ASYMPTOTICALLY SELF-SIMILAR SOLUTIONS FOR THE PARABOLIC SYSTEM MODELLING CHEMOTAXIS

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Abstract. We consider a nonlinear parabolic system modelling chemotaxis
\[ \frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u\nabla v), \quad \frac{\partial v}{\partial t} = \Delta v + u \]
in \( \mathbb{R}^2, \ t > 0 \). We first prove the existence of time-global solutions, including self-similar solutions, for small initial data, and then show the asymptotically self-similar behavior for a class of general solutions.

1. Introduction. We are concerned with the large time behavior of solutions to the Cauchy problem for the following system of partial differential equations:

\[
\begin{cases}
\frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u\nabla v), & x \in \mathbb{R}^2, \ t > 0, \\
\frac{\partial v}{\partial t} = \Delta v + u, & x \in \mathbb{R}^2, \ t > 0.
\end{cases}
\]

On the system we impose initial conditions

\[ u(x, 0) = u_0, \quad v(x, 0) = v_0, \quad x \in \mathbb{R}^2, \]

where \( u_0 \geq 0 \) and \( v_0 \geq 0 \).

The system (1.1) is a mathematical model describing chemotaxis, that is, the directed movement of an organism in response to gradients of a chemical attractant (see [6, 12, 4]). The function \( u(x, t) \geq 0 \) corresponds to the population of the organism at the place \( x \in \mathbb{R}^2 \) and time \( t > 0 \), and \( v(x, t) \geq 0 \) to the concentration of the chemical.

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The existence of local and global in time solutions, including self-similar solutions, of the problem (1.1)–(1.2) has been studied by Biler [3]. In this paper we show the asymptotically self-similar behavior for a class of general solutions of (1.1)–(1.2).

We write (1.1)–(1.2) in the form of the integral equation

$$\begin{cases}
  u(t) = e^{t \Delta} u_0 - \int_0^t (\nabla e^{(t-s) \Delta}) \cdot (u(s) \nabla v(s)) \, ds, \\
  v(t) = e^{t \Delta} v_0 + \int_0^t e^{(t-s) \Delta} u(s) \, ds,
\end{cases}$$

(1.3)

where

$$(e^{t \Delta} f)(x) = \int_{\mathbb{R}^2} G(x - y, t) f(y) \, dy$$

and $G(x, t)$ is the heat kernel

$$G(x, t) = \frac{1}{4\pi t} \exp \left( -\frac{|x|^2}{4t} \right).$$

In what follows, $\| \cdot \|_p$ represents the norm of $L^p(\mathbb{R}^2)$ for $1 \leq p \leq \infty$. For $u : \mathbb{R}^2 \to \mathbb{R}$, we use the notations $\nabla u = (\partial_1 u, \partial_2 u)$ and $\|\nabla u\|_p = \|\partial_1 u\|_p + \|\partial_2 u\|_p$, where $\partial_j = \partial / \partial x_j$.

We look for mild solutions $(u, v)$ of (1.3) in the class $u \in X_p$ with $p \in (4/3, 2)$, where $X_p$ is the set of Bochner measurable functions $u : (0, \infty) \to L^p(\mathbb{R}^2)$ such that $\sup_{t>0} t^{1-\frac{1}{p}} \|u(t)\|_p < \infty$. We will obtain $v$ according to the second formula in (1.3) for $u \in X_p$. Define $\| \cdot \|_{X_p}$ by

$$\|u\|_{X_p} = \sup_{t>0} t^{1-\frac{1}{p}} \|u(t)\|_p.$$

Throughout this paper, $p \in (4/3, 2)$ is fixed. First we show the existence of time global solution of (1.3) with $u_0 \in L^1(\mathbb{R}^2)$ and $\nabla v_0 \in L^2(\mathbb{R}^2)$.

**Theorem 1.** Assume that constants $M > 0$, $\alpha_0 > 0$, and $\beta_0 \geq 0$ satisfy the inequalities

$$\frac{\alpha_0}{(4\pi)^{1-\frac{1}{p}} M} + \tilde{C}_0 C_1 (\beta_0 + M) \leq 1 \quad \text{and} \quad \tilde{C}_0 C_1 (\beta_0 + 2M) < 1,$$

where positive constants $\tilde{C}_0$ and $C_1$ are given below in Lemma 2.2. Suppose that $u_0 \in L^1(\mathbb{R}^2)$ and $\nabla v_0 \in L^2(\mathbb{R}^2)$ satisfy $\|u_0\|_1 \leq \alpha_0$ and $\|\nabla v_0\|_2 \leq \beta_0$. Then there exists a unique global solution $(u, v)$ of (1.3) such that $\|u\|_{X_p} \leq M$.

The system (1.1) is invariant under the similarity transformation

$$u_\lambda(x, t) = \lambda^2 u(\lambda x, \lambda^2 t) \quad \text{and} \quad v_\lambda(x, t) = v(\lambda x, \lambda^2 t)$$

for $\lambda > 0$, that is, if $(u, v)$ is a solution of (1.1) then so is $(u_\lambda, v_\lambda)$. A solution $(u, v)$ is said to be self-similar, when the solution is invariant under this transformation, that is, $u(x, t) = u_\lambda(x, t)$ and $v(x, t) = v_\lambda(x, t)$ for all $\lambda > 0$. Letting $\lambda = 1/\sqrt{t}$, and putting $\phi(x) = u(x, 1)$ and $\psi(x) = v(x, 1)$, we find that the self-similar solution $(u, v)$ has the form

$$u(x, t) = \frac{1}{t} \phi \left( \frac{x}{\sqrt{t}} \right) \quad \text{and} \quad v(x, t) = \psi \left( \frac{x}{\sqrt{t}} \right)$$

for $x \in \mathbb{R}^2$ and $t > 0$. 
Let us consider the problem

\[
\begin{aligned}
&\frac{\partial u}{\partial t} = \alpha G(\cdot, t) - \int_0^t (\nabla e^{(t-s)\Delta}) \cdot (u(s) \nabla v(s)) \, ds, \\
&v(t) = \int_0^t e^{(t-s)\Delta} u(s) \, ds,
\end{aligned}
\tag{1.6}
\]

where \(\alpha\) is a positive constant and \(G\) is the heat kernel. We show the existence of the self-similar solutions of (1.6).

**Theorem 2.** Assume that constants \(M > 0\) and \(\alpha_0 > 0\) satisfy the inequalities

\[
\frac{\alpha_0}{(4\pi)^{1-\frac{1}{p}}} + \tilde{C}_0 C_1 M \leq 1 \quad \text{and} \quad 2\tilde{C}_0 C_1 M < 1,
\tag{1.7}
\]

where positive constants \(\tilde{C}_0\) and \(C_1\) are the same as in Theorem 1. Then, for \(\alpha \in (0, \alpha_0]\), there exists a unique self-similar solution \((u_\alpha, v_\alpha)\) of (1.6) such that \(\|u_\alpha\|_{X_p} \leq M\).

**Remark 1.** (i) We do not know the uniqueness of self-similar solution of (1.6) without the assumption \(\|u_\alpha\|_{X_p} \leq M\). Concerning the non-uniqueness of self-similar solutions for semilinear heat equations, we refer to [11].

(ii) For the properties of self-similar solutions to (1.1), we refer to [3, 9]. We also refer to [1, 2, 10], where the self-similar solutions to the parabolic-elliptic problem have been studied.

(iii) It is clear that, if \(M, \alpha_0,\) and \(\beta_0\) satisfy (1.4), then (1.7) holds. Thus, for a solution \((u, v)\), constructed by Theorem 1, there exists a self-similar solution \((u_\alpha, v_\alpha)\) of (1.6) with \(\alpha = \|u_0\|_1\).

Let \((u_\alpha, v_\alpha)\) be a self-similar solution of (1.6), constructed by Theorem 2. By the argument above, \((u_\alpha, v_\alpha)\) has the form

\[
u_\alpha(x, t) = \frac{1}{t} \phi_\alpha \left( \frac{x}{\sqrt{t}} \right) \quad \text{and} \quad v_\alpha(x, t) = \psi_\alpha \left( \frac{x}{\sqrt{t}} \right)
\]

for \(x \in \mathbb{R}^2\) and \(t > 0\). We note here that

\[
t^{1-\frac{1}{p}} \|u_\alpha(\cdot, t)\|_p = \|\phi_\alpha\|_p \quad \text{for} \ t > 0.
\]

From \(\|u_\alpha\|_{X_p} \leq M\), it follows that \(\phi_\alpha \in L^p(\mathbb{R}^2)\).

We consider the asymptotic behavior of solutions of (1.3) constructed by Theorem 1.

**Theorem 3.** Let \((u, v)\) be a solution of (1.3) constructed by Theorem 1. Assume, in addition, that \(u_0\) and \(v_0\) satisfy \((1 + |x|^2)u_0 \in L^1(\mathbb{R}^2)\) and \(\nabla v_0 \in L^1(\mathbb{R}^2)\). Let \((u_\alpha, v_\alpha)\) be a self-similar solution of (1.6) with \(\alpha = \|u_0\|_1\), constructed by Theorem 2. Then there exists \(\sigma \in (0, 1/2)\) such that

\[
t^{1-\frac{1}{p}} \|u(\cdot, t) - u_\alpha(\cdot, t)\|_p = O(t^{-\sigma}) \quad \text{as} \ t \to \infty.
\tag{1.8}
\]

In particular, \(\|tu(\cdot, \sqrt{t}, t) - \phi_\alpha(\cdot)\|_p = O(t^{-\sigma}) \quad \text{as} \ t \to \infty\).
It is interesting to compare the results for the problem (1.1) and the problem
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot (\nabla u - u \nabla v), \quad x \in \mathbb{R}^2, \ t > 0, \\
\frac{\partial v}{\partial t} &= \Delta v - v + u, \quad x \in \mathbb{R}^2, \ t > 0.
\end{align*}
\]

(1.9)

It has been shown by Nagai [7, 8] that every bounded solution of the problem (1.9) on \( \mathbb{R}^2 \times [0, \infty) \) decays to zero and behaves like a constant multiple of the heat kernel as \( t \to \infty \). The large time behavior for higher dimensional case is also studied in [8].

Theorems 1 and 2 are proven by employing the contraction mapping argument in suitable function spaces. We prove Theorem 3 by estimating the terms in the integral equations. We will give the proofs of Theorems 1 and 2 in Section 2, and we prove Theorem 3 in Section 3.

2. Proofs of Theorems 1 and 2. First we recall \( L^p - L^q \) estimates for the heat semigroup, which are proved by Young’s inequality for convolution.

**Lemma 2.1** Let \( 1 \leq q \leq p \leq \infty \) and \( f \in L^q(\mathbb{R}^2) \). Then
\[
\| e^{t\Delta} f \|_p \leq (4\pi t)^{-\left(\frac{1}{2} - \frac{1}{q}\right)} \| f \|_q,
\]
\[
\| \partial_j e^{t\Delta} f \|_p \leq C(p, q) t^{-\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{2}} \| f \|_q, \quad j = 1, 2,
\]
where \( C(p, q) \) is a positive constant depending only on \( p \) and \( q \).

Let
\[
X_p = \{ u : (0, \infty) \to L^p(\mathbb{R}^2) : \sup_{t > 0} t^{1 - \frac{1}{p}} \| u(t) \|_p < \infty \}
\]
with \( p \in (4/3, 2) \). Define \( \| \cdot \|_{X_p} \) by
\[
\| u \|_{X_p} = \sup_{t > 0} t^{1 - \frac{1}{p}} \| u(t) \|_p.
\]

For \( u \in X_p \) and \( \nabla v_0 \in L^2(\mathbb{R}^2) \), we define \( v \) by
\[
v(t) = e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta} u(s) \, ds,
\]
and then \( \Phi(u) \) by
\[
\Phi(u)(t) = \int_0^t (\nabla e^{(t-s)\Delta} \cdot (u(s) \nabla v(s))) \, ds.
\]

In what follows, we put \( q = p/(p - 1) \) for fixed \( p \in (4/3, 2) \). We denote by \( B(p, q) \) the beta function.

**Lemma 2.2.** (i) We have
\[
\sup_{t > 0} t^{\frac{1}{2} - \frac{1}{q}} \| \nabla v(t) \|_q \leq \tilde{C}_0 (\| \nabla v_0 \|_2 + \| u \|_{X_p}),
\]
where
\[
\tilde{C}_0 = \max\{ (4\pi)^{-\left(\frac{1}{2} - \frac{1}{q}\right)}, \ C_0 \}, \quad C_0 = 2C(q, p)B\left(\frac{3}{2} - \frac{2}{p}, \frac{1}{p}\right).
\]
\( \| \Phi(u) \|_{L^p} \leq \tilde{C}_0 C_1 (\| \nabla v_0 \|_2 + \| u \|_{X_p}) \| u \|_{X_p}, \)

where

\( C_1 = C(p, 1)B(\frac{1}{p} - \frac{1}{2}, \frac{1}{2}). \)

**Proof.** (i) From (2.3) we have

\[ \partial_j v(t) = e^{t \Delta} \partial_j v_0 + \int_0^t \partial_j e^{(t-s)\Delta} u(s) \, ds \]

for \( j = 1, 2. \) From (2.1) and (2.2) we obtain

\[
\| \partial_j v(t) \|_q \leq \| e^{t \Delta} \partial_j v_0 \|_q + \int_0^t \| \partial_j e^{(t-s)\Delta} u(s) \|_q \, ds \\
\leq (4\pi t)^{\frac{1}{2} - \frac{1}{p}} \| \partial_j v_0 \|_2 + C(q, p) \int_0^t (t-s)^{\frac{1}{2} - \frac{1}{p} - \frac{1}{2}} \| u(s) \|_p \, ds.
\]

Note that

\[
\int_0^t (t-s)^{\frac{1}{2} - \frac{1}{p} - \frac{1}{2}} \| u(s) \|_p \, ds \leq \int_0^t (t-s)^{\frac{1}{2} - \frac{1}{p} - \frac{1}{2}} s^{\frac{1}{p} - 1} \, ds \| u \|_{X_p} \\
= t^{\frac{1}{2} - \frac{1}{p}} B\left(\frac{3}{2} - \frac{2}{p}, \frac{1}{p}\right) \| u \|_{X_p}.
\]

By the definition \( \| \nabla u \|_q = \| \partial_1 u \|_q + \| \partial_2 u \|_q, \) we obtain

\[
t^{\frac{1}{2} - \frac{1}{q}} \| \nabla v(t) \|_q \leq (4\pi)^{-\left(\frac{1}{2} - \frac{1}{p}\right)} \| \nabla v_0 \|_2 + 2C(q, p)B\left(\frac{3}{2} - \frac{2}{p}, \frac{1}{p}\right) \| u \|_{X_p}.
\]

This implies that (2.5) holds.

(ii) By using of (2.2) and the Hölder inequality, we have

\[
\| \Phi(u)(t) \|_p \leq \int_0^t \| (\nabla e^{(t-s)\Delta}) \cdot (u(s) \nabla v(s)) \|_p \, ds \\
\leq C(p, 1) \int_0^t (t-s)^{\frac{1}{2} - \frac{1}{p}} \| u(s) \nabla v(s) \|_1 \, ds \\
\leq C(p, 1) \int_0^t (t-s)^{\frac{1}{2} - \frac{1}{p}} \| u(s) \|_p \| \nabla v(s) \|_q \, ds \equiv C(p, 1) I.
\]

Note that

\[
I \leq \int_0^t (t-s)^{\frac{1}{2} - \frac{1}{p}} s^{\frac{1}{2}} \| u \|_{X_p} \left(\sup_{t>0} t^{\frac{1}{2} - \frac{1}{p}} \| \nabla v \|_q\right) \\
= t^{\frac{1}{2} - \frac{1}{p}} B\left(\frac{1}{p} - \frac{1}{2}, \frac{1}{2}\right) \| u \|_{X_p} \left(\sup_{t>0} t^{\frac{1}{2} - \frac{1}{q}} \| \nabla v \|_q\right).
\]

Thus we obtain

\[
t^{1 - \frac{1}{p}} \| \Phi(u)(t) \|_p \leq C(p, 1) B\left(\frac{1}{p} - \frac{1}{2}, \frac{1}{2}\right) \| u \|_{X_p} \left(\sup_{t>0} t^{\frac{1}{2} - \frac{1}{q}} \| \nabla v \|_q\right).
\]

From (2.5) we obtain (2.7).
For \( u \in X_p \) and \( \nabla \nu_0 \in L^2(\mathbb{R}^2) \), define \( v \) and \( \Phi(\nu) \) by (2.3) and (2.4), respectively. For \( \tilde{u} \in X_p \), define

\[
\tilde{v}(t) = e^{t\Delta} \nu_0 + \int_0^t e^{(t-s)\Delta} \tilde{u}(s) \, ds
\]

and then \( \Phi(\tilde{u}) \) by

\[
\Phi(\tilde{u})(t) = \int_0^t (\nabla e^{(t-s)\Delta}) \cdot (\tilde{u}(s)\nabla \tilde{v}(s)) \, ds.
\]

We obtain the following estimates.

**Lemma 2.3.** (i) We have

\[
\sup_{t>0} t^{\frac{1}{2} - \frac{1}{q}} \| \nabla v(t) - \nabla \tilde{v}(t) \|_q \leq \tilde{C}_0 \| u - \tilde{u} \|_{X_p},
\]

where \( \tilde{C}_0 \) is the constant given by (2.6).

(ii) We have

\[
\| \Phi(u) - \Phi(\tilde{u}) \|_{X_p} \leq \tilde{C}_0 C_1 (\| \nabla \nu_0 \|_2 + \| u \|_{X_p} + \| \tilde{u} \|_{X_p}) \| u - \tilde{u} \|_{X_p},
\]

where \( C_1 \) is the constant given by (2.8).

**Proof.** (i) By the definition of \( v \) and \( \tilde{v} \), we see that

\[
\partial_j v(t) - \partial_j \tilde{v}(t) = \int_0^t \partial_j e^{(t-s)\Delta} (u(s) - \tilde{u}(s)) \, ds
\]

for \( j = 1, 2 \). By (2.2) we have

\[
\| \partial_j v(t) - \partial_j \tilde{v}(t) \|_q \leq C(q, p) \int_0^t (t-s)^{\frac{1}{2} - \frac{1}{p}} \| u(s) - \tilde{u}(s) \|_p \, ds.
\]

By a similar argument as in the proof of (i) of Lemma 2.2, we obtain

\[
\| \partial_j v(t) - \partial_j \tilde{v}(t) \|_q \leq t^{\frac{1}{2} - \frac{1}{p} - \frac{1}{2}} C(q, p) B(\frac{\frac{3}{2} - 2}{p}, \frac{1}{p}) \| u - \tilde{u} \|_{X_p}.
\]

Thus we obtain

\[
t^{\frac{1}{2} - \frac{1}{p}} \| \nabla v(t) - \nabla \tilde{v}(t) \|_q \leq 2C(q, p) B(\frac{\frac{3}{2} - 2}{p}, \frac{1}{p}) \| u - \tilde{u} \|_{X_p} = C_0 \| u - \tilde{u} \|_{X_p}.
\]

In particular, (2.9) holds.

(ii) We see that

\[
\Phi(u)(t) - \Phi(\tilde{u})(t) = \int_0^t (\nabla e^{(t-s)\Delta}) \cdot (u(s)\nabla v(s) - \tilde{u}(s)\nabla \tilde{v}(s)) \, ds
\]

\[
= \int_0^t (\nabla e^{(t-s)\Delta}) \cdot ((u(s) - \tilde{u}(s))\nabla v(s) - \tilde{u}(s)(\nabla v(s) - \nabla \tilde{v}(s)) \, ds.
\]

Then

\[
\| \Phi(u)(t) - \Phi(\tilde{u})(t) \|_p = \int_0^t \| (\nabla e^{(t-s)\Delta}) \cdot ((u(s) - \tilde{u}(s))\nabla v(s)) \|_p \, ds
\]

\[
+ \int_0^t \| (\nabla e^{(t-s)\Delta}) \cdot \tilde{u}(s)(\nabla v(s) - \nabla \tilde{v}(s)) \|_p \, ds \equiv I_1 + I_2.
\]
By a similar argument as in the proof of (ii) of Lemma 2.2 we obtain
\[ I_1 \leq t^{\frac{1}{\beta} - 1} C(p, 1) B(\frac{1}{p} - \frac{1}{2}, \frac{1}{2}) \|u - \tilde{u}\|_{X_p} (\sup_{t > 0} t^{\frac{3}{2} - \frac{1}{\beta}} \|\nabla v\|_q) \]
and
\[ I_2 \leq t^{\frac{1}{\beta} - 1} C(p, 1) B(\frac{1}{p} - \frac{1}{2}, \frac{1}{2}) \|\tilde{u}\|_{X_p} (\sup_{t > 0} t^{\frac{3}{2} - \frac{1}{\beta}} \|\nabla v(s) - \nabla \tilde{v}(s)\|_q). \]

From (i) of Lemma 2.2 and (i) of this lemma, it follows that
\[ I_1 \leq t^{\frac{1}{\beta} - 1} \tilde{C}_0 C_1 \|u - \tilde{u}\|_{X_p} (\|\nabla v_0\|_2 + \|u\|_{X_p}) \quad \text{and} \quad I_2 \leq t^{\frac{1}{\beta} - 1} \tilde{C}_0 C_1 \|\tilde{u}\|_{X_p} \|u - \tilde{u}\|_{X_p}, \]
where \( C_1 \) is the constant given by (2.8). Thus (2.10) holds. \( \blacksquare \)

**Proof of Theorem 1.** We will show the existence of global solutions of the problem (1.3) by applying the contraction mapping principle. We remark that \( X_p \) is a Banach space endowed with the metric \( \| \cdot \|_{X_p} \). Define
\[ (2.11) \quad X_{p,M} = \{ u \in X_p : \|u\|_{X_p} \leq M \}. \]

For \( u \in X_{p,M} \), we define \( v \) and \( \Phi(u) \) by (2.3) and (2.4), respectively, and define the operator \( \Psi(u) \) by
\[ \Psi(u)(t) = e^{t \Delta} u_0 - \Phi(u)(t). \]

For \( u \in X_{p,M} \), we have \( \|\Psi(u)\|_{X_p} \leq \|e^{t \Delta} u_0\|_{X_p} + \|\Phi(u)\|_{X_p} \). By (2.1) we have
\[ \|e^{t \Delta} u_0\|_{X_p} = \sup_{t > 0} t^{1 - \frac{1}{\beta}} \|e^{t \Delta} u_0\|_p \leq (4\pi)^{-(1 - \frac{1}{p})} \|u_0\|_1 \leq (4\pi)^{-(1 - \frac{1}{p})} \alpha_0. \]
From (ii) of Lemma 2.2 we obtain \( \|\Phi(u)\|_{X_p} \leq \tilde{C}_0 C_1 (\beta_0 + M) M \). Then it follows from the first part of (1.4) that
\[ \|\Psi(u)\|_{X_p} \leq (4\pi)^{-(1 - \frac{1}{p})} \alpha_0 + \tilde{C}_0 C_1 (\beta_0 + M) M \leq M. \]
This implies that \( \Psi(u) \in X_{p,M} \) for all \( u \in X_{p,M} \).

Let \( u, \tilde{u} \in X_{p,M} \). From (ii) of Lemma 2.3, we have
\[ \|\Psi(u) - \Psi(\tilde{u})\|_{X_p} = \|\Phi(u) - \Phi(\tilde{u})\|_{X_p} \leq \tilde{C}_0 C_1 (\beta_0 + 2M) \|u - \tilde{u}\|_{X_p}. \]

From the second part of (1.4), \( \Psi \) is contractive on \( X_{p,M} \). Then, by the contractive fixed point theorem, there exists an element \( u \in X_{p,M} \) such that \( u = \Psi(u) \). Define \( v \) by (2.3). Then it follows that \( (u, v) \) is a unique solution of (1.3) such that \( \|u\|_{X_p} \leq M \). \( \blacksquare \)

**Proof of Theorem 2.** Define the set \( X_{p,M} \) by (2.11). For \( u \in X_{p,M} \), we define \( v \) by
\[ (2.12) \quad v(t) = \int_0^t e^{(t - s)\Delta} u(s) \, ds. \]

Define the operators \( \Phi(u) \) and \( \Psi_\alpha(u) \) with \( \alpha > 0 \), respectively, by (2.4) and
\[ \Psi_\alpha(u)(t) = \alpha G(\cdot, t) - \Phi(u)(t). \]
From the fact that \( \|G(\cdot, t)\|_p \leq (4\pi t)^{-(1 - \frac{1}{p})} \), we have
\[ \|\alpha G(\cdot, t)\|_{X_p} = \alpha \sup_{t > 0} t^{1 - \frac{1}{p}} \|G(\cdot, t)\|_p \leq (4\pi)^{-(1 - \frac{1}{p})} \alpha_0. \]
Then, from (ii) of Lemma 2.2 and the first part of (1.7), we obtain
\[ \|\Psi_\alpha(u)\|_{X_p} \leq (4\pi)^{-(1 - \frac{1}{p})} \alpha_0 + \tilde{C}_0 C_1 M^2 \leq M. \]
This implies that $\Psi X_{p,M} \subset X_{p,M}$. By a similar argument as in the proof of Theorem 1, we see that $\Psi_\lambda$ is contractive on $X_{p,M}$. Then, by the contractive fixed point theorem, there exists an element $u \in X_{p,M}$ such that $u = \Psi(u)$. Define $v$ by (2.12). Then it follows that $(u, v)$ is a solution of (1.6) and is unique in the class $\|u\|_{X_p} \leq M$.

For $\lambda > 0$, define $(u_\lambda, v_\lambda)$ by (1.5). Then we easily see that $(u_\lambda, v_\lambda)$ satisfies the problem (1.6). Furthermore, from the fact $\|u_\lambda(t)\|_p = \lambda^{1 - \frac{1}{p}} \|u(\lambda^2 t)\|_p$, we have $\|u_\lambda\|_{X_p} = \|u\|_{X_p} \leq M$ for all $\lambda > 0$. By the uniqueness, we obtain $u \equiv u_\lambda$ and $v \equiv v_\lambda$ for all $\lambda > 0$. This implies that $(u, v)$ is a self-similar solution of (1.6). ■

3. Proof of Theorem 3. First we show the following lemma.

**Lemma 3.1.** Let $\sigma \in (0, 1/2)$. Then

$$\sup_{t>0} t^{\frac{1}{2} - \frac{1}{q}} (1 + t)^\sigma \|\nabla e^{t \Delta} v_0\|_q < \infty. \tag{3.1}$$

**Proof.** From (2.1) we have

$$\|\partial_j e^{t \Delta} v_0\|_q = \|e^{t \Delta} \partial_j v_0\|_q \leq C t^{-(\frac{1}{2} - \frac{1}{q})} \|\partial_j v_0\|_2$$

for $j = 1, 2$. Then

$$\lim_{t \to 0} t^{\frac{1}{2} - \frac{1}{q}} (1 + t)^\sigma \|\partial_j e^{t \Delta} v_0\|_q < \infty.$$

From $\nabla v_0 \in L^1(\mathbb{R}^2)$ and (2.1), we have

$$\|\partial_j e^{t \Delta} v_0\|_q = \|e^{t \Delta} \partial_j v_0\|_q \leq C t^{-(\frac{1}{2} - \frac{1}{q})} \|\partial_j v_0\|_1.$$ 

Then

$$\lim_{t \to \infty} t^{\frac{1}{2} - \frac{1}{q}} (1 + t)^\sigma \|\partial_j e^{t \Delta} v_0\|_q < \infty.$$

Thus we obtain (3.1). ■

Throughout this section, we put

$$A_{0,\sigma} = \sup_{t>0} t^{\frac{1}{2} - \frac{1}{q}} (1 + t)^\sigma \|\nabla e^{t \Delta} v_0\|_q.$$ 

Let $\sigma \in (0, 1/2)$. For $u \in X_p$ and $t > 0$, we define

$$\|u\|_{X_p^\sigma(t)} = \sup_{0 < s \leq t} s^{\frac{1}{2} - \frac{1}{q}} (1 + s)^\sigma \|u(s)\|_p.$$ 

Let $(u, v)$ and $(u_\alpha, v_\alpha)$ be solutions of (1.3) and (1.6), respectively.

**Lemma 3.2.** Let $\sigma \in (0, 1/2)$. For $t > 0$, we have

$$\sup_{0 < s \leq t} s^{\frac{1}{2} - \frac{1}{q}} (1 + s)^\sigma \|\nabla v(s) - \nabla v_\alpha(s)\|_q \leq A_{0,\sigma} + C_{0,\sigma} \|u - u_\alpha\|_{X_p^\sigma(t)}, \tag{3.2}$$

where

$$C_{0,\sigma} = 2C(q, p) B(\frac{3}{2} - \frac{2}{p}, \frac{1}{p} - \sigma). \tag{3.3}$$

**Proof.** We see that

$$\partial_j v(t) - \partial_j v_\alpha(t) = \partial_j e^{t \Delta} v_0 + \int_0^t \partial_j e^{(t-s) \Delta} (u(s) - u_\alpha(s)) \, ds.$$
for $j = 1, 2$. Then it follows from (2.2) that
\[
\|\partial_j v(t) - \partial_j v_\alpha(t)\|_q \leq \|\partial_j e^{t\Delta} v_0\|_q + \int_0^t \|\partial_j e^{(t-s)\Delta} (u(s) - u_\alpha(s))\|_q ds \\
\leq \|\partial_j e^{t\Delta} v_0\|_q + C(q, p) \int_0^t (t-s)^{-\frac{1}{p} + \frac{1}{q} - \frac{1}{2}} \|u(s) - u_\alpha(s)\|_p ds.
\]

We observe that
\[
\int_0^t (t-s)^{-\frac{1}{p} + \frac{1}{q} - \frac{1}{2}} \|u(s) - u_\alpha(s)\|_p ds \leq \int_0^t (t-s)^{\frac{1}{2} - \frac{2}{p} s^{\frac{1}{p}-1}(1+s)^{-\sigma}} ds \|u - u_\alpha\|_{X_p^p(t)}.
\]

From the fact that
\[
(3.4) \quad \frac{1 + t}{1 + s} \leq \frac{t}{s} \quad \text{for } t \geq s,
\]
we obtain
\[
\int_0^t (t-s)^{\frac{1}{2} - \frac{2}{p} s^{\frac{1}{p}-1}(1+s)^{-\sigma}} ds \leq t^{\sigma} (1+t)^{-\sigma} \int_0^t (t-s)^{\frac{1}{2} - \frac{2}{p} s^{\frac{1}{p}-1-\sigma}} ds \\
= t^{-\frac{1}{2} + \frac{1}{q}} (1 + t)^{-\sigma} B\left(\frac{3}{2}, \frac{1}{p}, \frac{1}{p} - \sigma\right).
\]

Then
\[
\|\partial_j v(t) - \partial_j v_\alpha(t)\|_q \leq \|\partial_j e^{t\Delta} v_0\|_q + t^{-\frac{1}{2} + \frac{1}{q}} (1+t)^{-\sigma} C(q, p) B\left(\frac{3}{2}, \frac{1}{p}, \frac{1}{p} - \sigma\right) \|u - u_\alpha\|_{X_p^p(t)}.
\]

Thus we obtain (3.2). \(\blacksquare\)

**Lemma 3.3.** Let $\sigma \in (0, 1/2)$. For $t > 0$, we have
\[
(3.5) \quad \|\Phi(u) - \Phi(u_\alpha)\|_{X_p^p(t)} \leq A_{0, \sigma} C_{1, \sigma} \|u_\alpha\|_{X_p} + A_{1, \sigma} C_{1, \sigma} \|u - u_\alpha\|_{X_p^p(t)},
\]

where
\[
C_{1, \sigma} = C(p, 1) B\left(\frac{1}{p}, \frac{1}{2}, \frac{1}{2} - \sigma\right),
\]

and
\[
(3.6) \quad A_{1, \sigma} = \tilde{C}_0 (\|\nabla v_0\|_2 + \|u\|_{X_p}) + C_{0, \sigma} \|u_\alpha\|_{X_p}.
\]

In (3.6), $\tilde{C}_0$ and $C_{0, \sigma}$ are constants defined by (2.6) and (3.3), respectively.

**Proof.** We see that
\[
\|\Phi(u)(t) - \Phi(u_\alpha)(t)\|_p \\
\leq \int_0^t \|\nabla e^{(t-s)\Delta} \cdot (u(s)\nabla v(s) - u_\alpha(s)\nabla v_\alpha(s))\|_p ds \\
\leq \int_0^t \|\nabla e^{(t-s)\Delta} \cdot (u(s) - u_\alpha(s))\nabla v(s)\|_p ds \\
+ \int_0^t \|\nabla e^{(t-s)\Delta} \cdot (u_\alpha(s)(\nabla v(s) - \nabla v_\alpha(s))\|_p ds \equiv I_1 + I_2.
\]
By (2.2) and the Hölder inequality, we have
\[
I_1 \leq C(p, 1) \int_0^t (t-s)^{-\frac{3}{2} + \frac{1}{p}} \|(u(s) - u_\alpha(s))\nabla v(s)\|_1 \, ds
\]
\[
\leq C(p, 1) \int_0^t (t-s)^{-\frac{3}{2} + \frac{1}{p}} \|u(s) - u_\alpha(s)\|_p \|\nabla v(s)\|_q \, ds
\]
\[
\leq C(p, 1) \int_0^t (t-s)^{-\frac{3}{2} + \frac{1}{p}} s^{-\frac{1}{2}} (1 + s)^{-\sigma} \, ds \|u - u_\alpha\|_{X_p^s(t)} (\sup_{t>0} t^{\frac{1}{p} - \frac{1}{q}} \|\nabla v(t)\|). 
\]
From (3.4) it follows that
\[
\int_0^t (t-s)^{-\frac{3}{2} + \frac{1}{p}} s^{-\frac{1}{2}} (1 + s)^{-\sigma} \, ds \leq t^\sigma (1 + t)^{-\sigma} \int_0^t (t-s)^{-\frac{3}{2} + \frac{1}{p}} s^{-\frac{1}{2}} ds
\]
\[
= t^{-\frac{1}{2} + \frac{1}{q}} (1 + t)^{-\sigma} B\left(\frac{1}{p} - \frac{1}{2}, \frac{1}{2} - \sigma\right).
\]
From (i) of Lemma 2.2, we have
\[
(3.7) \quad I_1 \leq t^{-\frac{1}{2} + \frac{1}{q}} (1 + t)^{-\sigma} C(p, 1) B\left(\frac{1}{p} - \frac{1}{2}, \frac{1}{2} - \sigma\right) \|u - u_\alpha\|_{X_p^s(t)} (\sup_{t>0} t^{\frac{1}{p} - \frac{1}{q}} \|\nabla v(t)\|)
\]
\[
\leq t^{-\frac{1}{2} + \frac{1}{q}} (1 + t)^{-\sigma} C_{1, \sigma} C_0 (\|\nabla v\|_2 + \|u\|_{X_p}) \|u - u_\alpha\|_{X_p^s(t)}.
\]
By (2.2) and the Hölder inequality, we have
\[
I_2 \leq C(p, 1) \int_0^t (t-s)^{-\frac{3}{2} + \frac{1}{p}} \|u_\alpha(s)(\nabla v(s) - \nabla \alpha v(s))\|_1 \, ds
\]
\[
\leq C(p, 1) \int_0^t (t-s)^{-\frac{3}{2} + \frac{1}{p}} \|u_\alpha(s)\|_p \|\nabla v(s) - \nabla \alpha v(s)\|_q \, ds
\]
\[
\leq C(p, 1) \int_0^t (t-s)^{-\frac{3}{2} + \frac{1}{p}} s^{-\frac{1}{2}} (1 + s)^{-\sigma} \, ds \|u_\alpha\|_{X_p A_{2, \sigma}},
\]
where
\[
A_{2, \sigma} = (\sup_{t>0} t^{\frac{1}{p} - \frac{1}{q}} (1 + t)^\sigma) \|\nabla v(t) - \nabla v_\alpha(t)\|.
\]
It follows from (3.4) that
\[
I_2 \leq t^{-\frac{1}{2} + \frac{1}{q}} (1 + t)^{-\sigma} C(p, 1) B\left(\frac{1}{p} - \frac{1}{2}, \frac{1}{2} - \sigma\right) \|u_\alpha\|_{X_p A_{2, \sigma}}.
\]
From Lemma 3.2 we obtain
\[
(3.8) \quad I_2 \leq t^{-\frac{1}{2} + \frac{1}{q}} (1 + t)^{-\sigma} C_{1, \sigma} \|u_\alpha\|_{X_p} (A_{0, \sigma} + C_{0, \sigma} \|u - u_\alpha\|_{X_p^s(t)}).
\]
Combining (3.7) and (3.8), we obtain (3.5). ■

Proof of Theorem 3. We see that
\[
\|u(s) - u_\alpha(s)\|_p = \|e^{t\Delta} u_0 - \alpha G(\cdot, t)\|_p + \|\Phi(u) - \Phi(u_\alpha)\|_p.
\]
By the arguments in the proofs of Theorems 1 and 2, we obtain
\[
\sup_{t>0} t^{\frac{1}{p} + 1} \|e^{t\Delta} u_0 - \alpha G(\cdot, t)\|_p \leq \sup_{t>0} t^{\frac{1}{p} + 1} \|e^{t\Delta} u_0\|_p + \sup_{t>0} t^{\frac{1}{p} + 1} \|\alpha G(\cdot, t)\|_p < \infty.
\]
By [5, Lemma 2.1] we have
\[
\|e^{t\Delta} u_0 - \alpha G(\cdot, t)\|_p \leq Ct^{\frac{1}{p} - \frac{2}{3}} \|(\nu)^2 + 1\| u_0 \|_1.
\]
Then we have
\[ \sup_{t>0} t^{1+\frac{1}{p}} (1 + t) \sigma \| e^t \Delta u_0 - \alpha G(\cdot, t) \|_p \equiv A_{2, \sigma} < \infty. \]

From Lemma 3.3 we obtain
\[ \| u - u_\alpha \|_{X^\sigma_p(t)} \leq A_{2, \sigma} + \| \Phi(u) - \Phi(u_\alpha) \|_{X^\sigma_p(t)} \]
(3.9)
\[ \leq A_{2, \sigma} + A_{0, \sigma} C_{1, \sigma} \| u_\alpha \|_{X^\sigma_p} + A_{1, \sigma} C_{1, \sigma} \| u - u_\alpha \|_{X^\sigma_p(t)}. \]

We note here that \( C_{0, \sigma} \to C_0 \) and \( C_{1, \sigma} \to C_1 \) as \( \sigma \to 0 \), where \( C_0 \) and \( C_1 \) are constants defined by (2.6) and (2.8), respectively. Then, from (3.6) and \( C_0 \leq \bar{C}_0 \), we obtain
\[ \lim_{\sigma \to 0} A_{1, \sigma} = \bar{C}_0 (\| \nabla v_0 \|_2 + \| u \|_{X^\sigma_p}) + C_0 \| u_\alpha \|_{X^\sigma_p} \leq \bar{C}_0 (\beta_0 + 2M). \]

From the second part of (1.4), we find that \( C_{1, \sigma} A_{1, \sigma} < 1 \) for sufficient small \( \sigma > 0 \). Then it follows from (3.9) that
\[ \| u - u_\alpha \|_{X^\sigma_p(t)} \leq \frac{A_{2, \sigma} + A_{0, \sigma} C_{1, \sigma} \| u_\alpha \|_{X^\sigma_p}}{1 - A_{1, \sigma} C_{1, \sigma}} < \infty \quad \text{for } t > 0. \]

This implies that \( \| u - u_\alpha \|_{X^\sigma_p(t)} \) is bounded for all \( t > 0 \). Thus we obtain
\[ \| u - u_\alpha \|_{X^\sigma_p} \leq C (1 + t)^\sigma \quad \text{for all } t > 0 \]
with some constant \( C > 0 \). In particular, we conclude that (1.8) holds. \( \blacksquare \)

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**References**


