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ASYMPTOTICALLY SELF-SIMILAR SOLUTIONS FOR THE PARABOLIC SYSTEM MODELLING CHEMOTAXIS

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Abstract. We consider a nonlinear parabolic system modelling chemotaxis

$$u_t = \nabla \cdot (\nabla u - u\nabla v), \quad v_t = \Delta v + u$$

in $\mathbf{R}^2$, $t > 0$. We first prove the existence of time-global solutions, including self-similar solutions, for small initial data, and then show the asymptotically self-similar behavior for a class of general solutions.

1. Introduction. We are concerned with the large time behavior of solutions to the Cauchy problem for the following system of partial differential equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u\nabla v), & x \in \mathbf{R}^2,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v + u, & x \in \mathbf{R}^2,\ t > 0. \end{cases} \tag{1.1}$$

On the system we impose initial conditions

$$u(x,0) = u_0, \quad v(x,0) = v_0, \quad x \in \mathbf{R}^2, \tag{1.2}$$

where $u_0 \geq 0$ and $v_0 \geq 0$.

The system (1.1) is a mathematical model describing chemotaxis, that is, the directed movement of an organism in response to gradients of a chemical attractant (see [6, 12, 4]). The function $u(x,t) \geq 0$ corresponds to the population of the organism at the place $x \in \mathbf{R}^2$ and time $t > 0$, and $v(x,t) \geq 0$ to the concentration of the chemical.

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The existence of local and global in time solutions, including self-similar solutions, of the problem (1.1)–(1.2) has been studied by Biler [3]. In this paper we show the asymptotically self-similar behavior for a class of general solutions of (1.1)–(1.2).

We write (1.1)–(1.2) in the form of the integral equation

$$\begin{cases} u(t) = e^{t\Delta}u_0 - \displaystyle\int_0^t (\nabla e^{(t-s)\Delta}) \cdot (u(s)\nabla v(s))\, ds, \\ v(t) = e^{t\Delta}v_0 + \displaystyle\int_0^t e^{(t-s)\Delta}u(s)\, ds, \end{cases} \tag{1.3}$$

where

$$(e^{t\Delta}f)(x) = \int_{\mathbf{R}^2} G(x-y,t)f(y)dy$$

and $G(x,t)$ is the heat kernel

$$G(x,t) = \frac{1}{4\pi t}\exp\left(-\frac{|x|^2}{4t}\right).$$

In what follows, $\|\cdot\|_p$ represents the norm of $L^p(\mathbf{R}^2)$ for $1 \le p \le \infty$. For $u : \mathbf{R}^2 \to \mathbf{R}$, we use the notations $\nabla u = (\partial_1 u, \partial_2 u)$ and $\|\nabla u\|_p = \|\partial_1 u\|_p + \|\partial_2 u\|_p$, where $\partial_j = \partial/\partial x_j$.

We look for mild solutions (u,v) of (1.3) in the class $u \in X_p$ with $p \in (4/3, 2)$, where X_p is the set of Bochner measurable functions $u : (0,\infty) \to L^p(\mathbf{R}^2)$ such that $\sup_{t>0} t^{1-\frac{1}{p}}\|u(t)\|_p < \infty$. We will obtain v according to the second formula in (1.3) for $u \in X_p$. Define $\|\cdot\|_{X_p}$ by

$$\|u\|_{X_p} = \sup_{t>0} t^{1-\frac{1}{p}}\|u(t)\|_p.$$

Throughout this paper, $p \in (4/3, 2)$ is fixed. First we show the existence of time global solution of (1.3) with $u_0 \in L^1(\mathbf{R}^2)$ and $\nabla v_0 \in L^2(\mathbf{R}^2)$.

Theorem 1. *Assume that constants $M > 0$, $\alpha_0 > 0$, and $\beta_0 \ge 0$ satisfy the inequalities*

$$\frac{\alpha_0}{(4\pi)^{1-\frac{1}{p}}M} + \tilde{C}_0 C_1(\beta_0 + M) \le 1 \quad \textit{and} \quad \tilde{C}_0 C_1(\beta_0 + 2M) < 1, \tag{1.4}$$

where positive constants $\tilde{C}_0$ and C_1 are given below in Lemma 2.2. Suppose that $u_0 \in L^1(\mathbf{R}^2)$ and $\nabla v_0 \in L^2(\mathbf{R}^2)$ satisfy $\|u_0\|_1 \le \alpha_0$ and $\|\nabla v_0\|_2 \le \beta_0$. Then there exists a unique global solution (u,v) of (1.3) such that $\|u\|_{X_p} \le M$.

The system (1.1) is invariant under the similarity transformation

$$u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t) \quad \text{and} \quad v_\lambda(x,t) = v(\lambda x, \lambda^2 t) \tag{1.5}$$

for $\lambda > 0$, that is, if (u,v) is a solution of (1.1) then so is (u_λ, v_λ). A solution (u,v) is said to be *self-similar*, when the solution is invariant under this transformation, that is, $u(x,t) \equiv u_\lambda(x,t)$ and $v(x,t) \equiv v_\lambda(x,t)$ for all $\lambda > 0$. Letting $\lambda = 1/\sqrt{t}$, and putting $\phi(x) = u(x,1)$ and $\psi(x) = v(x,1)$, we find that the self-similar solution (u,v) has the form

$$u(x,t) = \frac{1}{t}\phi\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad v(x,t) = \psi\left(\frac{x}{\sqrt{t}}\right)$$

for $x \in \mathbf{R}^2$ and $t > 0$.

Let us consider the problem

$$
\begin{cases} u(t) = \alpha G(\cdot,t) - \displaystyle\int_0^t (\nabla e^{(t-s)\Delta}) \cdot (u(s)\nabla v(s))\,ds, \\ v(t) = \displaystyle\int_0^t e^{(t-s)\Delta} u(s)\,ds, \end{cases} \tag{1.6}
$$

where α is a positive constant and G is the heat kernel. We show the existence of the self-similar solutions of (1.6).

THEOREM 2. *Assume that constants $M > 0$ and $\alpha_0 > 0$ satisfy the inequalities*

$$
\frac{\alpha_0}{(4\pi)^{1-\frac{1}{p}} M} + \tilde{C}_0 C_1 M \le 1 \quad \textit{and} \quad 2\tilde{C}_0 C_1 M < 1, \tag{1.7}
$$

where positive constants $\tilde{C}_0$ and C_1 are the same as in Theorem 1. Then, for $\alpha \in (0, \alpha_0]$, there exists a unique self-similar solution (u_α, v_α) of (1.6) such that $\|u_\alpha\|_{X_p} \le M$.

REMARK 1. (i) We do not know the uniqueness of self-similar solution of (1.6) without the assumption $\|u_\alpha\|_{X_p} \le M$. Concerning the non-uniqueness of self-similar solutions for semilinear heat equations, we refer to [11].

(ii) For the properties of self-similar solutions to (1.1), we refer to [3, 9]. We also refer to [1, 2, 10], where the self-similar solutions to the parabolic-elliptic problem have been studied.

(iii) It is clear that, if M, α_0, and β_0 satisfy (1.4), then (1.7) holds. Thus, for a solution (u, v), constructed by Theorem 1, there exists a self-similar solution (u_α, v_α) of (1.6) with $\alpha = \|u_0\|_1$.

Let (u_α, v_α) be a self-similar solution of (1.6), constructed by Theorem 2. By the argument above, (u_α, v_α) has the form

$$
u_\alpha(x,t) = \frac{1}{t}\phi_\alpha\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad v_\alpha(x,t) = \psi_\alpha\left(\frac{x}{\sqrt{t}}\right)
$$

for $x \in \mathbf{R}^2$ and $t > 0$. We note here that

$$
t^{1-\frac{1}{p}} \|u_\alpha(\cdot,t)\|_p = \|\phi_\alpha\|_p \quad \text{for } t > 0.
$$

From $\|u_\alpha\|_{X_p} \le M$, it follows that $\phi_\alpha \in L^p(\mathbf{R}^2)$.

We consider the asymptotic behavior of solutions of (1.3) constructed by Theorem 1.

THEOREM 3. *Let (u, v) be a solution of (1.3) constructed by Theorem 1. Assume, in addition, that u_0 and v_0 satisfy $(1+|x|^2)u_0 \in L^1(\mathbf{R}^2)$ and $\nabla v_0 \in L^1(\mathbf{R}^2)$. Let (u_α, v_α) be a self-similar solution of (1.6) with $\alpha = \|u_0\|_1$, constructed by Theorem 2. Then there exists $\sigma \in (0, 1/2)$ such that*

$$
t^{1-\frac{1}{p}} \|u(\cdot,t) - u_\alpha(\cdot,t)\|_p = O(t^{-\sigma}) \quad \textit{as } t \to \infty. \tag{1.8}
$$

In particular, $\|tu(\cdot\sqrt{t},t) - \phi_\alpha(\cdot)\|_p = O(t^{-\sigma})$ as $t \to \infty$.

It is interesting to compare the results for the problem (1.1) and the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u\nabla v), & x \in \mathbf{R}^2,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u, & x \in \mathbf{R}^2,\ t > 0. \end{cases} \tag{1.9}$$

It has been shown by Nagai [7, 8] that every bounded solution of the problem (1.9) on $\mathbf{R}^2 \times [0, \infty)$ decays to zero and behaves like a constant multiple of the heat kernel as $t \to \infty$. The large time behavior for higher dimensional case is also studied in [8].

Theorems 1 and 2 are proven by employing the contraction mapping argument in suitable function spaces. We prove Theorem 3 by estimating the terms in the integral equations. We will give the proofs of Theorems 1 and 2 in Section 2, and we prove Theorem 3 in Section 3.

2. Proofs of Theorems 1 and 2. First we recall L^p–L^q estimates for the heat semigroup, which are proved by Young's inequality for convolution.

LEMMA 2.1 *Let* $1 \le q \le p \le \infty$ *and* $f \in L^q(\mathbf{R}^2)$. *Then*

$$\|e^{t\Delta} f\|_p \le (4\pi t)^{-(\frac{1}{q} - \frac{1}{p})} \|f\|_q, \tag{2.1}$$

$$\|\partial_j e^{t\Delta} f\|_p \le C(p,q) t^{-(\frac{1}{q} - \frac{1}{p}) - \frac{1}{2}} \|f\|_q, \quad j = 1, 2, \tag{2.2}$$

where $C(p,q)$ *is a positive constant depending only on* p *and* q.

Let

$$X_p = \{u : (0,\infty) \to L^p(\mathbf{R}^2) :\ \sup_{t>0} t^{1-\frac{1}{p}} \|u(t)\|_p < \infty\}$$

with $p \in (4/3, 2)$. Define $\|\cdot\|_{X_p}$ by

$$\|u\|_{X_p} = \sup_{t>0} t^{1-\frac{1}{p}} \|u(t)\|_p.$$

For $u \in X_p$ and $\nabla v_0 \in L^2(\mathbf{R}^2)$, we define v by

$$v(t) = e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta} u(s)\, ds, \tag{2.3}$$

and then $\Phi(u)$ by

$$\Phi(u)(t) = \int_0^t (\nabla e^{(t-s)\Delta}) \cdot (u(s) \nabla v(s))\, ds. \tag{2.4}$$

In what follows, we put $q = p/(p-1)$ for fixed $p \in (4/3, 2)$. We denote by $B(p,q)$ the beta function.

LEMMA 2.2. (i) *We have*

$$\sup_{t>0} t^{\frac{1}{2} - \frac{1}{q}} \|\nabla v(t)\|_q \le \tilde{C}_0 (\|\nabla v_0\|_2 + \|u\|_{X_p}), \tag{2.5}$$

where

$$\tilde{C}_0 = \max\{(4\pi)^{-(\frac{1}{2} - \frac{1}{q})},\ C_0\}, \quad C_0 = 2C(q,p) B(\tfrac{3}{2} - \tfrac{2}{p}, \tfrac{1}{p}). \tag{2.6}$$

(ii) *We have*

$$\|\Phi(u)\|_{X_p} \le \tilde{C}_0 C_1(\|\nabla v_0\|_2 + \|u\|_{X_p})\|u\|_{X_p}, \tag{2.7}$$

where

$$C_1 = C(p,1)B(\tfrac{1}{p} - \tfrac{1}{2}, \tfrac{1}{2}). \tag{2.8}$$

Proof. (i) From (2.3) we have

$$\partial_j v(t) = e^{t\Delta}\partial_j v_0 + \int_0^t \partial_j e^{(t-s)\Delta} u(s)\,ds$$

for $j = 1, 2$. From (2.1) and (2.2) we obtain

$$\begin{aligned}\|\partial_j v(t)\|_q &\le \|e^{t\Delta}\partial_j v_0\|_q + \int_0^t \|\partial_j e^{(t-s)\Delta} u(s)\|_q\,ds \\ &\le (4\pi t)^{\frac{1}{q}-\frac{1}{2}}\|\partial_j v_0\|_2 + C(q,p)\int_0^t (t-s)^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}\|u(s)\|_p\,ds.\end{aligned}$$

Note that

$$\begin{aligned}\int_0^t (t-s)^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}\|u(s)\|_p\,ds &\le \int_0^t (t-s)^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}} s^{\frac{1}{p}-1}\,ds\,\|u\|_{X_p} \\ &= t^{\frac{1}{q}-\frac{1}{2}} B(\tfrac{3}{2} - \tfrac{2}{p}, \tfrac{1}{p})\|u\|_{X_p}.\end{aligned}$$

By the definition $\|\nabla u\|_q = \|\partial_1 u\|_q + \|\partial_2 u\|_q$, we obtain

$$t^{\frac{1}{2}-\frac{1}{q}}\|\nabla v(t)\|_q \le (4\pi)^{-(\frac{1}{2}-\frac{1}{q})}\|\nabla v_0\|_2 + 2C(q,p)B(\tfrac{3}{2} - \tfrac{2}{p}, \tfrac{1}{p})\|u\|_{X_p}.$$

This implies that (2.5) holds.

(ii) By using of (2.2) and the Hölder inequality, we have

$$\begin{aligned}\|\Phi(u)(t)\|_p &\le \int_0^t \|(\nabla e^{(t-s)\Delta})\cdot(u(s)\nabla v(s))\|_p\,ds \\ &\le C(p,1)\int_0^t (t-s)^{\frac{1}{p}-\frac{3}{2}}\|u(s)\nabla v(s)\|_1\,ds \\ &\le C(p,1)\int_0^t (t-s)^{\frac{1}{p}-\frac{3}{2}}\|u(s)\|_p\|\nabla v(s)\|_q\,ds \equiv C(p,1)I.\end{aligned}$$

Note that

$$\begin{aligned}I &\le \int_0^t (t-s)^{\frac{1}{p}-\frac{3}{2}} s^{-\frac{1}{2}}\ \|u\|_{X_p}(\sup_{t>0} t^{\frac{1}{2}-\frac{1}{q}}\|\nabla v\|_q) \\ &= t^{\frac{1}{p}-1}B(\tfrac{1}{p} - \tfrac{1}{2}, \tfrac{1}{2})\|u\|_{X_p}(\sup_{t>0} t^{\frac{1}{2}-\frac{1}{q}}\|\nabla v\|_q).\end{aligned}$$

Thus we obtain

$$t^{1-\frac{1}{p}}\|\Phi(u)(t)\|_p \le C(p,1)B(\tfrac{1}{p} - \tfrac{1}{2}, \tfrac{1}{2})\|u\|_{X_p}(\sup_{t>0} t^{\frac{1}{2}-\frac{1}{q}}\|\nabla v\|_q).$$

From (2.5) we obtain (2.7). ∎

For $u \in X_p$ and $\nabla v_0 \in L^2(\mathbf{R}^2)$, define v and $\Phi(u)$ by (2.3) and (2.4), respectively. For $\tilde{u} \in X_p$, define

$$\tilde{v}(t) = e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta} \tilde{u}(s)\, ds$$

and then $\Phi(\tilde{u})$ by

$$\Phi(\tilde{u})(t) = \int_0^t (\nabla e^{(t-s)\Delta}) \cdot (\tilde{u}(s) \nabla \tilde{v}(s))\, ds.$$

We obtain the following estimates.

LEMMA 2.3. (i) *We have*

$$\sup_{t>0} t^{\frac{1}{2}-\frac{1}{q}} \|\nabla v(t) - \nabla \tilde{v}(t)\|_q \le \tilde{C}_0 \|u - \tilde{u}\|_{X_p}, \tag{2.9}$$

where $\tilde{C}_0$ *is the constant given by* (2.6).

(ii) *We have*

$$\|\Phi(u) - \Phi(\tilde{u})\|_{X_p} \le \tilde{C}_0 C_1 (\|\nabla v_0\|_2 + \|u\|_{X_p} + \|\tilde{u}\|_{X_p}) \|u - \tilde{u}\|_{X_p}, \tag{2.10}$$

where C_1 *is the constant given by* (2.8).

Proof. (i) By the definition of v and $\tilde{v}$, we see that

$$\partial_j v(t) - \partial_j \tilde{v}(t) = \int_0^t \partial_j e^{(t-s)\Delta} (u(s) - \tilde{u}(s))\, ds$$

for $j = 1, 2$. By (2.2) we have

$$\|\partial_j v(t) - \partial_j \tilde{v}(t)\|_q \le C(q,p) \int_0^t (t-s)^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}} \|u(s) - \tilde{u}(s)\|_p\, ds.$$

By a similar argument as in the proof of (i) of Lemma 2.2, we obtain

$$\|\partial_j v(t) - \partial_j \tilde{v}(t)\|_q \le t^{\frac{1}{q}-\frac{1}{2}} C(q,p) B(\tfrac{3}{2} - \tfrac{2}{p}, \tfrac{1}{p}) \|u - \tilde{u}\|_{X_p}.$$

Thus we obtain

$$t^{\frac{1}{2}-\frac{1}{q}} \|\nabla v(t) - \nabla \tilde{v}(t)\|_q \le 2C(q,p) B(\tfrac{3}{2} - \tfrac{2}{p}, \tfrac{1}{p}) \|u - \tilde{u}\|_{X_p} = C_0 \|u - \tilde{u}\|_{X_p}.$$

In particular, (2.9) holds.

(ii) We see that

$$\begin{aligned}\Phi(u)(t) - \Phi(\tilde{u})(t) &= \int_0^t (\nabla e^{(t-s)\Delta}) \cdot (u(s)\nabla v(s) - \tilde{u}(s) \nabla \tilde{v}(s))\, ds \\ &= \int_0^t (\nabla e^{(t-s)\Delta}) \cdot ((u(s) - \tilde{u}(s)) \nabla v(s) - \tilde{u}(s)(\nabla v(s) - \nabla \tilde{v}(s))\, ds.\end{aligned}$$

Then

$$\begin{aligned}\|\Phi(u)(t) - \Phi(\tilde{u})(t)\|_p &= \int_0^t \|(\nabla e^{(t-s)\Delta}) \cdot ((u(s) - \tilde{u}(s))\, \nabla v(s)\|_p\, ds \\ &\quad + \int_0^t \|(\nabla e^{(t-s)\Delta}) \cdot \tilde{u}(s)\, (\nabla v(s) - \nabla \tilde{v}(s))\,\|_p\, ds \equiv I_1 + I_2.\end{aligned}$$

By a similar argument as in the proof of (ii) of Lemma 2.2 we obtain

$$I_1 \le t^{\frac{1}{p}-1} C(p,1) B(\tfrac{1}{p} - \tfrac{1}{2}, \tfrac{1}{2}) \|u - \tilde{u}\|_{X_p} (\sup_{t>0} t^{\frac{1}{2}-\frac{1}{q}} \|\nabla v\|_q)$$

and

$$I_2 \le t^{\frac{1}{p}-1} C(p,1) B(\tfrac{1}{p} - \tfrac{1}{2}, \tfrac{1}{2}) \|\tilde{u}\|_{X_p} (\sup_{t>0} t^{\frac{1}{2}-\frac{1}{q}} \|\nabla v(s) - \nabla \tilde{v}(s)\|_q).$$

From (i) of Lemma 2.2 and (i) of this lemma, it follows that

$$I_1 \le t^{\frac{1}{p}-1} \tilde{C}_0 C_1 \|u - \tilde{u}\|_{X_p} (\|\nabla v_0\|_2 + \|u\|_{X_p}) \quad \text{and} \quad I_2 \le t^{\frac{1}{p}-1} \tilde{C}_0 C_1 \|\tilde{u}\|_{X_p} \|u - \tilde{u}\|_{X_p},$$

where C_1 is the constant given by (2.8). Thus (2.10) holds. ■

Proof of Theorem 1. We will show the existence of global solutions of the problem (1.3) by applying the contraction mapping principle. We remark that X_p is a Banach space endowed with the metric $\|\cdot\|_{X_p}$. Define

$$X_{p,M} = \{u \in X_p : \|u\|_{X_p} \le M\}. \tag{2.11}$$

For $u \in X_{p,M}$, we define v and $\Phi(u)$ by (2.3) and (2.4), respectively, and define the operator $\Psi(u)$ by

$$\Psi(u)(t) = e^{t\Delta} u_0 - \Phi(u)(t).$$

For $u \in X_{p,M}$, we have $\|\Psi(u)\|_{X_p} \le \|e^{t\Delta} u_0\|_{X_p} + \|\Phi(u)\|_{X_p}$. By (2.1) we have

$$\|e^{t\Delta} u_0\|_{X_p} = \sup_{t>0} t^{1-\frac{1}{p}} \|e^{t\Delta} u_0\|_p \le (4\pi)^{-(1-\frac{1}{p})} \|u_0\|_1 \le (4\pi)^{-(1-\frac{1}{p})} \alpha_0.$$

From (ii) of Lemma 2.2 we obtain $\|\Phi(u)\|_{X_p} \le \tilde{C}_0 C_1 (\beta_0 + M) M$. Then it follows from the first part of (1.4) that

$$\|\Psi(u)\|_{X_p} \le (4\pi)^{-(1-\frac{1}{p})} \alpha_0 + \tilde{C}_0 C_1 (\beta_0 + M) M \le M.$$

This implies that $\Psi(u) \in X_{p,M}$ for all $u \in X_{p,M}$.

Let $u, \tilde{u} \in X_p$. From (ii) of Lemma 2.3, we have

$$\|\Psi(u) - \Psi(\tilde{u})\|_{X_p} = \|\Phi(u) - \Phi(\tilde{u})\|_{X_p} \le \tilde{C}_0 C_1 (\beta_0 + 2M) \|u - \tilde{u}\|_{X_p}.$$

From the second part of (1.4), Ψ is contractive on $X_{p,M}$. Then, by the contractive fixed point theorem, there exists an element $u \in X_{p,M}$ such that $u = \Psi(u)$. Define v by (2.3). Then it follows that (u, v) is a unique solution of (1.3) such that $\|u\|_{X_p} \le M$. ■

Proof of Theorem 2. Define the set $X_{p,M}$ by (2.11). For $u \in X_{p,M}$, we define v by

$$v(t) = \int_0^t e^{(t-s)\Delta} u(s)\, ds. \tag{2.12}$$

Define the operators $\Phi(u)$ and $\Psi_\alpha(u)$ with $\alpha > 0$, respectively, by (2.4) and

$$\Psi_\alpha(u)(t) = \alpha G(\cdot, t) - \Phi(u)(t).$$

From the fact that $\|G(\cdot,t)\|_p \le (4\pi t)^{-(1-\frac{1}{p})}$, we have

$$\|\alpha G(\cdot,t)\|_{X_p} = \alpha \sup_{t>0} t^{1-\frac{1}{p}} \|G(\cdot,t)\|_p \le (4\pi)^{-(1-\frac{1}{p})} \alpha_0.$$

Then, from (ii) of Lemma 2.2 and the first part of (1.7), we obtain

$$\|\Psi_\alpha(u)\|_{X_p} \le (4\pi)^{-(1-\frac{1}{p})} \alpha_0 + \tilde{C}_0 C_1 M^2 \le M.$$

This implies that $\Psi X_{p,M} \subset X_{p,M}$. By a similar argument as in the proof of Theorem 1, we see that Ψ_α is contractive on $X_{p,M}$. Then, by the contractive fixed point theorem, there exists an element $u \in X_{p,M}$ such that $u = \Psi(u)$. Define v by (2.12). Then it follows that (u, v) is a solution of (1.6) and is unique in the class $\|u\|_{X_p} \leq M$.

For $\lambda > 0$, define (u_λ, v_λ) by (1.5). Then we easily see that (u_λ, v_λ) satisfies the problem (1.6). Furthermore, from the fact $\|u_\lambda(t)\|_p = \lambda^{1-\frac{1}{p}}\|u(\lambda^2 t)\|_p$, we have $\|u_\lambda\|_{X_p} = \|u\|_{X_p} \leq M$ for all $\lambda > 0$. By the uniqueness, we obtain $u \equiv u_\lambda$ and $v \equiv v_\lambda$ for all $\lambda > 0$. This implies that (u, v) is a self-similar solution of (1.6). ■

3. Proof of Theorem 3. First we show the following lemma.

LEMMA 3.1. *Let $\sigma \in (0, 1/2)$. Then*

$$\sup_{t>0} t^{\frac{1}{2}-\frac{1}{q}}(1+t)^\sigma \|\nabla e^{t\Delta} v_0\|_q < \infty. \tag{3.1}$$

Proof. From (2.1) we have

$$\|\partial_j e^{t\Delta} v_0\|_q = \|e^{t\Delta}\partial_j v_0\|_q \leq C t^{-(\frac{1}{2}-\frac{1}{q})}\|\partial_j v_0\|_2$$

for $j = 1, 2$. Then

$$\lim_{t\to 0} t^{\frac{1}{2}-\frac{1}{q}}(1+t)^\sigma \|\partial_j e^{t\Delta} v_0\|_q < \infty.$$

From $\nabla v_0 \in L^1(\mathbf{R}^2)$ and (2.1), we have

$$\|\partial_j e^{t\Delta} v_0\|_q = \|e^{t\Delta}\partial_j v_0\|_q \leq C t^{-(1-\frac{1}{q})}\|\partial_j v_0\|_1.$$

Then

$$\lim_{t\to \infty} t^{\frac{1}{2}-\frac{1}{q}}(1+t)^\sigma \|\partial_j e^{t\Delta} v_0\|_q < \infty.$$

Thus we obtain (3.1). ■

Throughout this section, we put

$$A_{0,\sigma} = \sup_{t>0} t^{\frac{1}{2}-\frac{1}{q}}(1+t)^\sigma \|\nabla e^{t\Delta} v_0\|_q.$$

Let $\sigma \in (0, 1/2)$. For $u \in X_p$ and $t > 0$, we define

$$\|u\|_{X_p^\sigma(t)} = \sup_{0<s\leq t} s^{1-\frac{1}{p}}(1+s)^\sigma \|u(s)\|_p.$$

Let (u, v) and (u_α, v_α) be solutions of (1.3) and (1.6), respectively.

LEMMA 3.2. *Let $\sigma \in (0, 1/2)$. For $t > 0$, we have*

$$\sup_{0<s\leq t} s^{\frac{1}{2}-\frac{1}{q}}(1+s)^\sigma \|\nabla v(s) - \nabla v_\alpha(s)\|_q \leq A_{0,\sigma} + C_{0,\sigma}\|u - u_\alpha\|_{X_p^\sigma(t)}, \tag{3.2}$$

where

$$C_{0,\sigma} = 2C(q,p)B(\tfrac{3}{2} - \tfrac{2}{p}, \tfrac{1}{p} - \sigma). \tag{3.3}$$

Proof. We see that

$$\partial_j v(t) - \partial_j v_\alpha(t) = \partial_j e^{t\Delta} v_0 + \int_0^t \partial_j e^{(t-s)\Delta}(u(s) - u_\alpha(s))\,ds$$

for $j=1,2$. Then it follows from (2.2) that

$$\begin{aligned}\|\partial_j v(t)-\partial_j v_\alpha(t)\|_q &\le \|\partial_j e^{t\Delta}v_0\|_q+\int_0^t \|\partial_j e^{(t-s)\Delta}(u(s)-u_\alpha(s))\|_q\,ds\\ &\le \|\partial_j e^{t\Delta}v_0\|_q+C(q,p)\int_0^t (t-s)^{-\frac{1}{p}+\frac{1}{q}-\frac{1}{2}}\|u(s)-u_\alpha(s)\|_p\,ds.\end{aligned}$$

We observe that

$$\int_0^t (t-s)^{-\frac{1}{p}+\frac{1}{q}-\frac{1}{2}}\|u(s)-u_\alpha(s)\|_p\,ds \le \int_0^t (t-s)^{\frac{1}{2}-\frac{2}{p}}s^{\frac{1}{p}-1}(1+s)^{-\sigma}\,ds\|u-u_\alpha\|_{X_p^\sigma(t)}.$$

From the fact that

$$\frac{1+t}{1+s}\le\frac{t}{s}\quad\text{for } t\ge s, \tag{3.4}$$

we obtain

$$\begin{aligned}\int_0^t (t-s)^{\frac{1}{2}-\frac{2}{p}}s^{\frac{1}{p}-1}(1+s)^{-\sigma}\,ds &\le t^\sigma(1+t)^{-\sigma}\int_0^t (t-s)^{\frac{1}{2}-\frac{2}{p}}s^{\frac{1}{p}-1-\sigma}\,ds\\ &= t^{-\frac{1}{2}+\frac{1}{q}}(1+t)^{-\sigma}B(\tfrac{3}{2}-\tfrac{2}{p},\tfrac{1}{p}-\sigma).\end{aligned}$$

Then

$$\|\partial_j v(t)-\partial_j v_\alpha(t)\|_q\le\|\partial_j e^{t\Delta}v_0\|_q+t^{-\frac{1}{2}+\frac{1}{q}}(1+t)^{-\sigma}C(q,p)B(\tfrac{3}{2}-\tfrac{2}{p},\tfrac{1}{p}-\sigma)\|u-u_\alpha\|_{X_p^\sigma(t)}.$$

Thus we obtain (3.2). ∎

LEMMA 3.3. *Let $\sigma\in(0,1/2)$. For $t>0$, we have*

$$\|\Phi(u)-\Phi(u_\alpha)\|_{X_p^\sigma(t)}\le A_{0,\sigma}C_{1,\sigma}\|u_\alpha\|_{X_p}+A_{1,\sigma}C_{1,\sigma}\|u-u_\alpha\|_{X_p^\sigma(t)}, \tag{3.5}$$

where

$$C_{1,\sigma}=C(p,1)B(\tfrac{1}{p}-\tfrac{1}{2},\tfrac{1}{2}-\sigma),$$

and

$$A_{1,\sigma}=\tilde{C}_0(\|\nabla v_0\|_2+\|u\|_{X_p})+C_{0,\sigma}\|u_\alpha\|_{X_p}. \tag{3.6}$$

In (3.6), $\tilde{C}_0$ and $C_{0,\sigma}$ are constants defined by (2.6) and (3.3), respectively.

Proof. We see that

$$\begin{aligned}&\|\Phi(u)(t)-\Phi(u_\alpha)(t)\|_p\\ &\le\int_0^t \|(\nabla e^{(t-s)\Delta})\cdot(u(s)\nabla v(s)-u_\alpha(s)\nabla v_\alpha(s))\|_p\,ds\\ &\le\int_0^t \|(\nabla e^{(t-s)\Delta})\cdot(u(s)-u_\alpha(s))\nabla v(s)\|_p\,ds\\ &\quad+\int_0^t \|(\nabla e^{(t-s)\Delta})\cdot(u_\alpha(s)(\nabla v(s)-\nabla v_\alpha(s))\|_p\,ds\equiv I_1+I_2.\end{aligned}$$

By (2.2) and the Hölder inequality, we have

$$\begin{aligned} I_1 &\le C(p,1)\int_0^t (t-s)^{-\frac{3}{2}+\frac{1}{p}}\|(u(s)-u_\alpha(s))\nabla v(s)\|_1\,ds \\ &\le C(p,1)\int_0^t (t-s)^{-\frac{3}{2}+\frac{1}{p}}\|u(s)-u_\alpha(s)\|_p\|\nabla v(s)\|_q\,ds \\ &\le C(p,1)\int_0^t (t-s)^{-\frac{3}{2}+\frac{1}{p}}s^{-\frac{1}{2}}(1+s)^{-\sigma}\,ds\|u-u_\alpha\|_{X_p^\sigma(t)}(\sup_{t>0} t^{\frac{1}{2}-\frac{1}{q}}\|\nabla v(t)\|_q). \end{aligned}$$

From (3.4) it follows that

$$\begin{aligned} \int_0^t (t-s)^{-\frac{3}{2}+\frac{1}{p}}s^{-\frac{1}{2}}(1+s)^{-\sigma}\,ds &\le t^\sigma(1+t)^{-\sigma}\int_0^t (t-s)^{-\frac{3}{2}+\frac{1}{p}}s^{-\frac{1}{2}-\sigma}\,ds \\ &= t^{-\frac{1}{2}+\frac{1}{q}}(1+t)^{-\sigma}B(\tfrac{1}{p}-\tfrac{1}{2},\tfrac{1}{2}-\sigma). \end{aligned}$$

From (i) of Lemma 2.2, we have

$$\begin{aligned} \text{(3.7)}\quad I_1 &\le t^{-\frac{1}{2}+\frac{1}{q}}(1+t)^{-\sigma}C(p,1)B(\tfrac{1}{p}-\tfrac{1}{2},\tfrac{1}{2}-\sigma)\|u-u_\alpha\|_{X_p^\sigma(t)}(\sup_{t>0} t^{\frac{1}{2}-\frac{1}{q}}\|\nabla v(t)\|_q) \\ &\le t^{-\frac{1}{2}+\frac{1}{q}}(1+t)^{-\sigma}C_{1,\sigma}\tilde{C}_0(\|\nabla v_0\|_2+\|u\|_{X_p})\|u-u_\alpha\|_{X_p^\sigma(t)}. \end{aligned}$$

By (2.2) and the Hölder inequality, we have

$$\begin{aligned} I_2 &\le C(p,1)\int_0^t (t-s)^{-\frac{3}{2}+\frac{1}{p}}\|u_\alpha(s)(\nabla v(s)-\nabla_\alpha v(s))\|_1\,ds \\ &\le C(p,1)\int_0^t (t-s)^{-\frac{3}{2}+\frac{1}{p}}\|u_\alpha(s)\|_p\|\nabla v(s)-\nabla_\alpha v(s)\|_q\,ds \\ &\le C(p,1)\int_0^t (t-s)^{-\frac{3}{2}+\frac{1}{p}}s^{-\frac{1}{2}}(1+s)^{-\sigma}\,ds\|u_\alpha\|_{X_p}A_{2,\sigma}, \end{aligned}$$

where

$$A_{2,\sigma} = (\sup_{t>0} t^{\frac{1}{2}-\frac{1}{q}}(1+t)^\sigma\|\nabla v(t)-\nabla v_\alpha(t)\|_q).$$

It follows from (3.4) that

$$I_2 \le t^{-\frac{1}{2}+\frac{1}{q}}(1+t)^{-\sigma}C(p,1)B(\tfrac{1}{p}-\tfrac{1}{2},\tfrac{1}{2}-\sigma)\|u_\alpha\|_{X_p}A_{2,\sigma}.$$

From Lemma 3.2 we obtain

$$\text{(3.8)}\qquad I_2 \le t^{-\frac{1}{2}+\frac{1}{q}}(1+t)^{-\sigma}C_{1,\sigma}\|u_\alpha\|_{X_p}(A_{0,\sigma}+C_{0,\sigma}\|u-u_\alpha\|_{X_p^\sigma(t)}).$$

Combining (3.7) and (3.8), we obtain (3.5). ∎

Proof of Theorem 3. We see that

$$\|u(s)-u_\alpha(s)\|_p = \|e^{t\Delta}u_0-\alpha G(\cdot,t)\|_p + \|\Phi(u)-\Phi(u_\alpha)\|_p.$$

By the arguments in the proofs of Theorems 1 and 2, we obtain

$$\sup_{t>0} t^{1+\frac{1}{p}}\|e^{t\Delta}u_0-\alpha G(\cdot,t)\|_p \le \sup_{t>0} t^{1+\frac{1}{p}}\|e^{t\Delta}u_0\|_p + \sup_{t>0} t^{1+\frac{1}{p}}\|\alpha G(\cdot,t)\|_p < \infty.$$

By [5, Lemma 2.1] we have

$$\|e^{t\Delta}u_0-\alpha G(\cdot,t)\|_p \le Ct^{\frac{1}{p}-\frac{3}{2}}\|(|x|^2+1)u_0\|_1.$$

Then we have

$$\sup_{t>0} t^{1+\frac{1}{p}}(1+t)^{\sigma}\|e^{t\Delta}u_0 - \alpha G(\cdot,t)\|_p \equiv A_{2,\sigma} < \infty.$$

From Lemma 3.3 we obtain

$$\begin{aligned}\|u-u_\alpha\|_{X_p^\sigma(t)} &\le A_{2,\sigma} + \|\Phi(u)-\Phi(u_\alpha)\|_{X_p^\sigma(t)}\\ &\le A_{2,\sigma} + A_{0,\sigma}C_{1,\sigma}\|u_\alpha\|_{X_p} + A_{1,\sigma}C_{1,\sigma}\|u-u_\alpha\|_{X_p^\sigma(t)}.\end{aligned} \tag{3.9}$$

We note here that $C_{0,\sigma} \to C_0$ and $C_{1,\sigma} \to C_1$ as $\sigma \to 0$, where C_0 and C_1 are constants defined by (2.6) and (2.8), respectively. Then, from (3.6) and $C_0 \le \tilde{C}_0$, we obtain

$$\lim_{\sigma\to 0} A_{1,\sigma} = \tilde{C}_0(\|\nabla v_0\|_2 + \|u\|_{X_p}) + C_0\|u_\alpha\|_{X_p} \le \tilde{C}_0(\beta_0 + 2M).$$

From the second part of (1.4), we find that $C_{1,\sigma}A_{1,\sigma} < 1$ for sufficient small $\sigma > 0$. Then it follows from (3.9) that

$$\|u-u_\alpha\|_{X_p^\sigma(t)} \le \frac{A_{2,\sigma} + A_{0,\sigma}C_{1,\sigma}\|u_\alpha\|_{X_p}}{1 - A_{1,\sigma}C_{1,\sigma}} < \infty \quad \text{for } t > 0.$$

This implies that $\|u-u_\alpha\|_{X_p^\sigma(t)}$ is bounded for all $t > 0$. Thus we obtain

$$\|u-u_\alpha\|_{X_p} \le C(1+t)^{\sigma} \quad \text{for all } t > 0$$

with some constant $C > 0$. In particular, we conclude that (1.8) holds. ∎

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