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SINGULAR FOLIATIONS WITH EHRESMANN CONNECTIONS AND THEIR HOLONOMY GROUPOIDS

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Abstract. We introduce topological $\mathcal{Q}$-holonomy groupoids for singular foliations $(M, \mathcal{F})$ with an Ehresmann connection $\mathcal{Q}$ using $\mathcal{Q}$-holonomy groups, which have a global character. We show advantage of our groupoids over known ones.

1. Introduction. The concept of Ehresmann connection for foliations was introduced by Blumenthal and Hebda [3] as a natural generalization of Ehresmann connections for submersions.

The construction of Ehresmann connection is based on the existence of transfers of horizontal curves along vertical curves through the vertical horizontal homotopy. Note that this homotopy has already been used in global differential geometry (see, e. g., [8], [10], [19]).

We extend the concept of Ehresmann connection to foliations $(M, \mathcal{F})$ with singularities in the sense of Stefan [20] and Sussmann [21] (Section 3).

Transversally complete singular Riemannian and totally geodesic foliations with complete induced Riemannian metrics on the leaves have natural Ehresmann connections.

We define an Ehresmann connection $\mathcal{Q}$ of singular foliations $(M, \mathcal{F})$ as a generalized distribution $\mathcal{Q}$ on M, which is transverse to $\mathcal{F}$, and a vertical horizontal property is satisfied [34]. This property allows to transfer an integral curve σ of $\mathcal{Q}$ (called horizontal) along admissible curves (called vertical) lying in the leaf $L = L(\sigma(0))$ of $\mathcal{F}$. Unlike the regular case this transfer is not unique in general (Remark 1).

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Therefore we define the $*\mathcal{Q}$-holonomy group $*H_{\mathcal{Q}}(L,a)$ of an arbitrary leaf L of $(M,\mathcal{F})$ as the transformation group of a quotient set Ω_a/ρ of the set Ω_a of horizontal curves with the origin in a. We stress that this group has a global character (Section 4).

For a singular Riemannian foliation this transfer keeps lengths of horizontal curves. We use this property and prove a criterion of local stability of leaves of singular foliations (Section 11, Theorem 5).

The concept of a topological holonomy groupoid was introduced by C. Ehresmann [6]. A new approach to the concept of the holonomy groupoid was found by J. Pradines [16], [17]. He used germs of local transverse isomorphisms of foliation and defined the notion of a differentiable holonomy groupoid. H. E. Winkelnkemper [23] suggested an equivalent construction and called it the graph $G(\mathcal{F})$ of the foliation $\mathcal{F}$.

Under some assumptions holonomy groups of singular foliations were defined by M. Bauer [1] and P. Dazord [4]. For Stefan foliation with only tractable singular leaves a topological holonomy groupoid was defined by H. Suzuki [22]. B. Bigonnet and J. Pradines [2] indicated another construction of a graph of a singular foliation.

We introduce a graph $G(\mathcal{F},Q)$ for regular foliation $(M,\mathcal{F})$ with an Ehresmann connection $\mathcal{Q}$ [28] (see also [30], [31]) and prove that this graph has important advantages over the usual graph $G(\mathcal{F})$. In particular, our graph is always a Hausdorff manifold, and its projections onto M form a locally trivial fibration. It allows to apply this graph in investigation of a topology of regular foliations admitting an Ehresmann connection.

R. A. Wolak [24] proved that a regular foliation $\mathcal{F}$ with an Ehresmann connection has no vanishing cycles. Relations between sets with Hausdorff and non-Hausdorff graphs of suspended foliations were investigated by G. Chubarov and the author [32].

The main goal of this paper is a construction of a topological groupoid of a singular foliation with an Ehresmann connection. The global character of the $*\mathcal{Q}$-holonomy group $*H_{\mathcal{Q}}(L,a)$ allows us to introduce a topological holonomy groupoid $*G_{\mathcal{Q}}(\mathcal{F})$ using our $*\mathcal{Q}$-holonomy groups (Section 6, Proposition 3).

We define a generalized Ehresmann connection of a singular foliation and $\mathcal{Q}$-holonomy group $H_{\mathcal{Q}}(L,a)$ of a leaf L, and by contrast to the group $*H_{\mathcal{Q}}(L,a)$ the set of horizontal curves is formed by curves for which only endpoints may be singular (Section 8).

Using $\mathcal{Q}$-holonomy groups $H_{\mathcal{Q}}(L,a)$ we define a holonomy groupoid $G_{\mathcal{Q}}(\mathcal{F})$ of a singular foliation $(M,\mathcal{F},Q)$ with generalized Ehresmann connection $\mathcal{Q}$. Under natural assumptions we prove some properties of the groupoid $G_{\mathcal{Q}}(\mathcal{F})$, in particular the property of the topological space of $G_{\mathcal{Q}}(\mathcal{F})$ to be Hausdorff (Section 9, Theorem 2).

As corollary, $*G_{\mathcal{Q}}(\mathcal{F}) = G_{\mathcal{Q}}(\mathcal{F}) = G(\mathcal{F},Q)$ is Hausdorff for regular foliation with an Ehresmann connection $(M,\mathcal{F},Q)$, where $G(\mathcal{F},Q)$ is the graph mentioned above.

In particular we prove that the holonomy groupoid $G_{\mathcal{Q}}(\mathcal{F})$ of an orbit-like transversally complete singular Riemannian foliation $(M,\mathcal{F},Q)$ with a natural Ehresmann connection $\mathcal{Q} = (T\mathcal{F})^{\perp}$ is Hausdorff (Section 12, Corollary 4).

We hope that the groupoid $G_{\mathcal{Q}}(\mathcal{F})$ will be applied to singular foliations with transversal geometric structures. Nowadays the most studied class of these foliations is the class of Riemannian foliations ([12]). R.A. Wolak [25] introduces and investigates singular foliations on stratified manifolds admitting adapted linear connections.

In recent work of C. Debord [5] a Lie groupoid of a singular foliation was constructed by a new method using local Lie groupoids and their equivalences without holonomy groups. As indicated in [5] the foliation $\mathcal{F}$ of $\mathbf{R}^3$ by concentric spheres with unique singular leaf 0 has no holonomy groupoid in the sense of C. Debord. This foliation $(\mathbf{R}^3, \mathcal{F})$ is a Riemannian foliation with the natural Ehresmann connection $\mathcal{Q} = (T\mathcal{F})^{\perp}$. Thus the holonomy groupoids ${}*G_{\mathcal{Q}}(\mathcal{F})$ and $G_{\mathcal{Q}}(\mathcal{F})$ are well defined, and according to Corollary 4 the topological space $G_{\mathcal{Q}}(\mathcal{F})$ is Hausdorff.

A. Piątkowski [14] constructed the holonomy group of a leaf of a singular foliation which can be considered as an analogous of a germ holonomy group of a regular foliation. He proved a local transverse stability of a compact leaf with the trivial holonomy group [14].

Under an additional assumption the local stability in the sense of Ehresmann of a compact leaf with a finite holonomy group was proved by P. Dazord [4].

We state a theorem about global stability of a compact leaf L of maximal dimension with a finite holonomy group $H_{\mathcal{Q}}(L)$ (or a finite fundamental group $\pi_1(L)$), which belongs to the author [34] (Section 10).

2. Foliations with singularities. We assume that all manifolds under consideration are of class C^r, $r \geq 1$, connected, second countable, and Hausdorff, unless otherwise stated. All neighbourhoods are open. Paths, curves and maps are piecewise differentiable.

Let M be an n-dimensional manifold. A map $\mathcal{T}$ assigning to each point $x \in M$ a $p(x)$-dimensional subspace $\mathcal{T}_x$ in the tangent space T_xM, is called a *generalized distribution* on M. The distribution $\mathcal{T}$ is said to be *differentiable* if for any vector $Y \in \mathcal{T}_x$ there exists a differentiable vector field X in a neighbourhood U_x of x such that $X_x = Y$, $X_y \in \mathcal{T}_y$ for any $y \in U_x$.

An immersed submanifold L of M is called an *integral manifold* of a generalized distribution $\mathcal{T}$ if $T_xL = \mathcal{T}_x$ for any $x \in L$. An integral manifold L is said to be *maximal* if L is connected and coincides with each connected integral manifold containing L. If through each point $x \in M$ an integral manifold passes, then the generalized distribution $\mathcal{T}$ is said to be *integrable*.

The family $\mathcal{F} = \{L_\alpha \mid \alpha \in J\}$ of maximal integral manifolds L_α of a generalized integrable distribution $\mathcal{T}$ is called a *differentiable foliation with singularities*. L_α considered as a submanifold in M is called a *leaf* of the foliation $\mathcal{F}$.

H. Sussmann proved that a C^∞-smooth generalized distribution $\mathcal{T}$ is integrable if and only if $\mathcal{T}$ is involutive (a generalization of Frobenius' theorem) [21].

Let $(M, \mathcal{F})$ be a differentiable foliation with singularities. It is known [4] that at each point $x \in M$ there exists a chart (W, φ) such that

- (F_1) $\varphi(W) = U \times V$, where U is a neighbourhood of the origin in $\mathbf{R}^p$, V is a neighbourhood of the origin in $\mathbf{R}^q$, $p = p(x)$ is the dimension of the central leaf $L(x)$ passing through x, and $q = n - p$;
- (F_2) $\varphi(x) = (0, 0) \in U \times V$;
- (F_3) for any leaf $L \in \mathcal{F}$, $\varphi(L \cap W) = U \times l$, where $l := \{v \in V \mid \varphi^{-1}(0, v) \in L\}$.

A chart (W, φ) satisfying (F_1)–(F_3) is said to be *fibred* at x, and W is called a *fibred neighbourhood*. Also we call x the *center of this chart*.

J. Kubarski [11] shown that Stefan's definition of a singular foliation is equivalent to the existence of a partition $\mathcal{F} = \{L_\alpha \mid \alpha \in J\}$ of M into connected immersed submanifolds L_α such that for each $x \in M$, there is a fibred chart at x with respect to $\mathcal{F}$. From this fact and Dazord's result [4] follows the equivalence of Stefan's and Sussmann's approaches to the notion of a C^∞-smooth singular foliation.

Let (W, φ) be a fibred chart, $\varphi(W) = U \times V$. Without loss of generality, we can assume that U and V are neighbourhoods homeomorphic to $\mathbf{R}^p$ and $\mathbf{R}^q$, respectively. In this work we assume that charts under consideration are fibred.

The arc-wise connected component of $W \cap L_\alpha$ will be called a *local leaf* of the leaf L_α in chart or in the neighbourhood W and will be denoted by l_y^W, where $y \in W \cap L_\alpha$.

Note that, in the fibred neighbourhood W_x of x, the dimension $p(y)$ of any leaf $L(y)$, $y \in W_x$ is not less than $p(x)$. Hence the leaf dimension function $p(x)$, $x \in M$ is lower semi-continuous.

Since $p(x) \leq n$, $x \in M$, where $n = \dim M$, there exists the maximal dimension p_0 of leaves of foliation $\mathcal{F}$ with singularities. A point x is said to be *regular* if a leaf of dimension p_0 passes through x. If a point $x \in M$ is not regular, x is said to be *singular*. If $\dim L = p_0$, the leaf L is said to be *regular*, otherwise L is called a *singular leaf*. A foliation whose leaves have constant dimension is called *regular*. Therefore regular foliations form a subclass of the class of foliations with singularities.

Let us denote by M^0 the union of leaves of maximal dimension p_0. If $x \in M^0$, and (W, φ) is a fibred chart at x, then for any $y \in W$ the leaf $L(y)$ also has dimension p_0, hence $W \subset M^0$. Thus the union M^0 of regular leaves is an open subset in M.

3. Ehresmann connection for foliations with singularities. A generalized distribution $\mathcal{Q}$ on the manifold M is said to be *transversal* to the foliation $\mathcal{F}$ if on M there exists a Riemannian metric g such that $\mathcal{Q}_x$ is the orthogonal complement to the space $\mathcal{T}_x$ tangent to the leaf $L(x)$ in the Euclidean vector space (T_xM, g_x), i.e., for any $x \in M$

$$T_xM = \mathcal{T}_x \oplus \mathcal{Q}_x,$$

where $\oplus$ is the orthogonal sum. The subspace $\mathcal{Q}_x$, $x \in M$, and vectors in $\mathcal{Q}$ will be called horizontal. A piecewise differentiable curve σ is said to be horizontal if its tangent vectors are horizontal. The distribution $\mathcal{T}$ tangent to the leaves of $\mathcal{F}$ is said to be vertical. A curve h is said to be vertical if h lies in a leaf of $\mathcal{F}$.

A *vertical horizontal homotopy* (v.h.h.) is a piecewise differentiable map $H : I_1 \times I_2 \to M$, where $I_1 = I_2 = [0, 1]$ is such that for any $(s, t) \in I_1 \times I_2$ the curve $H|_{I_1 \times \{t\}}$ is horizontal and the curve $H|_{\{s\} \times I_2}$ is vertical (see Fig. 1). The pair of curves $(H|_{I_1 \times \{0\}}, H|_{\{0\} \times I_2})$ is called the *base* of the v.h.h. A pair of paths (σ, h) such that $\sigma(0) = h(0)$, σ is a horizontal path, and h is a vertical path, is said to be *admissible*.

DEFINITION 1. A generalized distribution $\mathcal{Q}$ transversal to the foliation $\mathcal{F}$ with singularities is called an *Ehresmann connection for* $\mathcal{F}$ if for any admissible pair of paths (σ, h)

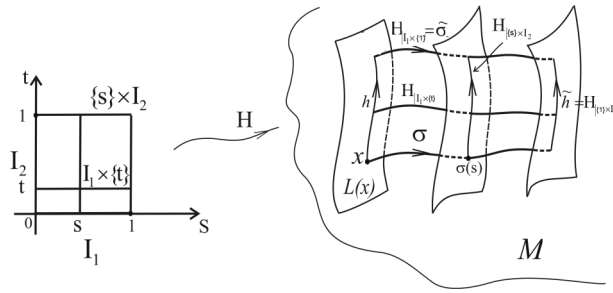

Fig. 1. A vertical horizontal homotopy with the base (σ, h)

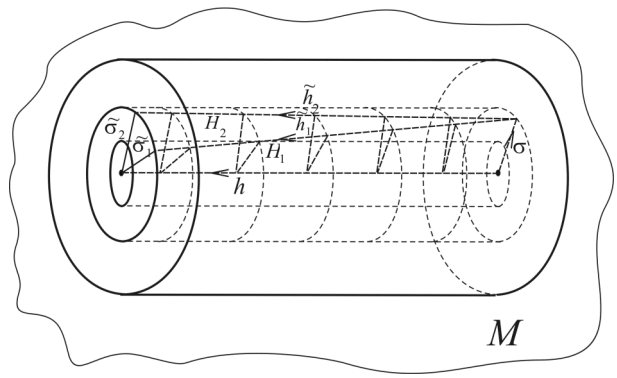

Fig. 2. Different vertical horizontal homotopies H_1 and H_2 of the singular Riemannian foliation with the same base (σ, h)

there exists a v.h.h. with the base (σ, h). A foliation $(M, \mathcal{F})$ with singularities admitting an Ehresmann connection $\mathcal{Q}$ is denoted by $(M, \mathcal{F}, \mathcal{Q})$.

Let H be a v.h.h. with the base (σ, h), and $\tilde{\sigma} := H|_{I_1 \times \{1\}}$, $\tilde{h} := H|_{\{1\} \times I_2}$. We say that the curve $\tilde{\sigma}$ is obtained from σ by *transfer along* h via the v.h.h. H and denote it by $\sigma \xrightarrow[h]{H}> \tilde{\sigma}$ or by $\sigma \xrightarrow{H}> \tilde{\sigma}$ and similarly $h \xrightarrow[\sigma]{H}> \tilde{h}$ or $h \xrightarrow{H}> \tilde{h}$.

Note that, in general, an Ehresmann connection $\mathcal{Q}$ is not a differentiable generalized distribution. However, $\mathcal{Q}$ possesses a "generalized differentiability" property because $\mathcal{Q}$ is the orthogonal complement to the differentiable generalized distribution $\mathcal{T}$ tangent to $\mathcal{F}$ (with respect to a Riemannian metric g on M).

Also note that our definition of Ehresmann connection for foliations with singularities naturally generalizes the definition of Ehresmann connection for regular foliations [3].

REMARK 1. If $(M, \mathcal{F}, \mathcal{Q})$ is a regular foliation with an Ehresmann connection, it is well known that there exists a *unique* vertical horizontal homotopy H with the given base (σ, h). This is not true in general for singular foliations $(M, \mathcal{F}, \mathcal{Q})$. Hence transfer of σ along h is a multivalued function (see Fig. 2).

4. $*\mathcal{Q}$-holonomy group for foliation with singularities. For any foliation with singularities admitting an Ehresmann connection $\mathcal{Q}$ we introduce an $*\mathcal{Q}$-holonomy group.

Let Ω_a be the set of horizontal curves with initial point a. Take an equivalence relation ρ on Ω_a as follows. Curves σ and σ' in Ω_a are said to be ρ-equivalent if there exist a vertical loop h_0 at a homotopic to the constant loop e_a in $L(a)$ and a v.h.h. H with the base (σ, h_0) such that $H|_{I_1\times\{1\}} = \sigma'$ (Fig. 3).

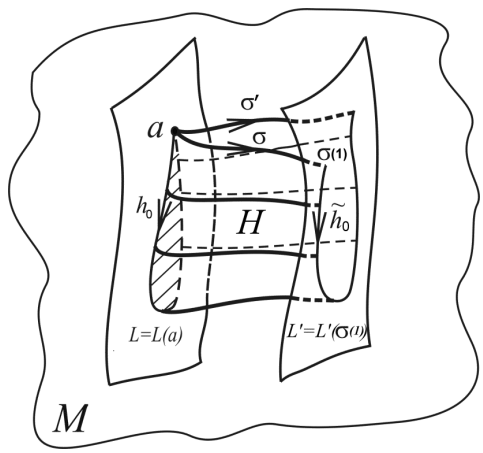

Fig. 3. The curves σ and σ' are ρ-equivalent

One can directly check that ρ is an equivalence relation in Ω_a. Let us denote by $[\sigma]_\rho$ the ρ-equivalence class of a curve σ, and by Ω_a/ρ the set of equivalence classes.

PROPOSITION 1. *The map*

$$\Phi_a\colon \Omega_a/\rho \times \pi_1(L,a) \to \Omega_a/\rho\colon ([\sigma]_\rho, [h]) \mapsto [\tilde{\sigma}]_\rho,$$

where $[h] \in \pi_1(L,a)$, *H is a v.h.h. with the base* (σ, h), *and* $\sigma \xrightarrow{H}> \tilde{\sigma}$, *defines a right action of the fundamental group* $\pi_1(L,a)$ *of* $L(a)$ *on the quotient set* Ω_a/ρ.

Proof. Let us demonstrate that Φ_a is correctly defined, i.e., does not depend on

1) the choice of the loop h in the class $[h] \in \pi_1(L,a)$ and the choice of the v.h.h. H with the base (σ, h);

2) the choice of σ in the class $[\sigma]_\rho$.

1. In a leaf L let us take paths h and h' joining a with b. Suppose that h is homotopic to h' in L, and σ is a curve in Ω_a. Let H, H' be v. h. hs with the bases (σ, h) and (σ, h'), respectively, and $\sigma \xrightarrow{H}> \tilde{\sigma}$, $\sigma \xrightarrow{H'}> \sigma'$. Let us prove that $[\tilde{\sigma}]_\rho = [\sigma']_\rho$. Since h is homotopic to h' in L, the loop $\phi_0 := h^{-1}h'$ is homotopic to the constant path e_b in L. Then we set

$$K(s,t) := \begin{cases} H(s, 1-2t), & \text{if } (s,t) \in I_1 \times [0, \frac{1}{2}]; \\ H'(s, 2t-1), & \text{if } (s,t) \in I_1 \times [\frac{1}{2}, 1], \end{cases}$$

and obtain that K is a v.h.h. with the base $(\tilde{\sigma}, \phi_0)$ such that $K|_{I_1\times\{1\}} = \sigma'$. Hence $[\tilde{\sigma}]_\rho = [\sigma']_\rho$.

2. Let $\sigma \in \Omega_a$, h be a vertical loop at a, and H be a v.h.h. with the base (σ, h). Take any path $\sigma^* \in [\sigma]_\rho$. Then there exists a v.h.h. K with the base (σ^*, h_0), where

$[h_0] = [e_a] \in \pi_1(L, a)$, which provides the ρ-equivalence between σ^* and σ, i.e., $K(s,0) = \sigma^*(s)$, $K(s,1) = \sigma(s)$, $s \in I_1$, and $K(0,t) = h_0(t)$, $t \in I_2$. Let us define the map $H^* : I_1 \times I_2 \to M$:

$$H^*(s,t) := \begin{cases} K(s,2t), & \text{if } (s,t) \in I_1 \times [0, \frac{1}{2}]; \\ H(s,2t-1), & \text{if } (s,t) \in I_1 \times [\frac{1}{2}, 1]. \end{cases}$$

Since $K(s,1) = H(s,0) = \sigma(s)$, $s \in I$, we have that H^* is a v.h.h. as well as K and H. The pair of paths $(\sigma^*, h_0 h)$ is the base of H^*. In addition, $\sigma^* \xrightarrow{H^*} \tilde{\sigma}$. Since $[h_0 h] = [h] \in \pi_1(L, a)$, from the definition of Φ_a we get that $\Phi_a([\sigma^*]_\rho, [h]) = [\tilde{\sigma}]_\rho$. Thus the map Φ_a is correctly defined.

One can easily check that Φ_a defines a right action of $\pi_1(L, a)$ on Ω_a/ρ. ∎

Definition 2. The kernel of the action Φ_a, $*K_{\mathcal{Q}}(L, x) := \ker \Phi_a = \{[h] \in \pi_1(L, a) \mid (\forall [\sigma]_\rho \in \Omega_a/\rho)\ \Phi_a([\sigma]_\rho, [h]) = [\sigma]_\rho\}$, is a normal subgroup in $\pi_1(L, a)$. Hence we have the quotient group

$$*H_{\mathcal{Q}}(L, a) := \pi_1(L, a)/\ker \Phi_a.$$

Let us call the group $*H_{\mathcal{Q}}(L, a)$ the *$*\mathcal{Q}$-holonomy group for the singular foliation with an Ehresmann connection* $(M, \mathcal{F}, \mathcal{Q})$.

As in [15], the $*$ indicates that $*H_{\mathcal{Q}}(L, a)$ is a holonomy group of foliation with singularities, which differs from the known holonomy groups. For a regular foliation each equivalence class $[\sigma]_\rho$ consists of one horizontal curve σ, hence the set Ω_a/ρ is bijective to the set Ω_a. In this case $*H_{\mathcal{Q}}(L, a)$ coincides with the group $H_{\mathcal{Q}}(L, a)$, which was defined in [3] (and was called the $\mathcal{Q}$-holonomy group in [28], [30]). It is not difficult to show that the following statement takes place.

Proposition 2. *For any points a and b in a leaf L of the foliation $(M, \mathcal{F}, \mathcal{Q})$ there exists an isomorphism $*H_{\mathcal{Q}}(L, a) \to *H_{\mathcal{Q}}(L, b)$, which is unique up to inner automorphisms of these groups.*

By Proposition 2 we can speak about the $*\mathcal{Q}$-holonomy group of a leaf L with understanding that this group is the algebraic group $*H_{\mathcal{Q}}(L)$ which does not depend on $a \in L$. We say that a leaf L has finite $*\mathcal{Q}$-holonomy group if this group is finite.

5. Subordinated charts. Let $(M, \mathcal{F}, Q)$ be a singular foliation with Ehresmann connection. Let us consider a fibred chart (W, φ) at a point a, $\varphi(W) = U \times V$. Then $\mathcal{F}_W := \{\varphi^{-1}(U \times \{y\}) \mid y \in V\}$ is a simple foliation in W, and let $\pi \colon W \to W/\mathcal{F}_W$ be the projection on the space of leaves. We say that the distribution $\mathcal{Q}$ has the property of *local transversal projectability* if at an arbitrary point $a \in M$ there exists a fibred chart (W, φ) with center a such that, for an arbitrary curve $\sigma \in \Omega_a$ in W at each point y in the local leaf l_a^W there exists a curve $\sigma_y \in \Omega_y$ smoothly depending on y and satisfying $\pi \circ \sigma_y = \pi \circ \sigma$.

After this we assume that for a foliation $(M, \mathcal{F}, Q)$ the Ehresmann connection $\mathcal{Q}$ has the property of local transversal projectability.

Definition 3. Let (W, φ) and (V, ψ) be two fibred charts with center in a and $b \in L(a)$ respectively of singular foliation $(M, \mathcal{F})$. We say that *chart (V, ψ) is subordinated to the*

chart (W, φ) if the following conditions are satisfied:

(i) $\varphi(W) = U_1 \times V_1$, $\psi(V) = U_2 \times V_2$, with $0 \in V_2 \subset V_1 \subset \mathbf{R}^q$, $q = n - p$, $p = p(a)$.
(ii) Let L be an arbitrary leaf of $\mathcal{F}$ such that $L \cap V \neq \emptyset$. Then for any connected component $c(L \cap V)$ of $L \cap V$ the set $\varphi^{-1}(\psi(c(L \cap V)))$ belongs to some connected component of the intersection $L \cap W$.

For regular foliations the notion of subordinated charts was introduced by R. Palais [13].

Let a, b be points of the same leaf $L \in \mathcal{F}$. Connect a with b by a path h in L and cover the set $h([0,1])$ by a finite chain of charts (W_k, φ_k) with center at $a_k = h(t_k)$, $t_k \in [0,1]$, $k = 0, \ldots, m$, where $(W_0, \varphi_0) = (W, \varphi)$, $a_0 = a$, $a_m = b$. Without loss of generality we can assume that the sets $W_k \cap W_{k+1} \cap h([0,1])$ are connected and an arbitrary local leaf of W_{k+1} intersects some local leaf of W_k.

By analogy with [13] using a local transversal containing a point $h(\tau_1) \in W_1 \cap W_2$, $\tau_1 \in [0,1]$, we change coordinates in the neighbourhood W_1 such that (W_1, φ_1) becomes a subordinated chart to (W_0, φ_0). If necessary we decrease W_1 to $V_1 \subset W_1$ such that (V_1, ψ_1) becomes a subordinated chart to (W_0, φ_0) with center a_1.

After this we construct a chart (V_2, ψ_2) with center a_2 subordinated to (V_1, ψ_1). Remark that the relation of subordination is transitive, so (V_2, ψ_2) is subordinated to (W, φ). By analogy in several steps we construct a chart (V, ψ) with center b subordinated to (W, φ). Thus we have the following theorem.

THEOREM 1. *Let $(M, \mathcal{F})$ be a singular foliation and (W, φ) be a fibred chart with center $a \in M$. For an arbitrary point $b \in L(a)$ there exists a fibred chart (V, ψ) with center b subordinated to the chart (W, φ).*

The analogous theorem for a regular foliation was proved by R. Palais [13].

Moving along h^{-1} from b to a we can construct fibred charts (V, χ) and (V', χ') with centers a and b such that each of them is subordinated to the other. In this case we shall call (V, χ) and (V', χ') *mutually subordinated charts.*

REMARK 2. Let (U, φ) and (V, ψ) be mutually subordinated charts. From the property of transversal projectibility and construction of a subordinated chart it follows that for any admissible pair (σ, h) with σ lying in U, where $\sigma(0) = h(0)$ is the center of U, there exists such v.h.h. H that $\sigma \xrightarrow{H} \tilde{\sigma}$, and $\tilde{\sigma}$ is contained in V.

6. Groupoid of $*\mathcal{Q}$-holonomy. Let $(M, \mathcal{F}, Q)$ be an arbitrary singular foliation with an Ehresmann connection $\mathcal{Q}$. We denote by $[h]$ the homotopic class of vertical paths containing h. Let x, y be both in a leaf L of $\mathcal{F}$. Denote by $A(x, y)$ the set of all piecewise smooth path from x to y in L. We use notations introduced above. Two paths $h_1, h_2 \in A(x, y)$ are called equivalent if and only if $[h_1 \cdot h_2^{-1}] \in *K_{\mathcal{Q}}(L, x)$, where $[h_1 \cdot h_2^{-1}] \in \pi_1(L, x)$.

In other words $h_1 \sim h_2$ if and only if the homotopy class $[h_1 \cdot h_2^{-1}]$ of the loop $h_1 \cdot h_2^{-1}$ acts on the factor set $\widetilde{\Omega}_x := \Omega_x / \rho$ in the trivial way. We denote by $\langle h \rangle$ the equivalence class containing h.

DEFINITION 4. We call the set

$$*G_{\mathcal{Q}}(\mathcal{F}) := \{(x, \langle h\rangle, y) \mid x \in M \ \wedge \ y \in L(x) \ \wedge \ h \in A(x,y)\}$$

a *graph of singular foliation* $(M, \mathcal{F})$ *with an Ehresmann connection* $\mathcal{Q}$.

The equality

$$(x, \langle h\rangle, y) \circ (z, \langle g\rangle, x) := (z, \langle g \cdot h\rangle, y),$$

where $g \cdot h$ is the product of paths g and h, defines a partial multiplication $\circ$ in $*G_{\mathcal{Q}}(\mathcal{F})$ relative to which $*G_{\mathcal{Q}}(\mathcal{F})$ is a groupoid called the $*\mathcal{Q}$-*holonomy groupoid* of this foliation.

The maps $p_1 \colon *G_{\mathcal{Q}}(\mathcal{F}) \to M \colon (x, \langle h\rangle, y) \mapsto x$ and $p_2 \colon *G_{\mathcal{Q}}(\mathcal{F}) \to M \colon (x, \langle h\rangle, y) \mapsto y$ are called *the source* and *range maps* or *canonical projections.*

Now we introduce a topology in $*G_{\mathcal{Q}}(\mathcal{F})$. Let $c = (a, \langle h\rangle, b)$ be any point in $*G_{\mathcal{Q}}(\mathcal{F})$ and (W, φ) and (W', φ') be arbitrary fibred neighbourhoods with centers a and b, respectively. Suppose that there is a horizontal curve σ in W, $\sigma \in \Omega_a$, and a transfer $\sigma \xrightarrow[h]{H}> \tilde{\sigma}$, such that there exists a curve $\sigma' \in [\tilde{\sigma}]_\rho \in \widetilde{\Omega}_b$, $\sigma'(I) \subset W'$. It is equivalent to the existence of a path h' homotopic to h and v.h.h. K such that $\sigma \xrightarrow[h']{K}> \sigma'$, $\sigma'(I) \subset W'$. Consider two arbitrary points x, y in local leaves $l^W_{\sigma(1)}$ and $l^{W'}_{\sigma'(1)}$ respectively. We take paths t_x in $l^W_{\sigma(1)}$ from x to $\sigma(1)$ and t_y in $l^{W'}_{\sigma'(1)}$ from y to $\sigma'(1)$. Let $h' \xrightarrow{K}> \tilde{h}$ and $g := t_x \cdot \tilde{h} \cdot t_y^{-1}$. We have an element $(x, \langle g\rangle, y)$ that belongs to $*G_{\mathcal{Q}}(\mathcal{F})$.

Denote by $\mathcal{V}(c, W, W')$ the set of all elements $(x, \langle g\rangle, y)$ of $*G_{\mathcal{Q}}(\mathcal{F})$ defined in the indicated way by c and the charts (W, φ) and (W', φ').

The following lemma will be useful later.

LEMMA 1. *If products of vertical paths* $h \cdot k$ *and* $k' \cdot g$ *have common ends, then the following equalities are equivalent*

$$\langle h \cdot k\rangle = \langle k' \cdot g\rangle \quad \textit{and} \quad \langle k'^{-1} \cdot h\rangle = \langle g \cdot k^{-1}\rangle.$$

PROPOSITION 3. *The set* $\Sigma := \{\mathcal{V}(c, W, W')\}$ *forms a base of some topology* τ *in* $*G_{\mathcal{Q}}(\mathcal{F})$ *relatively to which* $*G_{\mathcal{Q}}(\mathcal{F})$ *is a topological groupoid. A set* $\mathbf{L} := p_1^{-1}(L)$, $L \in \mathcal{F}$ *is open in the space* $*G_{\mathcal{Q}}(\mathcal{F})$ *if and only if* $\Omega_a = \{e_a\}$, *where* e_a *is the constant path at* $a \in L$.

Proof. Let $\mathcal{V}_1 := \mathcal{V}(c_1, W_1, W_1')$ and $\mathcal{V}_2 := \mathcal{V}(c_2, W_2, W_2')$ be two arbitrary sets from Σ, with $\mathcal{V}_1 \cap \mathcal{V}_2 \neq \emptyset$, and $c_1 := (a_1, \langle h\rangle, b_1)$, $c_2 := (a_2, \langle g\rangle, b_2)$. Consider an arbitrary point $z \in \mathcal{V}_1 \cap \mathcal{V}_2$. As $z \in \mathcal{V}_1$, so $z = (x, \langle t_x \cdot \tilde{h} \cdot t_y^{-1}\rangle, y)$ by construction of $\mathcal{V}_1$. As $z \in \mathcal{V}_2$, so $z = (x, \langle \tau_x \cdot \tilde{g} \cdot \tau_y^{-1}\rangle, y)$, with $x \in W_1 \cap W_2$, $y \in W_1' \cap W_2'$, there are $\sigma \in \Omega_{a_1}$, $\gamma \in \Omega_{a_2}$, $h' \in [h]$, $g' \in [g]$ and some transfers $h' \xrightarrow[\sigma]{H}> \tilde{h}$, $g' \xrightarrow[\gamma]{H'}> \tilde{g}$ (Fig. 4).

As six points $x, \tilde{h}(0), \tilde{h}(1), \tilde{g}(0), \tilde{g}(1)$ and y lie in the same leaf L' of $\mathcal{F}$, there exist pairwise mutually subordinated fibred neighbourhoods of these points contained in $W_1 \cap W_2$ and $W_1' \cap W_2'$, respectively. Denote the indicated neighbourhoods at x and y by $\tilde{W}_1 \ni x$ and $\tilde{W}_2 \ni y$.

Let us show the inclusion $\tilde{\mathcal{V}} := \mathcal{V}(z, \tilde{W}_1, \tilde{W}_2) \subset \mathcal{V}_1 \cap \mathcal{V}_2$. At first we check the inclusion $\tilde{\mathcal{V}} \subset \mathcal{V}_1$. Consider any point $w = (u, \langle t_u \cdot \tilde{k} \cdot t_v^{-1}\rangle, v)$ in $\tilde{\mathcal{V}}$, where $k := t_x \cdot \tilde{h} \cdot t_y^{-1}$ and there exists $\delta \in \Omega_x$ in $\tilde{W}_1$ such that $\tilde{k}$ is the result of some transfer $k' \xrightarrow[\delta]{H_1}> \tilde{k}$, $k' \in [k]$,

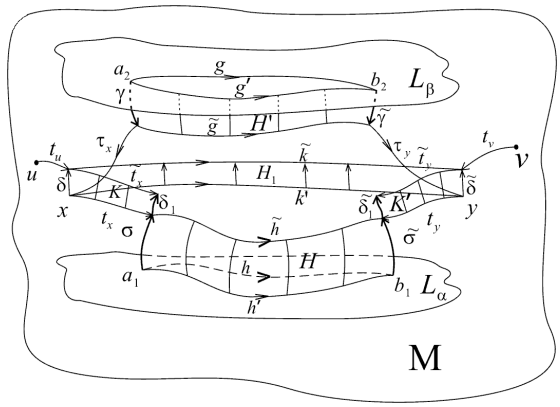

Fig. 4. A construction of $\tilde{\mathcal{V}} \subset \mathcal{V}_1 \cap \mathcal{V}_2$

with $\delta \xrightarrow{H_1}> \tilde{\delta}$, $\tilde{\delta} \in \Omega_y$ in $\tilde{W}_2$. According to the choice of $\tilde{W}_1$, $\tilde{W}_2$, there are some v.h.h. $\delta \xrightarrow[t_x]{K}> \delta_1$, $\tilde{\delta} \xrightarrow[t_y]{K'}> \tilde{\delta}_1$ in W_1 and W_2 respectively. Denote $t_x \xrightarrow{K}> \tilde{t}_x$, $t_y \xrightarrow{K'}> \tilde{t}_y$. We take Lemma 1 into account and have $\langle \tilde{h} \rangle = \langle t_x^{-1} \cdot k' \cdot t_y \rangle$. Hence we may put $\tilde{h}' := t_x^{-1} \cdot k' \cdot t_y$. Let us construct a v.h.h. $\tilde{K}$ by the formula

$$\tilde{K}(s,t) := \begin{cases} K(s, 1-4t), & t \in [0, \frac{1}{4}], \\ H_1(s, 4t-1), & t \in [\frac{1}{4}, \frac{1}{2}], \\ K'(s, 2t-1), & t \in [\frac{1}{2}, 1], \end{cases} \qquad s \in [0,1],$$

then $\delta_1 \xrightarrow{\tilde{K}}> \tilde{\delta}_1$ and $\tilde{h}' \xrightarrow{\tilde{K}}> \tilde{t}_x^{-1} \cdot \tilde{k} \cdot \tilde{t}_y = \hat{h}$ (Fig. 4).

So we have $w = (u, \langle \tau_u \cdot \hat{h} \cdot \tau_v^{-1} \rangle, v)$, where $\tau_u := t_u \cdot \tilde{t}_x$, $\tau_v := t_v \cdot \tilde{t}_y$. Therefore $w \in \mathcal{V}_1$.

The inclusion $\tilde{\mathcal{V}} \subset \mathcal{V}_2$ can be shown analogously. Thus Σ is a base of some topology τ in $*G_{\mathcal{Q}}(\mathcal{F})$.

The continuity of the partial multiplication in $*G_{\mathcal{Q}}(\mathcal{F})$ can be checked in the usual way.

Now we assume the existence of a point $a \in M$ such that $\Omega_a = \{e_a\}$. In this case for any $b \in L = L(a)$ and arbitrary fibred charts (W_1, φ_1) and (W_2, φ_2) with centers a, b, respectively, the set $\mathcal{V} = \mathcal{V}(c, W_1, W_2)$, $c = (a, \langle h \rangle, b)$, is formed by points $z = (x, \langle t_x \cdot \tilde{h} \cdot t_y^{-1} \rangle, y)$, with $x \in l_a^{W_1}$, $y \in l_b^{W_2}$, i.e. $\mathcal{V} \subset \mathbf{L} := p_1^{-1}(L)$. Therefore $\mathbf{L}$ is an open set in $*G_{\mathcal{Q}}(\mathcal{F})$.

Evidently, if $\mathbf{L}$ is an open set in $*G_{\mathcal{Q}}(\mathcal{F})$, then $\Omega_a = \{e_a\}$, $\forall a \in L$. ∎

Let $G_{\mathcal{Q}}(\mathcal{F})$ be the graph of a regular foliation $(M, \mathcal{F}, Q)$ with an Ehresmann connection introduced in [28] ([30], [33]).

PROPOSITION 4. *Let $(M, \mathcal{F}, Q)$ be a regular foliation with an Ehresmann connection. Then the holonomy groupoid $*G_{\mathcal{Q}}(\mathcal{F})$ coincides with $G_{\mathcal{Q}}(\mathcal{F})$.*

Proof. Let $\mathcal{V} = \mathcal{V}(c, W_1, W_2)$ be a neighbourhood of point $c = (a, \langle h \rangle, b)$ from the base Σ of topology in $*G_{\mathcal{Q}}(\mathcal{F})$. Consider an arbitrary point $z = (x, \langle t_x \cdot \tilde{h} \cdot t_y^{-1} \rangle, y)$ in $\mathcal{V}$, defined by a path $h' \in [h]$, a horizontal curve σ in W_1 and v.h.h. H, with $\sigma \xrightarrow[h']{H}> \tilde{\sigma}$. Recall that

t_x and t_y are paths in local leaves from x to $\sigma(1)$ and y to $\tilde{\sigma}(1)$ respectively. Let γ be another horizontal curve in W_1 and K be a v.h.h. with the base (γ, h''), $h'' \in [h]$, with $\gamma \overset{K}{\underset{h''}{\longrightarrow}} \tilde{\gamma}$, $h'' \overset{K}{\longrightarrow} \hat{h}$, and $z = (x, \langle \tau_x \cdot \hat{h} \cdot \tau_y^{-1} \rangle, y)$. In the regular case as all local leaves of fibred neighbourhoods are simply connected, the vertical paths $t_x \cdot \tilde{h} \cdot t_y^{-1}$ and $\tau_x \cdot \hat{h} \cdot \tau_y^{-1}$ are homotopy equivalent. Therefore the point z does not depend on the choice of the curves γ and t_x, t_y.

Thus in regular case the topology in $*G_{\mathcal{Q}}(\mathcal{F})$ defined by Proposition 3 coincides with the topology in the graph $G_{\mathcal{Q}}(\mathcal{F})$ introduced in the usual way [30]. ∎

7. Simple transversal bifibrations

DEFINITION 5. Let $p_i \colon M \to B_i$, $i = 1, 2$, be a submersion with connected fibres. Suppose that the universal covering manifold of M is a product of two manifolds $X_1 \times X_2$. If the universal covering map $\kappa \colon X_1 \times X_2 \to M$ transforms fibres of the trivial fibrations F_i, where $F_1 := \{X_1 \times \{v\} \mid v \in X_2\}$, $F_2 := \{\{u\} \times X_2 \mid u \in X_1\}$, to corresponding fibres of the submersion p_i then the quintuple (M, p_1, p_2, B_1, B_2) is called a *simple transversal bifibration* by Ya. Shapiro and the author [18].

Remark that two submersions $p_i \colon M \to B_i$, $i = 1, 2$, of M with connected fibres form a simple transversal bifibration if and only if the distribution $\mathcal{Q}_1$ tangent to fibres of $p_2 \colon M \to B_2$ is an Ehresmann connection for the submersion $p_1 \colon M \to B_1$.

It is well known that a diffeomorphism group Ψ of manifold N is a group of covering transformations of some regular covering map $f \colon N \to B$ onto the factor manifold $N/\Psi = B$ if and only if Ψ acts freely and properly discontinuously, i.e. for $x, y \in N$, where y does not belong to the orbit $\Psi \cdot x$ of x, there exist two neighbourhoods V and W of x and y respectively such that for any $\psi \in \Psi$, $\psi \neq \mathrm{id}_N$, $\psi(V) \cap W = \emptyset$ holds. The next proposition follows from the work of Ya. Shapiro and the author [18].

PROPOSITION 5. *Let (M, p_1, p_2, B_1, B_2) be a simple transversal bifibration. Let x be an arbitrary fixed point in M, $x_i := p_i(x) \in B_i$ and $Y_i := p_i^{-1}(x_i)$, where $i = 1, 2$. Then we have the following:*

(i) $p_i \colon M \to B_i$ is a locally trivial fibration with a standard fiber Y_i.

(ii) The map $p \colon M \to B_1 \times B_2 \colon x \mapsto (p_1(x), p_2(x))$ is an regular covering map with the group Ψ of covering transformations.

(iii) The restrictions $p_1|_{Y_2} \colon Y_2 \to B_1$ and $p_2|_{Y_1} \colon Y_1 \to B_2$ are regular covering maps with groups Ψ_2 and Ψ_1 of covering transformations respectively, and there exist group isomorphisms $\theta_i \colon \Psi \to \Psi_i$.

(iv) The action of Ψ on the product $Y_1 \times Y_2$ given by the rule

$$\psi(y_1, y_2) := (\theta_1(\psi) y_1, \theta_2(\psi) y_2), \ \forall (y_1, y_2) \in Y_1 \times Y_2,$$

is free and properly discontinuous. It defines a simple transversal bifibration $((Y_1 \times Y_2)/\Psi, f_1, f_2, B_1, B_2)$, where $f_i \colon \Psi \cdot (y_1, y_2) \mapsto \Psi_i \cdot y_i$, $(y_1, y_2) \in Y_1 \times Y_2$, and there is a diffeomorphism Ξ such that the following diagram is commutative

$$\begin{array}{ccccc} B_1 & \xleftarrow{p_1} & M & \xrightarrow{p_2} & B_2 \\ & {\scriptstyle f_1}\nwarrow & \downarrow \Xi & \nearrow{\scriptstyle f_2} & \\ & & (Y_1 \times Y_2)/\Psi. & & \end{array}$$

(v) The inclusion $j\colon Y_i \to M$ induces an isomorphism of the fundamental group $\pi_1(Y_i, x)$ onto a normal subgroup G_{ii} of $\pi_1(M, x)$, and Ψ is isomorphic to the factor group $\pi_1(M, x)/(G_{11} \times G_{22})$.

DEFINITION 6. The algebraic group Ψ satisfying conditions of Proposition 5 is called the *structure group* of this bifibration.

The above notion is well defined, i.e. it does not depend on the choice of the canonical bifibration $((Y_1 \times Y_2)/\Psi, f_1, f_2, B_1, B_2)$ satisfying the conditions of Proposition 5. Remark that each submersion p_i forms the fibration with the structure group Ψ.

DEFINITION 7. A bifibration (M, p_1, p_2, B_1, B_2) is called *symmetric* if the fibrations $p_i\colon M \to B_i$, $i = 1, 2$, are isomorphic (in the category of fibrations).

PROPOSITION 6. *Let $*G_{\mathcal{Q}}(\mathcal{F})$ be the holonomy groupoid of a singular foliation with an Ehresmann connection $(M, \mathcal{F}, Q)$ and $p_1, p_2\colon *G_{\mathcal{Q}}(\mathcal{F}) \to M$ be the canonical projections. Let x be an arbitrary point in M, $L = L(x)$ and $\mathbf{L} := p_1^{-1}(L)$. The quintuple $(\mathbf{L}, p_1|_{\mathbf{L}}, p_2|_{\mathbf{L}}, L, L)$ is a symmetric simple transversal bifibration with the structure group Ψ, that is isomorphic to the holonomy group $*H_{\mathcal{Q}}(L)$.*

Proof. Let us introduce a smooth structure in $\mathbf{L} := p_1^{-1}(L)$, $L \in \mathcal{F}$. Let $c = (a, \langle h\rangle, b)$ be an arbitrary point in $\mathbf{L}$. Let (W_1, φ_1) and (W_2, φ_2) be an arbitrary mutually subordinated charts with centers a, b respectively and $\mathcal{V} := \mathcal{V}(c, W_1, W_2)$. Denote $\mathcal{U} := \mathcal{V} \cap \mathbf{L}$, $p = \dim L$. The map $\chi\colon \mathcal{U} \to \mathbf{R}^p \times \mathbf{R}^p\colon (x, \langle t_x \cdot \tilde{h} \cdot t_y^{-1}\rangle, y) \mapsto (\varphi_1(x), \varphi_2(y))$ is a homeomorphism onto $\varphi_1(W_1) \times \varphi_2(W_2) = \mathbf{R}^p \times \mathbf{R}^p$. The set of all these pairs $\{(\mathcal{U}, \chi)\}$ forms a C^∞-atlas of $2p$-dimensional manifold in $\mathbf{L}$. The restrictions $p_1|_{\mathbf{L}}\colon \mathbf{L} \to L$ and $p_2|_{\mathbf{L}}\colon \mathbf{L} \to L$ become submersions onto L.

We show that fibres of p_i, $i = 1, 2$, are arc-wise connected. Take two arbitrary points $c_1 = (a, \langle h\rangle, b)$ and $c_2 = (a, \langle g\rangle, d)$ in the fibre $p_1^{-1}(a)$. As the formula

$$k_t(s) := \begin{cases} h(s(1-2t)), & t \in [0, \frac{1}{2}], \\ g(s(2t-1)), & t \in [\frac{1}{2}, 1], \end{cases} \qquad s \in [0, 1]$$

defines a homotopy of maps h and g in L, so $\mu(t) := (a, \langle k_t\rangle, (h^{-1} \cdot g)(t))$ is a path in $p_1^{-1}(a)$ from c_1 to c_2. Analogously, fibres of p_2 are also arc-wise connected.

Let $\nu(t)$ be an arbitrary path in L, and $\nu(0) = b$. An arbitrary point $z \in p_2^{-1}(b)$ has a form $z = (a, \langle h\rangle, b)$, $a \in L$. A homotopy $\kappa_t(s) := (h \cdot \nu|_{[0,t]})(s)$, $(t, s) \in I \times I$, $I = [0, 1]$, is situated in L. Hence $z(t) := (a, \langle \kappa_t\rangle, \nu(t))$ is a path in the fibre $p_1^{-1}(a)$ such that $z = z(0)$ and $p_2 \circ z(t) = \nu(t)$, $t \in I$. Therefore the distribution $\mathcal{P}$ formed by tangent spaces to leaves of the submersion $p_1|_{\mathbf{L}}\colon \mathbf{L} \to L$ is an integrable Ehresmann connection for the submersion $p_2|_{\mathbf{L}}$.

Thus $(\mathbf{L}, p_1|_{\mathbf{L}}, p_2|_{\mathbf{L}}, L, L)$ is a simple transversal bifibration. The symmetry of the projections $p_1, p_2\colon *G_{\mathcal{Q}}(\mathcal{F}) \to M$ gives rise to the symmetry of the restrictions $p_1|_{\mathbf{L}}$ and $p_2|_{\mathbf{L}}$.

According to Proposition 5 (ii) the map $p\colon \mathbf{L} \to L \times L\colon (x, \langle h\rangle, y) \mapsto (x, y)$ is a regular covering map. Using the definition of $\mathbf{L} \subset {*G_{\mathcal{Q}}}(\mathcal{F})$ it is not difficult to check that the group of the covering transformations Ψ of p is isomorphic to ${*H_{\mathcal{Q}}}(L)$. ∎

8. Generalized Ehresmann connection for foliation with singularities and its holonomy. Let $(M, \mathcal{F})$ be a foliation with singularities and M^0 be the union of leaves of maximal dimension. Let $\mathcal{Q}$ be a generalized distribution on M which is transversal to the foliation $\mathcal{F}$ (we use terminology introduced in Section 2). $\mathcal{Q}$ is called a *generalized Ehresmann connection* of $\mathcal{F}$ if for any admissible pair of paths (σ, h), where $\sigma((0,1)) \subset M^0$, there exists a v.h.h. with the base (σ, h). Thus, we extend the notion of Ehresmann connection for foliations with singularities by considering only horizontal paths without singular points except for the endpoints.

Let $(M, \mathcal{F}, \mathcal{Q})$ be a singular foliation with a generalized Ehresmann connection. Then the distribution $\mathcal{Q}^0 := \mathcal{Q}|_{M^0}$ is an Ehresmann connection for the regular foliation $(M^0, \mathcal{F}^0)$, where $\mathcal{F}^0 := \mathcal{F}|_{M^0}$. Let $\Omega_a^0 := \{\sigma \in \Omega_a \mid \sigma((0,1]) \subset M^0\}$ and $\Omega_a^* := \{\sigma \in \Omega_a \mid \sigma((0,1)) \subset M^0\}$ then $\Omega_a^0 \subset \Omega_a^*$. If $\sigma \in \Omega_a^0$, each path σ' which is ρ-equivalent to σ also lies in Ω_a^0. Therefore $\Omega_a^0/\rho \subset \Omega_a^*/\rho$. For any admissible pair of paths (σ, h), where $\sigma \in \Omega_a^*$, there exists a v.h.h. with the base (σ, h). Let $\Phi_a^*([\sigma]_\rho, [h]) := [\widetilde{\sigma}]_\rho$, where $[h] \in \pi_1(L, a)$, $\sigma \xrightarrow{H}> \widetilde{\sigma}$. Similar to the proof of Proposition 1, one can check that Φ_a^* defines a right action of the fundamental group $\pi_1(L, a)$ on the quotient set Ω_a^*/ρ. For a singular foliation with a generalized Ehresmann connection $(M, \mathcal{F}, \mathcal{Q})$, we call the quotient group $H_{\mathcal{Q}}(L, a) := \pi_1(L, a)/\ker \Phi_a^*$ the *$\mathcal{Q}$-holonomy group of the leaf L at $a \in L$.*

In the case of a regular foliation $(M, \mathcal{F})$ with an Ehresmann connection $\mathcal{Q}$ we have $M = M^0$ and the group $H_{\mathcal{Q}}(L_\alpha, x)$ coincides with the $\mathcal{Q}$-holonomy group of L_α at $x \in L_\alpha$.

As $\ker \Phi_a^* \subset \ker \Phi_a$, we have the following assertion.

PROPOSITION 7. *For any leaf L of a singular foliation $(M, \mathcal{F}, \mathcal{Q})$ with a generalized Ehresmann connection there exists a group epimorphism $\nu : {*H_{\mathcal{Q}}}(L, a) \to H_{\mathcal{Q}}(L, a)$ such that the diagram*

$$\begin{array}{ccc} & \pi_1(L,a) & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ {*H_{\mathcal{Q}}}(L,a) & \xrightarrow{\nu} & H_{\mathcal{Q}}(L,a), \end{array}$$

where α and β are the quotient maps, is commutative.

Since M^0 is an open (possibly disconnected) submanifold in M, for any leaf $L \subset M^0$ we have the germ holonomy group $\Gamma(L, a)$, $a \in L$, which is widely used in the foliation theory. Evidently, there exists a natural group epimorphism $\chi\colon H_{\mathcal{Q}}(L, a) \to \Gamma(L, a)$ such that $\chi \circ \beta = \gamma$, where $\gamma\colon \pi_1(L, a) \to \Gamma(L, a)$ is the projection, which maps $[h] \in \pi_1(L, a)$ to the germ of holonomy diffeomorphism corresponding to the path h. The following statement establishes a relation between different holonomy groups of a regular leaf L.

COROLLARY 1. *For any leaf $L \subset M^0$ of singular foliation with generalized Ehresmann connection the diagram*

$$\begin{array}{ccccc} & & \pi_1(L,a) & & \\ & \overset{\alpha}{\swarrow} & \downarrow \beta & \overset{\gamma}{\searrow} & \\ *H_{\mathcal{Q}}(L,a) & \xrightarrow{\nu} & H_{\mathcal{Q}}(L,a) & \xrightarrow{\chi} & \Gamma(L,a), \end{array}$$

where α, β, γ are the quotient maps, and ν, χ are the group epimorphisms, is commutative.

If for an admissible pair of paths (σ, h) there exists a *unique* v.h.h. H with the base (σ, h), $\sigma \xrightarrow{H}> \widetilde{\sigma}$, $h \xrightarrow{H}> \widetilde{h}$, we say that the curve $\widetilde{\sigma}$ is obtained by *transfer of σ along h*, the path $\widetilde{h}$ *is obtained by transfer of h along σ*, and denote it by $\sigma \xrightarrow{h}> \widetilde{\sigma}$, $h \xrightarrow{\sigma}> \widetilde{h}$, as in the regular case.

9. $\mathcal{Q}$-holonomy groupoid of foliations with generalized Ehresmann connections. Let $(M, \mathcal{F}, Q)$ be a foliation with generalized Ehresmann connection. If in the definition of $*G_{\mathcal{Q}}(\mathcal{F})$ we use the holonomy groups $H_{\mathcal{Q}}(L,x)$, $L \in \mathcal{F}$, instead of the groups $*H_{\mathcal{Q}}(L,x)$ we shall get a groupoid $G_{\mathcal{Q}}(\mathcal{F}) = \{(x, \{h\}, y)\}$, which will be called a $\mathcal{Q}$-*holonomy groupoid*. Topology in $G_{\mathcal{Q}}(\mathcal{F})$ is introduced similarly to the groupoid $*G_{\mathcal{Q}}(\mathcal{F})$.

PROPOSITION 8. *Let $(M, \mathcal{F}, Q)$ be a foliation with generalized Ehresmann connection. Then for any admissible pair (σ, h), where $\sigma([0,1)) \subset M^0$ there exists a unique vertical horizontal homotopy H with the base (σ, h).*

Proof. Given the horizontal curve $\sigma \in \Omega_b$ with $\sigma([0,1)) \subset M^0$, take a path h in the leaf $L(b)$ such that $h(0) = b$. Since $\mathcal{F}$ is a foliation with a generalized Ehresmann connection $\mathcal{Q}$, there exists a v.h.h. H with the base (σ, h). Suppose that there exists another v.h.h. K with the same base (σ, h). For any $\tau \in (0,1)$, we have $\sigma|_{[0,\tau]} \in \Omega_b^0$. In M^0 there exists the unique v.h.h. with the base $(\sigma|_{[0,\tau]}, h)$, hence $H|_{[0,\tau]\times I_2} = K|_{[0,\tau]\times I_2}$, and then $H|_{[0,1)\times I_2} = K|_{[0,1)\times I_2}$. Since H and K are continuous, and M is Hausdorff, we obtain that $H = K$. ∎

The following lemma is analogous to Lemma 1.

LEMMA 2. *If products of vertical paths $h \cdot k$ and $k' \cdot g$ have common ends, then the following equalities are equivalent:*

$$\{h \cdot k\} = \{k' \cdot g\} \quad \textit{and} \quad \{k'^{-1} \cdot h\} = \{g \cdot k^{-1}\}.$$

Let us consider the following conditions for a singular foliation $(M, \mathcal{F}, Q)$ with a generalized Ehresmann connection.

(P_1) The union M^0 of leaves of maximal dimension is connected and dense in M.

(P_2) For each point $c = (a, \{h\}, b)$ in $G_{\mathcal{Q}}(\mathcal{F})$ there is a neighbourhood $\mathcal{V} = \mathcal{V}(c, W_1, W_2)$ from the base Σ of a topology in $G_{\mathcal{Q}}(\mathcal{F})$ defined above satisfying the following conditions:

(i) There is a dense saturated subset M_0^0 in M^0 such that a point $z = (x, \{t_x \cdot \tilde{h} \cdot t_y^{-1}\}, y) \in \mathcal{V}$, where $x, y \in L_\alpha \subset M_0^0$ does not depend on the choice of paths σ, t_x, t_y and v.h.h. with the base (σ, h'), $h' \in [h]$ used by us in the construction of z, i.e. z is defined only by x, y and $\{h\}$.

(ii) There exists a fibred neighbourhood $\tilde{W}$ at point b, $W_2 \subset \tilde{W}$ with the following property. Let (δ^{-1}, k) be an arbitrary admissible pair of paths in W_2, where $\delta \in \Omega_b^0$ and $k([0,1]) \subset M_0^0$. Let $\tilde{k}$ be the result of the unique transfer $k \xrightarrow{\delta^{-1}}> \tilde{k}$ according to Proposition 8, with $\tilde{k}$ being a loop. Then the loop $\tilde{k}$ lies in $\tilde{W}$, i.e. $\tilde{k}(t) \in \tilde{W}$, $\forall t \in [0,1]$.

THEOREM 2. *Let $(M, \mathcal{F}, Q)$ be a foliation with generalized Ehresmann connection satisfying (P_1) and (P_2). Then the following properties are fulfilled:*

(i) The space of $\mathcal{Q}$-holonomy groupoid $G_{\mathcal{Q}}(\mathcal{F})$ is Hausdorff.

(ii) There exists a natural smooth structure on $\mathbf{L} := p_1^{-1}(L)$, $L \in \mathcal{F}$, and $\dim \mathbf{L} = 2p$, $p = \dim L$, and the quintuple $(\mathbf{L}, p_1|_{\mathbf{L}}, p_2|_{\mathbf{L}}, L, L)$ is a symmetric simple transversal bifibration, and its structure group Ψ is isomorphic to the holonomy group $H_{\mathcal{Q}}(L)$.

(iii) The map $f\colon *G_{\mathcal{Q}}(\mathcal{F}) \to G_{\mathcal{Q}}(\mathcal{F})\colon (x, \langle h \rangle, y) \mapsto (x, \{h\}, y)$ is a natural epimorphism of groupoids.

Proof. (i). Consider two different points $c_1 = (a_1, \{h_1\}, b_1)$ and $c_2 = (a_2, \{h_2\}, b_2)$ in $G_{\mathcal{Q}}(\mathcal{F})$. If $a_1 \neq a_2$ (or $b_1 \neq b_2$) there are fibred neighbourhoods $W_1 \ni a_1$, $W_1' \ni a_2$ such that $W_1 \cap W_1' = \emptyset$. Hence $\mathcal{V}(c_1, W_1, W_2) \cap \mathcal{V}(c_2, W_1', W_2') = \emptyset$.

Suppose that there are two different points $c_1 = (a, \{h\}, b)$ and $c_2 = (a, \{g\}, b)$ which are not separated. As by properties (P_1) and (i) of (P_2) the set M_0^0 is dense in M, there exists a sequence $\{z_n\} \subset p_1^{-1}(M_0^0)$ which converges both to c_1 and c_2. Hence for arbitrary fibred neighbourhoods $W_1 \ni a$ and $W_2 \ni b$ in M there is a member of the sequence $z_m \in \mathcal{V}_1 \cap \mathcal{V}_2 \cap p_1^{-1}(M_0^0)$, where $\mathcal{V}_1 := \mathcal{V}(c_1, W_1, W_2)$ and $\mathcal{V}_2 := \mathcal{V}(c_2, W_1, W_2)$. Consider $\mathcal{V}_1$ and $\mathcal{V}_2$ satisfying the assumption (P_2).

By the definition $\mathcal{V}_1$ point $z_m \in \mathcal{V}_1 \cap p_1^{-1}(M_0^0)$ if and only if the following conditions are satisfied: 1) there is a horizontal curve $\sigma \in \Omega_a^0$ in W_1; 2) there exists a vertical path $h' \in [h]$ and a v.h.h. H such that the result $\tilde{\sigma}$ of the transfer $\sigma \xrightarrow[h']{H}> \tilde{\sigma}$, $h' \xrightarrow[\sigma]{H}> \tilde{h}$; 3) $z_m = (x, \{t_x \cdot \tilde{h} \cdot t_y^{-1}\}, y)$. Similarly $z_m = (x, \{\tau_x \cdot \tilde{g} \cdot \tau_y^{-1}\}, y)$, where $\gamma \in \Omega_a^0$ in W_1 and there exist $g' \in [g]$ and v.h.h. H' with the base (γ, g') such that $\gamma \xrightarrow[g']{H'}> \tilde{\gamma}$, $g' \xrightarrow[\gamma]{H'}> \tilde{g}$.

By the property (i) of (P_2) without loss of generality we can assume that $\sigma = \gamma$ and $t_x = \tau_x$.

According to Lemma 2 the equality $\{t_x \cdot \tilde{h} \cdot t_y^{-1}\} = \{t_x \cdot \tilde{g} \cdot \tau_y^{-1}\}$ is equivalent to $\{\tilde{h}\} = \{\tilde{g} \cdot k\}$, where $k := \tau_y^{-1} \cdot t_y$.

By Proposition 8 there exists a unique v.h.h. H with the base $(\sigma^{-1}, \tilde{h})$. As $\sigma \in \Omega_a^0$, so the equivalent class $[\tilde{\sigma}^{-1}]_\rho$ contains only one curve $\tilde{\sigma}^{-1}$. From this fact and the equality $\{\tilde{h}\} = \{\tilde{g} \cdot k\}$ it follows that the results of the transfers σ^{-1} along $\tilde{h}$ and along $\tilde{g} \cdot k$ are the same and equal to $\tilde{\sigma}^{-1}$. Denote by K the unique v.h.h. with the base $(\tilde{\gamma}^{-1}, k)$. Let $k \xrightarrow[\gamma^{-1}]{K}> \tilde{k}$, then $\tilde{k}$ is a loop in b (see Fig. 5). According to the property (ii) of (P_2) the path $\tilde{k}$ lies in fibred neighbourhood $\tilde{W}$, hence $\tilde{k}$ is a loop in b belonging to the local leaf $l_b^{\tilde{W}}$. As $l_b^{\tilde{W}}$ is diffeomorphic to $\mathbf{R}^q$, the loop $\tilde{k}$ is homotopic to the constant path e_b and $\{g' \cdot \tilde{k}\} = \{g'\} = \{g\}$. Therefore we have $\tilde{g} \cdot k \xrightarrow{\sigma^{-1}}> g' \cdot \tilde{k}$. Denote $\hat{g} := g' \cdot \tilde{k}$. Consider

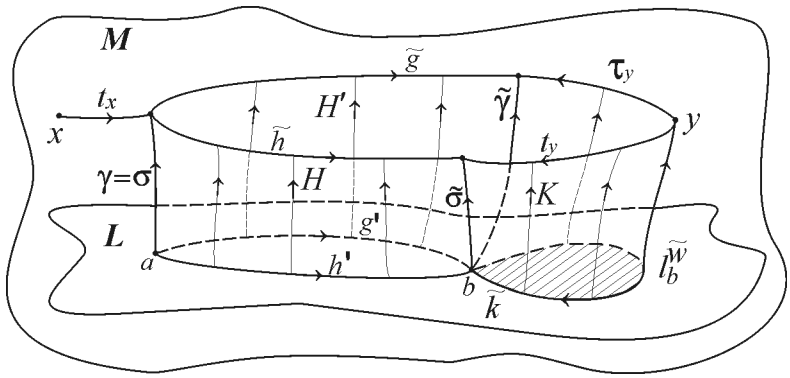

Fig. 5. $\{\tilde{k}\} = \{e_b\}$ and $\{g' \cdot \tilde{k}\} = \{g''\}$

an arbitrary curve $\delta \in \Omega_a$. Let $\delta \xrightarrow[h']{\Psi} \delta_1$, $\delta \xrightarrow[g' \cdot \tilde{k}]{\Phi} \delta_2$. V.h.hs Ψ and Φ induce transfers of the product of path $\sigma^{-1} \cdot \delta$ along $\tilde{h}$ and along $\tilde{g} \cdot k$ with results $\tilde{\sigma}^{-1} \cdot \delta_1$ and $\tilde{\sigma}^{-1} \cdot \delta_2$, respectively. As $\{\tilde{h}\} = \{\tilde{g} \cdot k\}$ so $[\tilde{\sigma}^{-1} \cdot \delta_1]_\rho = [\tilde{\sigma}^{-1} \cdot \delta_2]_\rho$, hence $[\delta_1]_\rho = [\delta_2]_\rho$. Therefore $\{h\} = \{g' \cdot \tilde{k}\} = \{g\}$, i.e. $c_1 = c_2$ that is in contradiction with the assumption. Thus $G_{\mathcal{Q}}(\mathcal{F})$ is Hausdorff.

The proof of the statement (ii) is similar to the proof of Proposition 6.

Applying Proposition 7 we get the assertion (iii). ∎

COROLLARY 2. *Let $(M, \mathcal{F}, Q)$ be a regular foliation with an Ehresmann connection. Then the manifold of the graph $G_{\mathcal{Q}}(\mathcal{F})$ is always Hausdorff.*

Proof. From the proof of Proposition 4 it follows that the condition (i) of (P_2) is fulfilled. In regular case in (ii) of (P_2) as $\tilde{k}$ is a loop, k is also a loop in $l_y^{W_2}$ and $\tilde{k}$ lies in $l_b^{W_2}$. Hence the condition (ii) of (P_2) is true for $\tilde{W} = W_2$. Thus the conditions (P_1) and (P_2) are satisfied. ∎

REMARK 3. The Hausdorff topological space $G_{\mathcal{Q}}(\mathcal{F})$ has a partition into manifolds $\{p_1^{-1}(L) \mid L \in \mathcal{F}\}$.

10. A global stability of compact leaves with finite $\mathcal{Q}$-holonomy. In this section we consider singular foliations with generalized Ehresmann connections $(M, \mathcal{F}, Q)$.

Consider the following condition:

(P_1^*) The set M^0 of regular points is connected and for each singular leaf L there is a horizontal curve $\sigma\colon [0,1] \to M$ such that $\sigma(0) \in L$ and $\sigma((0,1]) \subset M^0$.

Remark that the assumption (P_1^*) implies (P_1). The following theorem is proved by us in [34].

THEOREM 3. *Let $(M, \mathcal{F}, Q)$ be a singular foliation with generalized Ehresmann connection, satisfying the condition (P_1^*). If there exists a compact leaf L_0 in M^0 with a finite holonomy group $H_{\mathcal{Q}}(L_0)$ (or $\pi_1(L_0)$), then each leaf $L_\alpha \in \mathcal{F}$ is compact and has a finite holonomy group $H_{\mathcal{Q}}(L_\alpha)$ ($\pi_1(L_\alpha)$ respectively).*

This theorem can be considered as an analog of the famous Reeb theorem on global stability of a compact leaf L_0 with a finite group $\pi_1(L_0)$ of C^r-differentiable, $r \geq 2$, regular foliation of codimension one of a compact manifold.

11. Singular Riemannian foliations. Let $\mathcal{F}$ be a singular foliation on a Riemannian manifold (M, g). The foliation $\mathcal{F}$ is called a Riemannian foliation if any geodesic of g orthogonal to $\mathcal{F}$ at one point is orthogonal to $\mathcal{F}$ at any point of its domain. A foliated Riemannian manifold $(M, \mathcal{F}, g)$ is called transversally complete if the natural parameter on each horizontal geodesic changes over $(-\infty, +\infty)$.

The two following theorems belong to the author [29].

THEOREM 4. *Let $(M, \mathcal{F}, g)$ be a transversally complete singular Riemannian foliation. Then:*

1. The generalized orthogonal distribution $\mathcal{Q} = (T\mathcal{F})^{\perp}$ to $\mathcal{F}$ is an Ehresmann connection for $\mathcal{F}$.

2. The transfers with respect to $\mathcal{Q}$ keep the length of horizontal geodesics, i.e. if $\gamma \xrightarrow[h]{H} \tilde{\gamma}$ then $l(\tilde{\gamma}) = l(\gamma)$.

A singular leaf L is called *tractable* if L has a saturated neighbourhood W which satisfies the requirements:

1) W is isomorphic to a fibre bundle over L with the standard fibre V (transversal to $\mathcal{F}$) endowed with the induced foliation Δ_V;

2) the structure group of this fibre bundle is the group of automorphisms of the foliation Δ_V, and the foliation on W, which is given locally as product of the leaf L by Δ_V, coincides with $\mathcal{F}|_W$.

THEOREM 5. *For a leaf L of a transversally complete singular Riemannian foliation $\mathcal{F}$ the following conditions are equivalent:*

1. L is proper;

2. L is closed as subset of M;

3. L is tractable;

4. L is locally stable.

COROLLARY 3. *Let $\mathcal{F}$ be as in Theorem 4 and M be compact. Then each tractable leaf L of $\mathcal{F}$ is also compact.*

For regular Riemannian foliations Theorem 5 was proved by the author in [26]. Under an additional assumption about the existence of a complementary topological foliation $\mathcal{F}_1$, a local stability of a proper leaf had been proved by C. Ehresmann [6]. A parallel foliation on a Riemannian manifold admits the complementary orthogonal parallel foliation. Therefore such foliations are always regular and form a subclass of the class of previous foliations. For parallel foliations the equivalence of the conditions 1. – 3. was proved in [9], where the proof was considerably simpler, due to specificity of the case.

12. The $\mathcal{Q}$-holonomy groupoid of a singular Riemannian foliation. In this section let $(M, \mathcal{F}, g)$ be a transversally complete singular Riemannian foliation. It has a natural Ehresmann connection $\mathcal{Q} = (T\mathcal{F})^{\perp}$. Let us consider $\mathcal{Q}$ as a generalized Ehresmann connection, where piece-wise horizontal geodesics play the role of horizontal curves.

Let $(M, \mathcal{F})$ be an orbit-like foliation in the sense of Molino relative to a smooth action of the compact group $O(m)$. It follows from the theory of compact transformation groups that the manifold M^0 of regular points is open connected and dense in M. Therefore for any transversally complete orbit-like singular Riemannian foliation $(M, \mathcal{F}, Q)$ conditions (P_1) and (P_1^*) are fulfilled and Theorem 3 is true.

THEOREM 6. *Let $(M, \mathcal{F}, Q)$ be a singular transversally complete Riemannian foliation with a natural generalized Ehresmann connection $\mathcal{Q} = (T\mathcal{F})^{\perp}$, and suppose the condition (P_1^*) is satisfied. Then the $\mathcal{Q}$-holonomy groupoid $G_{\mathcal{Q}}(\mathcal{F})$ is Hausdorff.*

Proof. It is sufficient to check that property (P_2) is satisfied and to apply Theorem 2. Let us denote by M_0^0 the set of all leaves L_α of the regular foliation $\mathcal{F}^0 := \mathcal{F}|_{M^0}$ with the trivial germ holonomy group $\Gamma(L_\alpha, x)$. According to the result of Epstein–Millett–Tischler [7] M_0^0 is a dense G_δ-subset in M^0. As was shown by the author [27] there exists a canonical isomorphism of the groups $\Gamma(L_\alpha, x)$ and $H_{\mathcal{Q}^0}(L_\alpha, x)$ of the regular Riemannian foliation $(M^0, \mathcal{F}^0, \mathcal{Q}^0)$ with the Ehresmann connection $\mathcal{Q}^0 := \mathcal{Q}|_{M^0}$. Therefore the $\mathcal{Q}^0$-holonomy group $H_{\mathcal{Q}^0}(L_\alpha, x)$ is also trivial.

Consider any point $c = (a, \{h\}, b) \in G_{\mathcal{Q}}(\mathcal{F})$ and its neighbourhood $\mathcal{V} = \mathcal{V}(c, W_1, W_2)$ from the base Σ of the topology of $G_{\mathcal{Q}}(\mathcal{F})$. Let $z = (x, \{t_x \cdot \tilde{h} \cdot t_y^{-1}\}, y)$ be an arbitrary point in $\mathcal{V}$, and $x, y \in L_\alpha \subset M_0^0$. Here $\tilde{h}$ is the result of a transfer of $h' \in [h]$ along a horizontal curve σ respective to v.h.h. H, i.e. $h' \xrightarrow[\sigma]{H}> \tilde{h}$, Suppose that $z = (x, \{\tau_x \cdot \hat{h} \cdot \tau_y^{-1}\}, y)$, where $h'' \xrightarrow[\gamma]{K}> \hat{h}$, $h'' \in [h]$, K is a v.h.h. with the base (γ, h''). It is not difficult to show that a check of the condition (i) of (P_2) is reduced to the case $h'' = h'$. As $H(I_1 \times I_2) \cup K(I_1 \times I_2)$ is a simply connected subset in M, so there are two paths k_1 from $\sigma(1)$ to $\gamma(1)$ in the local leaf $l_x^{W_1}$ and k_2 from $\tilde{\sigma}(1)$ to $\tilde{\gamma}(1)$ in $l_y^{W_2}$, where $\sigma \xrightarrow[h']{H}> \tilde{\sigma}$, $\gamma \xrightarrow[h']{K}> \tilde{\gamma}$, $h' \xrightarrow[\gamma]{K}> \hat{h}$, such that $[\tilde{h}] = [k_1 \cdot \hat{h} \cdot k_2^{-1}]$. Remark that the paths k_1 and $t_x^{-1} \cdot \tau_x$ have common endpoints. By virtue of the equality $H_{\mathcal{Q}^0}(L_\alpha, x) = 0$ and Proposition 8 we have $\{k_1\} = \{t_x^{-1} \cdot \tau_x\}$. Analogously, $\{k_2\} = \{t_y^{-1} \cdot \tau_y\}$. Therefore $\{\tilde{h}\} = \{t_x^{-1} \cdot \tau_x \cdot \hat{h} \cdot \tau_y^{-1} \cdot t_y\}$ and by Lemma 2 $\{t_x \cdot \tilde{h} \cdot t_y^{-1}\} = \{\tau_x \cdot \hat{h} \cdot \tau_y^{-1}\}$, i.e. z depends only on x, y and $\{h\}$. Hence the condition (i) of (P_2) is true. Remark that the neighbourhoods W_1 and W_2 are as small as desired.

Let $(\tilde{W}, \tilde{\varphi})$ be a fibred chart at b containing open n-dimensional ball $D_{2\varepsilon}^n(b)$ with center b and the radius $2\varepsilon > 0$ in Riemannian manifold (M, g). Let (W_2, φ) be a fibred chart at b, and $W_2 \subset D_\varepsilon^n(b)$, $\varphi = \tilde{\varphi}|_{W_2}$. According to Theorem 4 transfers keep lengths of horizontal curves. Therefore the pair $W_2 \subset \tilde{W}$ satisfies to the condition (ii) of (P_2). ∎

COROLLARY 4. *Let $(M, \mathcal{F}, \mathcal{Q})$ be a singular transversally complete Riemannian foliation with a natural generalized Ehresmann connection $\mathcal{Q} = (T\mathcal{F})^{\perp}$. If $(M, \mathcal{F})$ is an orbit-like*

foliation in the sense of Molino or if each singular leaf of $(M, \mathcal{F})$ is isolated in M, then the $\mathcal{Q}$-holonomy grupoid $G_{\mathcal{Q}}(\mathcal{F})$ is Hausdorff.

COROLLARY 5 ([23]). *The manifold of the graph of a regular Riemannian foliation $(M, \mathcal{F})$ on a complete Riemannian manifold (M, g) with the bundle-like metric g is Hausdorff.*

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