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DEFORMATIONS OF STRUCTURES, EMBEDDING OF A RIEMANNIAN MANIFOLD IN A KÄHLERIAN ONE AND GEOMETRIC ANTIGRAVITATION

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Tubular neighborhoods play an important role in modern differential topology. The main aim of the paper is to apply these constructions to geometry of structures on Riemannian manifolds. Deformations of tensor structures on a normal tubular neighborhood of a submanifold in a Riemannian manifold are considered in section 1. In section 2, this approach is used to obtain a Kählerian structure on the corresponding normal tubular neighborhood of the null section in the tangent bundle TM of a smooth manifold M. In section 3, we consider a new deformation of a tensor structure on some neighborhood of a curve and introduce the so-called *geometric antigravitation.*

Some results of the paper were announced in [4], [5]. The work [3] is close to our discussion.

1. Deformations of structures on a tubular neighborhood of a submanifold

$\mathbf{1^0}$. Let (M', g') be a k-dimensional Riemannian manifold isometrically embedded in an n-dimensional Riemannian manifold (M, g). The restriction of g to M' coincides with g' and for any $p \in M'$

$$T_p(M) = T_p(M') \oplus T_p(M')^{\perp}.$$

So, we obtain a vector bundle $M' \to T(M')^{\perp} : p \mapsto T_p(M')^{\perp}$ over the submanifold M'. There exists a neighborhood $\widetilde{U}_0$ of the null section $O_{M'}$ in $T(M')^{\perp}$ such that the mapping

$$\pi \times \exp : v \mapsto (\pi(v), \exp_{\pi(v)} v),\ v \in \widetilde{U}_0,$$

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is a diffeomorphism of $\widetilde{U}_0$ onto an open subset $\widetilde{U} \subset M$. The subset $\widetilde{U}$ is called a *tubular neighborhood of the submanifold* M' in M. We assume that ε is a positive number such that for any $p \in M'$, the open geodesic balls $B(p; \frac{\varepsilon}{2}) \subset B(p; \varepsilon) \subset \widetilde{U}$. For example, such a number ε exists if M' is compact. We denote $\widetilde{U}_p = \exp(\widetilde{U}_0 \cap T_p(M')^{\perp})$, $D(p; \frac{\varepsilon}{2}) = B(p; \frac{\varepsilon}{2}) \cap \widetilde{U}_p$, $D(p; \varepsilon) = B(p; \varepsilon) \cap \widetilde{U}_p$. It is clear that $\dim \widetilde{U}_p = \dim D(p; \frac{\varepsilon}{2}) = \dim D(p; \varepsilon) = n - k$. For any point $o \in M'$, we can consider an orthonormal frame $(X_{1,o}, \ldots, X_{n,o})$ such that $T_o(M') = \mathrm{Lin}[X_{1,o}, \ldots, X_{k,o}]$ and $T_o(M')^{\perp} = \mathrm{Lin}[X_{k+1,o}, \ldots, X_{n,o}]$. There exist coordinates $x_1, \ldots, x_k$ in some neighborhood $\widetilde{V_0} \subset M'$ of the point o such that $\frac{\partial}{\partial x_i}|o = X_{i,o}$, $i = 1, \ldots, k$. We consider orthonormal vector fields $X_{k+1}, \ldots, X_n$ which are cross-sections of the vector bundle $p \mapsto T_p(M')^{\perp}$ over $\widetilde{V_0}$ and the neighborhood $\widetilde{W_0} = \bigcup_{p \in \widetilde{V_0}} \widetilde{U_p}$. The basis $\{X_{k+1,p}, \ldots, X_{n,p}\}$ defines the normal coordinates $x_{k+1}, \ldots, x_n$ on $\widetilde{U_p}$, [7]. For any point $x \in \widetilde{W_0}$, there exists the unique point $p \in \widetilde{V_0}$ such that $x = \exp_p(t\xi)$, $\|\xi\| = 1$, $\xi \in T_p(M')^{\perp}$. A point $x \in \widetilde{W_0}$ has the coordinates $x_1, \ldots, x_k,\ x_{k+1}, \ldots, x_n$ where $x_1, \ldots, x_k$ are coordinates of the point p in $\widetilde{V_0}$ and $x_{k+1}, \ldots, x_n$ are normal coordinates of x in $\widetilde{U_p}$. We denote $X_i = \frac{\partial}{\partial x_i}, i = \overline{1, n}$, on $\widetilde{W_0}$. Thus, we can consider normal *tubular neighborhoods* $Tb(M'; \frac{\varepsilon}{2}) = \bigcup_{p \in M'} D(p; \frac{\varepsilon}{2})$ and $Tb(M'; \varepsilon) = \bigcup_{p \in M'} D(p; \varepsilon)$ of the submanifold M'.

2⁰. Let K be a smooth tensor field ot type (r, s) on the manifold M and for $x \in \widetilde{W_0}$, let

$$K_x = \sum_{i_1, \ldots, i_r,\, j_1, \ldots, j_s} k^{i_1,\, \ldots, i_r}_{j_1, \ldots, j_s}(x) X_{i_1,x} \otimes \cdots \otimes X_{i_r,x} \otimes X_x^{j_1} \otimes \cdots \otimes X_x^{j_s},$$

where $\{X_x^1, \ldots, X_x^n\}$ is the dual basis of $T_x^*(M)$, $x = \exp_p(t\xi)$, $\|\xi\| = 1$, $\xi \in T_p(M')^{\perp}$. We define a tensor field $\overline{K}$ on M in the following way.

a) $x \in D(p; \frac{\varepsilon}{2})$, then

$$\overline{K_x} = \sum_{i_1, \ldots, i_r, j_1, \ldots, j_s} k^{i_1, \ldots, i_r}_{j_1, \ldots, j_s}(p) X_{i_1,x} \otimes \cdots \otimes X_{i_r,x} \otimes X_x^{j_1} \otimes \cdots \otimes X_x^{j_s};$$

b) $x \in D(p; \varepsilon) \backslash D(p; \frac{\varepsilon}{2})$, then

$$\overline{K_x} = \sum_{i_1, \ldots, i_r, j_1, \ldots, j_s} k^{i_1, \ldots, i_r}_{j_1, \ldots, j_s}(\exp_p((2t - \varepsilon)\xi)) X_{i_1,x} \otimes \cdots \otimes X_{i_r,x} \otimes X^{j_1 x} \otimes \cdots \otimes X_x^{j_s};$$

c) $x \in M \backslash \bigcup_{M'} D(p; \varepsilon)$, then

$$\overline{K_x} = K_x.$$

It is easy to see that the tensor field $\overline{K}$ is independent of the choice of coordinates in $\widetilde{W_0}$ for every point $o \in M'$.

Definition 1. The tensor field $\overline{K}$ is called a deformation of the tensor field K on the normal tubular neighborhood of a submanifold M'.

Remark. The obtained tensor field $\overline{K}$ is continuous but it is not smooth on the boundaries of the normal tubular neighborhoods $Tb(M'; \frac{\varepsilon}{2})$ and $Tb(M'; \varepsilon)$, $\overline{K}$ is smooth in other points of the manifold M.

3^0. We consider a deformation $\overline{g}$ of the Riemannian metric g on the normal tubular neighborhood $Tb(M';\varepsilon)$ of a submanifold M'. For $x \in \widetilde{W}_0$, $x = \exp_p(t\xi)$, $\|\xi\| = 1$, $\xi \in T_p(M')$, we define the Riemannian metric $\overline{g}$ in the following way.

a) $\overline{g_p} = g_p$ for any $p \in M'$;
b) $\overline{g_x}(X_i, X_j) = \overline{g_{ij}}(x) = g_{ij}(p)$, where $X_i = \frac{\partial}{\partial x_i}$, $i = \overline{1,n}$, $X_j = \frac{\partial}{\partial x_j}$, $j = \overline{1,n}$, on $\widetilde{W}_0$, $x \in D(p;\frac{\varepsilon}{2})$;
c) $\bar{g_x}(X_i, X_j) = \bar{g_{ij}}(x) = \bar{g_{ij}}(\exp_p((2t-\varepsilon)\xi))$, for any $x \in D(p;\varepsilon)/D(p;\frac{\varepsilon}{2})$;
d) $\bar{g_x} = g_x$ for each point $x \in M\backslash \bigcup_{p\in M'} D(p;\varepsilon)$.

The independence of $\bar{g}$ of a choice of local coordinates follows and the correctly defined Riemannian metric $\bar{g}$ on M has been obtained.

It is known from [8] that every autoparallel submanifold of M is a totally geodesic submanifold and a submanifold M' is autoparallel if and only if $\nabla_X Y \in T(M')$ for any $X, Y \in \chi(M')$, where ∇ is the Riemannian connection of g.

THEOREM 1. *Let M' be a submanifold of a Riemannian manifold (M, g) and $\bar{g}$ be the deformation of g on the normal tubular neighborhood $Tb(M';\varepsilon)$ of M' constructed above. Then M' is a totally geodesic submanifold of $(Tb(M';\frac{\varepsilon}{2}), \bar{g})$.*

Proof. For any point $x \in D(p;\frac{\varepsilon}{2}) \subset \widetilde{W}_0$ the functions $\bar{g}_{ij}(x) = g_{ij}(p)$ and $\frac{\partial \bar{g}_{ij}}{\partial x_l} = 0$, $l = \overline{k+1,n}$ on $D(p;\frac{\varepsilon}{2})$ because the vector fields $X_l = \frac{\partial}{\partial x_l}$ are tangent to $D(p;\frac{\varepsilon}{2})$. By the formula of the Riemannian connection $\bar{\nabla}$ of the Riemannian metric $\bar{g}$, [7], we obtain for $i, j = \overline{1,k}$, $l = \overline{k+1,n}$

$$
\begin{aligned}
(1.1) \qquad 2\bar{g}_p(\bar{\nabla}_{X_j}, X_j, X_l) &= X_{i_p}\bar{g}(X_j, X_l) + X_{j_p}\bar{g}(X_i, X_l) \\
&- X_{l_p}\bar{g}(X_i, X_j) + \bar{g}_p([X_i, X_j], X_l) + \bar{g}_p([X_l, X_i], X_j) \\
&+ \bar{g}_p(X_i, [X_l, X_j]) = -\frac{\partial \bar{g}_{ij}}{\partial x_l} = 0.
\end{aligned}
$$

Here we use the fact that $[X_i, X_j] = [X_l, X_i] = [X_l, X_j] = 0$ and that $\bar{g}(X_j, X_l) = \bar{g}(X_i, X_l) = 0$ because $X_l \in T(M')^{\perp}$. Thus, $\bar{\nabla}_{X_i} X_j \in T(M')$ and from the remarks above the theorem follows. ∎

COROLLARY 1. *Let $\bar{R}$ be the Riemannian curvature tensor field of $\bar{g}$. Then $\bar{R}$ vanishes on every $D(p;\frac{\varepsilon}{2})$ for $p \in M'$.*

Proof. From the formula (1.1) it is clear that $\bar{\nabla}_{X_l} X_m = 0$ for $l, m = \overline{k+1,n}$. The rest is obvious. ∎

4^0. Let (F, ξ, η, g) be an almost contact metric structure (acms) on M and $M' \cong S^1$ be a closed integral curve of the vector field ξ passing through a point $o \in M'$. We consider a tubular neighborhood $\widetilde{U}$ of the submanifold M' and for any point $o \in M'$ the coordinate neighborhood $\widetilde{W}_0 = \bigcup_{p\in\widetilde{V}_0} \widetilde{U}_p$ with the coordinates $x_1, \ldots, x_{2n}, x_{2n+1}$ where x_{2n+1} is the coordinate of the point $p \in \widetilde{V}_0 \subset M'$ and $x_1, \ldots, x_{2n}$ are normal coordinates of $x \in \widetilde{U}_0$, $x = \exp_p(t\nu)$, $\| \nu \| = 1$, $\nu \in T_p(M')^{\perp}$. If $X_i = \frac{\partial}{\partial x_i}, i = \overline{1, 2n+1}$ on $\widetilde{W}_0$, then $\frac{\partial}{\partial x_{2n+1}} = \xi$ and $\frac{\partial}{\partial x_{i|p}}, i = \overline{1, 2n}$, are orthonormal vectors for any $p \in \widetilde{V}_0$, $\{X_{1_p}, \ \ldots \ , X_{2np}\}$ define the normal coordinates $x_1, \dots, x_{2n}$ on $\widetilde{U}_p$, see $\mathbf{1^0}$. We can choose cross-sections $X_1, \dots, X_{2n}$ of the vector bundle $p \mapsto T_p(M')^\perp$ over $\tilde{V}_0$ in such a way that $FX_i = X_{n+i}$, $i = \overline{1, n}$; $FX_j = -X_{j-n}$, $j = \overline{n+1, 2n}$. A deformation $\bar{g}$ of the Riemannian metric g is defined on the normal tubular neighborhood $Tb(M';\ \varepsilon)$ of M', see $\mathbf{3^0}$. Further, a deformation $\bar{F}$ of the (1,1) tensor field F on M is defined in the following way.

a) $x \in D(p; \frac{\varepsilon}{2}) \subset \tilde{U}_p \subset \tilde{W}_0$,

$$\bar{F}_x X_i = F_p X_i,\ i = \overline{1, 2n};\ \bar{F}_x \xi = 0;$$

b) $x \in D(p; \varepsilon) \backslash D(p; \frac{\varepsilon}{2})$,

$$x = \exp_p(t\nu), \| \nu \| = 1,\ \nu \in T_p(M')^\perp;$$
$$\bar{F}_x X_i = F_{\exp_p((2t-\varepsilon)\nu)} X_i,\ i = \overline{1, 2n},\ \bar{F}_x \xi = 0;$$

c) $x \in M \backslash \bigcup_{p \in M'} D(p; \varepsilon)$,

$$\bar{F}_x = F_x.$$

It is clear that $\bar{g}\left(\bar{F}X, \bar{F}Y\right) = \bar{g}\left(X, Y\right)$ for $X, Y \in \mathrm{Lin}[\xi]^\perp$ in the cases a), c). In the case b) we obtain

$$\begin{aligned}\bar{g}_x(\bar{F}_x X_i,\ \bar{F}_x X_j) &= g_{\exp_p((2t-\varepsilon)\nu)}(F_{\exp_p((2t-\varepsilon)\nu)} X_i, F_{\exp_p((2t-\varepsilon)\nu)} X_j) \\ &= g_{\exp_p((2t-\varepsilon)\nu)}\left(X_i,\ X_j\right) = \bar{g}_x\left(X_i,\ X_j\right).\end{aligned}$$

If $\bar{\eta}(X) = \bar{g}(X, \xi)$ for $X \in \chi(M)$ then we have got the correctly defined acms $\left(\bar{F},\ \xi,\ \bar{\eta},\ \bar{g}\right)$ on M.

PROPOSITION 2. *The curve M' is a geodesic with respect to $\bar{\nabla}$ and $D(p; \frac{\varepsilon}{2})$ is a flat totally geodesic submanifold of $(Tb(M';\ \frac{\varepsilon}{2}),\ \bar{g})$ for any $p \in M'$. The structure $\left(\bar{F},\ \bar{g}\right)$ is a Kählerian one on every $D(p; \frac{\varepsilon}{2})$, $p \in M'$.*

Proof. The equality $\bar{\nabla}_{\xi_p} \xi = 0$ directly follows from (1.1). From Corollary 1 we see that $\bar{\nabla}_{X_i} X_j = 0$ on $D(p; \frac{\varepsilon}{2})$, $i,\ j = \overline{1, 2n}$, and the first conclusion of the Proposition 2 is fulfilled. Further, we have

$$\bar{\nabla}_{X_i} \bar{F} X_j - \bar{F} \bar{\nabla}_{X_i} X_j = \pm \nabla_{X_i} X_l = 0,$$

where $l = n + j$, $j = \overline{1,\ n}$; or $l = j - n$, $j = \overline{n+1,\ 2n}$. So we obtain $\bar{\nabla} \bar{F} = 0$ on $D(p; \frac{\varepsilon}{2})$ and $\left(\bar{F},\ \bar{g}\right)$ is a Kählerian structure on every $D(p; \frac{\varepsilon}{2})$, $p \in M'$. ■

2. Embedding of a Riemannian manifold into a Kählerian one

$\mathbf{1^0}$. Let (M, g) be an n-dimensional Riemannian manifold and TM be its tangent bundle. For the Riemannian connection ∇ we consider the connection mapping K of ∇ [1], [6], defined by the formula

$$\nabla_X Z = KZ * X, \tag{2.1}$$

where Z is considered as a mapping from M into TM and the right hand side means a vector field on M assigning to $p \in M'$ the vector $KZ_* X_p \in M_p = T_p(M)$.

If $U \in TM$, we denote by H_U the kernel of $K_{|TM_U}$ and this n-dimensional subspace of TM_U is called the horizontal subspace of $TM_U = T_U(TM)$.

Let π denote the natural projection of TM onto M, then π_* is a C^∞-map of TTM onto TM. If $U \in TM$, we denote by V_U the kernel of $\pi_{*|TM_U}$ and this n-dimensional subspace of TM_U is called the vertical subspace of TM_U ($\dim TM_U = 2\dim M = 2n$). The following maps are isomorphisms of the corresponding vector spaces ($p = \pi(U)$)

$$\pi_{*|TM_U} : H_U \to M_p, K_{|TM_U} : V_U \to M_p$$

and we have

$$TM_U = H_U \oplus V_U.$$

If $X \in \chi(M)$, then there exists exactly one vector field on TM called the "horizontal lift" (resp. "vertical lift") of X and denoted by $\bar{X}^h$ ($\bar{X}^\nu$), such that for all $U \in TM$

$$\pi_* \bar{X}^h_U = X_{\pi(U)},\ K\bar{X}^h_U = 0_{\pi(U)}, \tag{2.2}$$

$$\pi_* \bar{X}^v_U = 0_{\pi(U)},\ K\bar{X}^v_U = X_{\pi(U)}. \tag{2.3}$$

Let R be the curvature tensor field of ∇, then following [1] we have

$$[\bar{X}^\nu,\ \bar{Y}^\nu] = 0, \tag{2.4}$$

$$[\bar{X}^h,\ \bar{Y}^\nu] = \left(\overline{\nabla_X Y}\right)^\nu, \tag{2.5}$$

$$\pi_*([\bar{X}^h,\ \bar{Y}^h]_U) = [X, Y], \tag{2.6}$$

$$K([\bar{X}^h,\ \bar{Y}^h]_U) = R(X,Y)U. \tag{2.7}$$

For vector fields $\bar{X} = \bar{X}^h \oplus \bar{X}^\nu$ and $\bar{Y} = \bar{Y}^h \oplus \bar{Y}^\nu$ on TM, the natural Riemannian metric $\hat{g} =<,>$ is defined on TM by the formula

$$< \bar{X}, \bar{Y} >= g(\pi_*\bar{X}, \pi_*\bar{Y}) + g(K\bar{X}, K\bar{Y}). \tag{2.8}$$

It is clear that the subspaces H_U and V_U are orthogonal with respect to $<,>$.

It is easy to verify that $\bar{X}^h_1,\ \bar{X}^h_2, \ldots, \bar{X}^h_n,\ \ \bar{X}^v_1,\ \bar{X}^v_2, \ldots, \bar{X}^v_n$ are orthonormal vector fields on TM if so are $X_1,\ X_2, \ldots, X_n$ on M, i.e. $g(X_i, X_j) = \delta^i_j$.

$\mathbf{2^0}$. We define a tensor field J on TM by the equalities

$$J\bar{X}^h = \bar{X}^\nu,\ J\bar{X}^\nu = -\bar{X}^h,\ X \in \chi(M). \tag{2.9}$$

Since $J^2 = -I$ and $< J\bar{X}, J\bar{Y} >=< \bar{X}, \bar{Y} >$, $(TM, J, <,>)$ is an almost Hermitian manifold.

Further, we want to analyze the second fundamental tensor field h of the pari $(J, <,>)$, see (2.11) below, cf. [2]. It is obvious that $(J, <,>)$ is a Kählerian structure if and only if $h = 0$.

The Riemannian connection $\hat{\nabla}$ of the metric $\hat{g} =<,>$ on TM is defined by the formula (see [7])

$$\begin{aligned} < \hat{\nabla}_{\bar{X}}\bar{Y}, \bar{Z} > = \tfrac{1}{2}(&\bar{X} < \bar{Y}, \bar{Z} > + \bar{Y} < \bar{Z}, \bar{X} > \\ &- \bar{Z} < \bar{X}, \bar{Y} > + < \bar{Z}, [\bar{X},\ \bar{Y}] > + < \bar{Y}, [\bar{Z},\ \bar{X}] > \\ &+ \ < \bar{X},\ [\bar{Z},\ \bar{Y}] >), \qquad X, Y, Z \in \chi(M). \end{aligned} \tag{2.10}$$

For orthonormal vector fields $\bar{X}$, $\bar{Y}$, $\bar{Z}$ on TM we obtain

$$(2.11)\quad h_{\bar{X}\bar{Y}\bar{Z}} =< h_{\bar{X}}\bar{Y}, \bar{Z}> = \frac{1}{2} < \hat{\nabla}_{\bar{X}}\bar{Y} + J\hat{\nabla}_{\bar{X}}J\bar{Y}, \bar{Z} >$$
$$= \frac{1}{2}(< \hat{\nabla}_{\bar{X}}\bar{Y}, \bar{Z} > - < \hat{\nabla}_{\bar{X}}J\bar{Y}, J\bar{Z} >)$$
$$= \frac{1}{4}(< [\bar{X}, \bar{Y}], \bar{Z} > + < [\bar{Z}, \bar{X}], \bar{Y} > + < [\bar{Z}, \bar{Y}], \bar{X} >$$
$$- < [\bar{X}, J\bar{Y}], J\bar{Z}> - < [J\bar{Z}, \bar{X}], J\bar{Y}> - < [J\bar{Z}, J\bar{Y}], \bar{X}>).$$

Using (2.4)-(2.7) and (2.11) we consider the following cases for the tensor field h assuming all the vector fields to be orthonormal.

1.1^0) $h_{\bar{X}^h\bar{Y}^h\bar{Z}^h} = \frac{1}{4}(< [\bar{X}^h, \bar{Y}^h], \bar{Z}^h > + < [\bar{Z}^h, \bar{X}^h], \bar{Y}^h >$
$$+ < [\bar{Z}^h, \bar{Y}^h], \bar{X}^h > - < [\bar{X}^h, J\bar{Y}^h], J\bar{Z}^h > - < [J\bar{Z}^h, \bar{X}^h], J\bar{Y}^h >$$
$$- < [J\bar{Z}^h, J\bar{Y}^h], \bar{X}^h >) = \frac{1}{4}(g([X, Y], Z) + g([Z, X], Y)$$
$$+g([Z, Y], X) - < [\bar{X}^h, \bar{Y}^\nu], \bar{Z}^\nu > - < [\bar{Z}^\nu, \bar{X}^h], \bar{Y}^\nu >$$
$$- < [\bar{Z}^\nu, \bar{Y}^\nu], \bar{X}^h >) = \frac{1}{2}g(\nabla_X Y, Z) - \frac{1}{4}(g(\nabla_X Y, Z)$$
$$-g(\nabla_X Z, Y)) = \frac{1}{2}(g(\nabla_X Y, Z) - g(\nabla_X Y, Z)) = 0.$$

2.1^0) $h_{\bar{X}^h\bar{Y}^h\bar{Z}^\nu} = \frac{1}{4}(< [\bar{X}^h, \bar{Y}^h], \bar{Z}^\nu > + < [\bar{Z}^\nu, \bar{X}^h], \bar{Y}^h >$
$$+ < [\bar{Z}^\nu, \bar{Y}^h], \bar{X}^h > - < [\bar{X}^h, J\bar{Y}^h], J\bar{Z}^\nu > - < [J\bar{Z}^\nu, \bar{X}^h], J\bar{Y}^h >$$
$$- < [J\bar{Z}^\nu, J\bar{Y}^h], \bar{X}^h >) = \frac{1}{4}(g(R(X, Y)U, Z)$$
$$+ < [\bar{Z}^h, \bar{X}^h], \bar{Y}^\nu >) = \frac{1}{4}(g(R(X, Y)U, Z) + g(R(Z, X)U, Y))$$
$$= -\frac{1}{4}(g(R(X, Y)Z, U) + g(R(Z, X)Y, U)).$$

By similar arguments we obtain

3.1^0) $h_{\bar{X}^h\bar{Y}^\nu\bar{Z}^h} = -\frac{1}{4}(g(R(Z, X)Y, U) + g(R(X, Y)Z, U))$.
4.1^0) $h_{\bar{X}^\nu\bar{Y}^h\bar{Z}^h} = -\frac{1}{4}(g(R(Z, Y)X, U))$.
5.1^0) $h_{\bar{X}^\nu\bar{Y}^\nu\bar{Z}^\nu} = \frac{1}{4}(g(R(Z, Y)X, U))$.
6.1^0) $h_{\bar{X}^\nu\bar{Y}^\nu\bar{Z}^h} = 0$.
7.1^0) $h_{\bar{X}^\nu\bar{Y}^h\bar{Z}^\nu} = 0$.
8.1^0) $h_{\bar{X}^h\bar{Y}^\nu\bar{Z}^\nu} = 0$.

Thus the second fundamental tensor field h (of the structure $(J, <, >)$) strongly depends on the connection ∇.

3^0. We have constructed the almost Hermitian structure $(J, \hat{g} =<, >)$ on TM with the help of a Riemannian metric g on M, i.e. we have obtained an injective mapping

$$E : (M) \to (TM) : g \mapsto (J, \hat{g} =<, >),$$

where (M) is the set of Riemannian metrics on M and (TM) is the set of almost Hermitian structures on TM.

Definition 2. A smooth manifold M will be called an ε-manifold if there exists a normal tubular neighborhood $Tb(M, \varepsilon)$ in TM with respect to $\hat{g} =<, >= E(g)$ on TM induced by some $g \in (M)$, where M is considered as the null section O_M in TM ($M \ni p \leftrightarrow O_p \in O_M \subset TM$).

It is clear that any smooth compact manifold is an ε-manifold.

THEOREM 3. *Let M be an ε-manifold and $Tb(M,\varepsilon)$ be the corresponding normal tubular neighborhood in TM with respect to $\hat{g} =<,>= E(g)$ on TM. Then $M(O_M)$ is a totally geodesic submanifold of the Kählerian manifold $(Tb(M,\frac{\varepsilon}{2}),\bar{J},\bar{g})$, where the almost Hermitian structure $(\bar{J},\bar{g})$ is the deformation of the structure $(J,<,>)$ defined above.*

Proof. It follows from Theorem 1 that M is a totally geodesic submanifold of the Riemannian manifold $(Tb(M,\frac{\varepsilon}{2}),\bar{g})$.

Let $\tilde{W}_0$ be a coordinate neighborhood in TM considered in $\mathbf{1^0},\mathbf{1}$. A point $x \in \tilde{W}_0$ has the coordinates $x_1,\dots,x_n,\ x_{n+1},\dots,x_{2n}$ where $x_1,\dots,x_n$ are coordinates of the point p in $\tilde{V}_0 \subset M$ and $x_{n+1},\dots,x_{2n}$ are normal coordinates of x in $D(p;\frac{\varepsilon}{2})$.

Let denote $X_i = \frac{\partial}{\partial x_i}$, $i=\overline{1,\ 2n}$, $\hat{\nabla}_{X_i}X_j = \sum_k \hat{\Gamma}^k_{ij}X_k$, $\bar{\nabla}_{X_i}X_j = \sum_k \bar{\Gamma}^k_{ij}X_k$, $JX_j = \sum_k J^k_j X_k$, $\bar{J}X_j = \sum_k J^k_j X_k$, $\hat{g}_{ij} = \hat{g}(X_i,X_j)$, $\bar{g}_{ij} = \bar{g}(X_i,X_j)$ where $\hat{\nabla}$ and $\bar{\nabla}$ are Riemannian connections of metrics $\hat{g}$ and $\bar{g}$, respectively.

Using the construction of $\mathbf{2^0},\mathbf{1}$ we have $\bar{g}_{ij}(x) = \hat{g}_{ij}(p)$, $\bar{J}^i_j(x) = J^i_j(p)$ on $Tb(M,\frac{\varepsilon}{2})\cap \tilde{W}_0$. According to [7] we can write

$$\sum_l \bar{g}_{lk}\bar{\Gamma}^l_{ij} = \frac{1}{2}\left(\frac{\partial \bar{g}_{kj}}{\partial x_i} + \frac{\partial \bar{g}_{ik}}{\partial x_j} - \frac{\partial \bar{g}_{ij}}{\partial x_k}\right). \tag{2.12}$$

It follows from (2.12) that $\bar{\Gamma}^l_{ij}(x) = \bar{\Gamma}^l_{ij}(p)$ and $\bar{\Gamma}^l_{ij}(x) = 0$, i.e. $\bar{\nabla}_{X_i}X_j = 0$ for $i = \overline{n+1,\ 2n}$.

Further, we get

$$\begin{aligned}(\bar{\nabla}_{X_i}\bar{J})X_j &= \bar{\nabla}_{X_i}\bar{J}X_j - \bar{J}\bar{\nabla}_{X_i}X_j = \sum_k \bar{\nabla}_{X_i}\bar{J}^k_j X_k \\ -\bar{J}\left(\sum_k \bar{\Gamma}^k_{ij}X_k\right) &= \sum_k \left(\bar{J}^k_j \bar{\nabla}_{X_i}X_k + (X_i\bar{J}^k_j)X_k\right) \\ -\sum_{k,\,l}\bar{\Gamma}^l_{ij}\bar{J}^k_l X_k &= \sum_{k,\,l}\left(\bar{J}^l_j\bar{\Gamma}^k_{il} - \bar{\Gamma}^l_{ij}\bar{J}^k_l + X_i\bar{J}^k_j\right)X_k, \\ \left((\bar{\nabla}_{X_i}\bar{J})X_j\right)(x) &= \sum_{k,\,l}\left(\bar{J}^l_j\bar{\Gamma}^k_{il} - \bar{\Gamma}^l_{ij}\bar{J}^k_l + X_i\bar{J}^k_j\right)(x)X_{k|x} \\ &= \sum_{k,\,l}\left(\left(\bar{J}^l_j\bar{\Gamma}^k_{il} - \bar{\Gamma}^l_{ij}\bar{J}^k_l\right)(p) + \left(X_i\bar{J}^k_j\right)(x)\right)X_{k|x}.\end{aligned}$$

It follows that $\bar{\nabla}_{X_i}\bar{J} = 0$ for $i = \overline{n+1,\ 2n}$. For $i = \overline{1,\ n}$, $(X_i\bar{J}^k_j)(x) = (X_iJ^k_j)(p)$ and we obtain

$$((\bar{\nabla}_{X_i}\bar{J})X_j)(x) = \sum_{k,\,l}(J^l_j\hat{\Gamma}^k_{il} - \hat{\Gamma}^l_{ij}J^k_l + X_iJ^k_j)(p)X_{k|x}.$$

On the other hand we can write

$$((\hat{\nabla}_{X_i}J)X_j)(p) = \sum_{k,\,l}(J^l_j\hat{\Gamma}^k_{il} - \hat{\Gamma}^l_{ij}J^k_l + X_iJ^k_j)(p)X_{k|p}.$$

According to [2], we have $\left((\nabla_{X_i}J)X_j\right)(p) = \left(2h_{X_i}JX_j\right)(p)$ where the tensor field h is defined by (2.11). From $1.1^0)$-$8.1^0)$ it follows that $h_p = 0$ for any $p \in M'$ $(U = O_p \in O_M)$.

Finally, we have obtained that $\bar{\nabla}\bar{J} = 0$ and the structure $(\bar{J}, \bar{g})$ is a Kählerian one on $Tb(M, \frac{\varepsilon}{2})$. ■

Corollary 3. *Any compact smooth manifold of dimension n can be embedded in a Kählerian manifold of dimension 2n as a totally geodesic submanifold.*

4⁰. Let M be an arbitrary smooth Riemannian manifold. For any point $p \in M$ we can consider a set $\{\delta(p)\}$ of positive numbers such that the mapping $exp_{|U(\delta(p))}$ is defined and injective on $U(\delta(p)) \subset T_pM$. Let $\varepsilon(p) = sup\{\delta(p)\}$.

Lemma 4 ([6]). *The mapping $M \to \boldsymbol{R}_+ : p \mapsto \varepsilon(p)$ is continuous on M.*

From the constructions above it is evident that the following proposition is true.

Proposition 5. *There exists a function $\varepsilon(p)$ continuous on M such that the subset $Tb(M, \varepsilon(p)) = \bigcup_{p\in M} D(p; \varepsilon(p))$ of TM defined by the Riemannian metric $\widehat{g} = E(g)$ on TM induced by some $g \in R(M)$ is a normal tubular neighborhood of M considered as the null section O_M of TM.*

All the constructions considered above for the case $\varepsilon = const$ can be generalized to the case of a positive continuous function $\varepsilon(p)$ on M.

Theorem 6. *Let M be a smooth real manifold and $Tb(M, \varepsilon(p))$ be the corresponding normal tubular neighborhood in TM with respect to some $\widehat{g} = E(g)$ on TM. Then $M(O_M)$ is a totally geodesic submanifold of the Kählerian manifold $(Tb(M, \frac{\varepsilon(p)}{2}, \overline{J}, \overline{g}))$, where the almost Hermitian structure $(\overline{J}, \overline{g})$ is the deformation of the structure $(J, \widehat{g})$. So, any smooth real manifold of dimension n can be embedded into a Kählerian manifold of dimension 2n as a totally geodesic submanifold.*

3. Deformations of structures on a neighborhood of a curve and geometric antigravitation

1⁰. Let (M, g) be a Riemannian smooth n-dimensional manifold and $\gamma : p = p(s), 0 \leq s \leq s_1$, be a smooth curve isometrically embedded in M. We can consider a *balloon neighborhood* $Bl(\gamma, \varepsilon)$ consisting of the tubular neighborhood $Tb(\gamma, \varepsilon)$ and two geodesic semiballs $Bs(p(0); \varepsilon)$, $Bs(p(s_1); \varepsilon)$ attached to $Tb(\gamma, \varepsilon)$ at the ends. For any point $x \in Tb(\gamma, \varepsilon)$, there exist $0 \leq s \leq s_1$ and $0 \leq t < \varepsilon(p(s))$ such that $x = \exp_{p(s)}(t\xi)$, where $\| \xi \| = 1, g(\xi, \nu) = 0$, and $\nu = \dot{p}(s)$. For every point $x \in Bs(p(0); \varepsilon)$ (or $x \in Bs(p(s_1); \varepsilon)$) we have $x = \exp_{p_0}(t\xi)$, $\| \xi \| = 1$, $\xi \in T_{p_0}M$, $p_0 = p(0)$ (or $x = \exp_{p_1}(t\xi)$, $\| \xi \| = 1$, $\xi \in T_{p_1}M$, $p_1 = p(s_1)$). We can choose local coordinates $(x_1, x_2, \ldots, x_n)$ on $Bl(\gamma, \varepsilon)$ such that $\frac{\partial}{\partial x_1}|_p = \nu_p$. We denote $X_{i_x} = \frac{\partial}{\partial x_i}|_x$, $i = 1, 2, \ldots, n$.

2⁰. Let K be a tensor field of type (r, s) from the algebra of all smooth tensor fields on M and on $Bl(\gamma, \varepsilon)$ we have

$$K_x = \sum_{i_1,\ldots,i_r,j_1,\ldots,j_s} k^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}(x) X_{i_1,x} \otimes \cdots \otimes X_{i_r,x} \otimes X^{j_1}_x \otimes \cdots \otimes X^{j_s}_x$$

where $\{X^1_x, \ldots, X^n_x\}$ is the dual basis of T^*_xM.

A tensor field $\bar{K}$ is defined on M as follows:

a) $x \in Bl(\gamma; \frac{\varepsilon}{4})$,

$$\bar{K}_x = \sum_{i_1,\ldots,i_r,j_1,\ldots,j_s} k^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}(p_0) X_{i_{1_x}} \otimes \cdots \otimes X_{i_{r_x}} \otimes X^{j_1}_x \otimes \cdots \otimes X^{j_s}_x;$$

b) $x \in Bl(\gamma; \frac{\varepsilon}{2}) \backslash Bl(\gamma; \frac{\varepsilon}{4}),\ x = \exp_p(t\xi), p = p(s_p)$,

$$\bar{K}_x = \sum_{i_1,\ldots,i_r,j_1,\ldots,j_s} k^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}\left(p\left(\frac{2s_p}{\varepsilon}\left(2t - \frac{\varepsilon}{2}\right)\right)\right) X_{i_{1_x}} \otimes \cdots \otimes X_{i_{r_x}} \otimes X^{j_1}_x \otimes \cdots \otimes X^{j_s}_x;$$

c) $x \in Bl(\gamma, \varepsilon) \backslash Bl(\gamma, \frac{\varepsilon}{2})$, $x = \exp_p(t\xi), p = p(s_p)$,

$$\bar{K}_x = \sum_{i_1,\ldots,i_r,j_1,\ldots,j_s} k^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}\left(\exp_p\left((2t - \varepsilon)\,\xi\right)\right) X_{i_{1_x}} \otimes \cdots \otimes X_{i_{r_x}} \otimes X^{j_1}_x \otimes \cdots \otimes X^{j_s}_x;$$

d) $x \in M \backslash Bl(\gamma, \varepsilon)$,

$$\bar{K}_x = K_x.$$

Remark. The obtained tensor field $\bar{K}$ is continuous but is not smooth on the boundaries of the balloon neighborhoods, $\bar{K}$ is smooth at other points of M.

Problem. Can we get a good deformation $\tilde{K}$ of $\bar{K}$ such that $\tilde{K}$ is smooth and $\tilde{K} = \bar{K}$ on some balloon neighborhood of $\gamma, \tilde{K} = \bar{K}$ on $M \backslash Bl(\gamma, \varepsilon)$?

$\mathbf{3^0}$. Let $\Phi(X, Y) = g(PX, Y)$ be a pseudo-Riemannian metric on M, $P^2 = I,\ X,\ Y \in \chi(M)$. Using the construction presented in $\mathbf{2^0}$ we can obtain the deformations $\bar{g}$ of g and $\bar{P}$ of P on $Bl(\gamma, \varepsilon)$, i.e. a new pseudo-Riemannian metric $\bar{\Phi}$ where $\bar{\Phi}(X, Y) = \bar{g}(\bar{P}X, Y),\ \bar{P}^2 = I$.

Theorem 7. *Let $\bar{\nabla}$ be the canonical connection of $\bar{\Phi}$ on $Bl(\gamma; \frac{\varepsilon}{4})$. Then γ is geodesic with respect to $\bar{\nabla}$ and the curvature tensor field (gravitation) $\bar{R}$ vanishes on $Bl(\gamma; \frac{\varepsilon}{4})$.*

Proof. For any $x \in Bl(\gamma; \frac{\varepsilon}{4})$ we have

$$\bar{g}_x = \sum_{j,\,k} g_{jk}(p_0) X^j_x \otimes X^k_x, \quad \bar{P}_x = \sum_{i,\,l} f^i_l(p_0) X_{i_x} \otimes X^l_x$$

and the functions $g_{jk}(p_0),\ f^i_l(p_0)$ are constant on $Bl(\gamma; \frac{\varepsilon}{4})$. The formula of the canonical connection $\bar{\nabla}$ of $\bar{\Phi}$ is similar to (1.1)

$$\begin{aligned} 2\bar{\Phi}_x\left(\bar{\nabla}_{X_i} X_j,\ X_l\right) &= X_{i_x}\bar{\Phi}(X_j, X_l) + X_{j_x}\bar{\Phi}(X_i, X_l) - X_{l_x}\bar{\Phi}(X_i, X_j) \\ &+ \bar{\Phi}x([X_i, X_j], X_l) + \bar{\Phi}x([X_l, X_i], X_j) + \bar{\Phi}x(X_i, [X_l, X_j]) \\ &= X_{i_x}\bar{g}(\bar{P}X_j, X_l) + X_{j_x}\bar{g}(\bar{P}X_i, X_l) - X_{l_x}\bar{g}(\bar{P}X_i, X_j) = 0. \end{aligned}$$

So we obtain that $\bar{\nabla}_{X_i} X_j = 0$ and it follows that $\bar{R} = 0$ on $Bl(\gamma; \frac{\varepsilon}{4})$. From $\bar{\nabla}_{X_1} X_1 = 0$ we get that γ is a geodesic segment. ∎

Remark. We can also consider almost Hermitian manifolds and Riemannian G-structures, see [3] for details.

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