given by a global Riemannian submersion $\pi_\alpha \colon \Sigma_\alpha \to \bar{\Sigma}_\alpha$. Let denote by Sec_α the sectional curvature of $(\bar{\Sigma}_\alpha, \bar{g}_\alpha)$, where $\bar{g}_\alpha$ is the Riemannian metric induced by the bundle-like metric g. Locally, the submersion π_α factorizes as follows:

$$U \twoheadrightarrow V \twoheadrightarrow \bar{V}$$

where the first arrow is the projection of an open neighbourhood in Σ_α onto a transverse manifold of $\mathcal{F}_\alpha$, which is a Riemannian submersion. Moreover, the transverse sectional curvature on U is equal to the sectional curvature of the induced Riemannian metric. As the traces of the closures of leaves on V are orbits of the commuting sheaf, cf. [28], which consists of Killing vector fields, the second arrow is also a Riemannian submersion. Therefore

$$Sec_\alpha(\bar{\sigma}) \geq Sec^{\alpha}_{\mathcal{F}}(\sigma)$$

where σ is the horizontal lift of $\bar{\sigma} \subset T\bar{\Sigma}_\alpha$.

The above considerations lead us to the formulation of the following theorems:

Theorem 6. *If a Riemannian foliation of a compact manifold admits a transverse metric of positive sectional curvature, then any stratum of the leaf closure space is a Riemannian manifold of positive sectional curvature.*

Theorem 7. *Let $\mathcal{F}$ be a Riemannian foliation of a compact Riemannian manifold of positive sectional curvature. Then any stratum of the leaf closure space is a Riemannian manifold of positive sectional curvature.*

Remark. Similar results are true for transversally Kähler foliations and φ-sectional curvature.

The results of [24] yield the following corollaries.

Theorem 8. *Let $\mathcal{F}$ be a Riemannian foliation of codimension three of a compact Riemannian manifold. If the transverse Ricci curvature is positive definite, then on the strata of the leaf closure space there exist Riemannian metrics of positive sectional curvature.*

Theorem 9. *Let $\mathcal{F}$ be a Riemannian foliation of codimension four of a compact Riemannian manifold. If the transverse curvature operator is positive definite, then on the strata of the leaf closure space there exist Riemannian metrics of positive sectional curvature.*

References

[1] D. Alekseevsky, A. Kriegl, M. Losik and P. W. Michor, *The Riemannian geometry of orbit spaces, the metric, geodesics and integrable systems*, Publ. Math. Debrecen 62 (2003), 247–276.

[2] P. Baguis and M. Cahen, *A construction of symplectic connections through reduction*, Lett. Math. Phys. 57 (2001), 149–160.

[3] P. Baguis and M. Cahen, *Marsden-Weinstein reduction for symplectic connections*, Bull. Belg. Math. Soc. S. Stevin 10 (2003), 91–100.

[4] E. Bierstone, *The structure of orbit spaces and the singularities of equivariant mappings*, Monogr. Mat. 35, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1980.

[5] R. A. Blumenthal and J. Hebda, *Complementary distributions which preserve the leaf geometry and applications to totally geodesic foliations*, Quart. J. Math. Oxford 24 (1973), 385–395.

[6] R. A. Blumenthal and J. J. Hebda, *Ehresmann connections for foliations*, Indiana Un. Math. J. 33 (1984), 597–611.

[7] R. Bott, *Lectures on characteristic classes and foliations*, in: Springer LNM 279 (1972), 1–94.

[8] H. Boualem and P. Molino, *Modèle locaux saturés de feuilletages riemanniens singuliers*, C. R. Acad. Sci. Paris 316 (1993), 913–916.

[9] M. Davis, *G-smooth manifolds as collections of fibre bundles*, Pacific Math J. 77 (1978), 315–363.

[10] L. Di Terlizzi, J. Konderak, A. M. Pastore and R. Wolak, *K-structures and foliations*, Ann. Univ. Sci. Budapest 44 (2001), 171–182.

[11] L. Di Terlizzi, J. Konderak and R. Wolak, *A generalization of the Boothby-Wang theorem*, Tsukuba Math. J., to appear.

[12] Ch. Ehresmann, *Les connexions infinitésimales dans un espace fibré différentiable*, Colloq. Topologie Bruxelles (1950), 29–55.

[13] Th. Friedrich, *On types of non-integrable geometries, On types of non-integrable geometries*, in: Proceedings of the 22nd Winter School "Geometry and Physics" (Srní, 2002). Rend. Circ. Mat. Palermo (2) Suppl. No. 71 (2003), 99–113.

[14] M. Goresky and R. MacPherson, *Intersection homology theory*, Topology 19 (1980), 135–165.

[15] M. Goresky and R. MacPherson, *Stratified Morse Theory*, Ergeb. Math. Grenzgeb. 14, Springer-Verlag, Berlin, 1988.

[16] M. Hirsch, *Differential Topology*, Springer.

[17] J. Huebschmann, *Kähler reduction and quantizaton*, J. Reine Angew. Math. 591 (2006), 75–109.

[18] D. L. Johnson and A. M. Naveira, *A topological obstruction to the geodesibility of a foliation of odd dimension*, Geom. Dedicata 11 (1981), 347–352.

[19] F. Kamber and Ph. Tondeur, *Foliated Bundles and Characteristic Classes*, Springer LNM 494, 1975.

[20] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Interscience Publishers, 1963.

[21] P. Libermann, *Sur quelques propriétés des variétés symplectiques*, in: Proc. Conf. on Diff. Geom. 1980, Univerzita Karlova, Praha 1981.

[22] P. Libermann, *Sous-variétés et feuilletages symplectiquement réguliers*, in: Symplectic Geometry, A. Crumeyrolle and J. Grifone (eds.), Pitman, London, 1983, 81–106.

[23] P. Libermann and C.-M. Marle, *Symplectic Geometry and Analytical Mechanics*, D. Reidel, 1987.

[24] M. Lovrić, M. Min-Oo and E. A. Ruh, *Deforming transverse Riemannian metrics of foliations*, Asian J. Math. 4 (2000), 303–314.

[25] P. Michor, *Isometric Actions of Lie Groups and Invariants*, lecture notes, Vienna, 1997.

[26] V. Miquel and R. Wolak, *Minimal singular Riemannian foliations*, CRAS 342 (2006), 33–37.

[27] P. Molino, *Désingularisation des feuilletages riemanniens*, American J. Math. 106 (1984), 1091–1106.

[28] P. Molino, *Riemannian Foliations*, Progress in Math. 73, Birkhäuser, 1988.

[29] P. Molino, *Orbit–like foliations*, in: Geometric Study of Foliations, Proc. Tokyo 1993, World Scientific, 1994, 97–119.

[30] J.-P. Ortega and T. S. Ratiu, *Mamentum Maps and Hamiltonian Reduction*, Birkhäuser, 2004.

[31] M. Pflaum, *Smooth structures on stratified spacs*, in: Progress in Math. 198, Birkhäuser, 2001.

[32] M. Pflaum, *Analytic and Geometric Study of Stratified Spaces*, Springer LNM 1768, 2001.

[33] M. Pierrot, *Orbites des champs feuilletés pour un feuilletage riemannien sur une variété compacte*, C. R. Acad. Sc. Paris 301 (1985), 443–445.

[34] K. Richardson, *The transverse geometry of G-manifolds and Riemannian foliations*, Illinois J. Math. 45 (2001), 517–535.

[35] R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. Math. 134 (1991), 375-422.

[36] J.-M. Souriau, *Structure de systèmes dynamiques*, Dunod, Paris 1969.

[37] S. Sternberg, *Lectures on Differential Geometry*, Prentice-Hall, 1964.

[38] I. Vaisman, *Symplectic curvature tensors*, Mon. Math. 100 (1985), 299–327.

[39] I. Vaisman, *Connections under symplectic reduction*, Bull. Math. Soc. Sci. Math. Roumanie 43 (2000), 377–387.

[40] R. Wolak, *Geometric Structures on Foliated Manifolds*, Santiago de Compostela, 1989.

[41] R. Wolak, *Foliated G-structures and Riemannian foliations*, Manuscripta Math. 66 (1989), 45–59.

[42] R. Wolak, *The structure tensor of a transverse G–structure on a foliated manifold*, Boll. U. M. I. 4 (1990), 1–15.

[43] R. Wolak, *Pierrot's theorem for singular Riemannian foliations*, Publ. Matem. 38 (1994), 433–439.

[44] R. A. Wolak, *Graphs, Ehresmann connections and vanishing cycles*, in: Proc. Differential Geometry and Applications, Brno 1995, 345–352.

[45] R. A. Wolak, *Ehresmann connections for Lagrangian foliations*, J. Geom. and Phys. 17 (1995), 310–320.

[46] R. Wolak, *Connections for singular foliations on stratified manifolds*, Publ. Math. Debrecen 63 (2003), 623–633.

[47] N. I. Zhukova, *Ehresmann connection of foliation with singularities and global stability of leaves*, Iz. VUZ Mat. 10 (2004), 45–56.

[48] N. I. Zhukova, *Singular foliations with Ehresmann connections and their holonomy groupoids*, this volume, 471–490.