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A BLOWUP ANALYSIS OF THE MEAN FIELD EQUATION FOR ARBITRARILY SIGNED VORTICES

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Abstract. We study the noncompact solution sequences to the mean field equation for arbitrarily signed vortices and observe the quantization of the mass of concentration, using the rescaling argument.

1. Introduction. We continue the study [34] on the noncompact solution sequences to the mean field equation for arbitrarily signed vortices on a two-dimensional compact orientable Riemannian manifold (M, g) without boundary:

$$-\Delta_g v = \lambda_1\left(\frac{e^v}{\int_M e^v\,dv_g} - \frac{1}{|M|}\right) - \lambda_2\left(\frac{e^{-v}}{\int_M e^{-v}\,dv_g} - \frac{1}{|M|}\right),$$
$$\int_M v\,dv_g = 0, \tag{1}$$

where Δ_g, dv_g, and $|M|$ are the Laplace-Beltrami operator, the volume form, and the volume of M, respectively, and λ_1, λ_2 are nonnegative constants.

This equation is derived by Joyce and Montgomery [20] and Pointin and Lundgren [35] from different statistical arguments for describing the mean field of the equilibrium turbulence with arbitrarily signed vortices, see also [28, 12, 25, 30]. Here, these vortices are composed of positive and negative intensities with the same absolute value, and v and

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λ_1/λ_2 are associated with the stream function of the fluid and the ratio of the numbers of the signed vortices, respectively.

The equation (1) is the Euler-Lagrange equation of the functional

$$J_{\lambda_1,\lambda_2}(v) = \frac{1}{2}\int_M |\nabla_g v|^2 \, dv_g - \lambda_1 \log \int_M e^v \, dv_g - \lambda_2 \log \int_M e^{-v} \, dv_g$$

defined on

$$E = \left\{ w \in H^1(M) \mid \int_M w \, dv_g = 0 \right\},$$

which forms a Hilbert space with the inner product $\langle u, v \rangle = \int_M \nabla_g u \cdot \nabla_g v \, dv_g$. When $(\lambda_1, \lambda_2) = (\lambda, 0)$ or $(\lambda_1, \lambda_2) = (0, \lambda)$, this J_{λ_1,λ_2} is reduced to

$$I_\lambda(v) = \frac{1}{2}\int_M |\nabla_g v|^2 \, dv_g - \lambda \log \int_M e^v \, dv_g,$$

and it is associated with the Trudinger-Moser inequality [16] given by

$$\begin{aligned} &\inf_{v\in E} I_\lambda(v) > -\infty \quad \text{if } \lambda \in [0, 8\pi], \\ &\inf_{v\in E} I_\lambda(v) = -\infty \quad \text{if } \lambda > 8\pi. \end{aligned}$$

We have

$$J_{\lambda_1,\lambda_2}(v) = \frac{1}{2}\left(1 - \frac{\lambda_1}{8\pi} - \frac{\lambda_2}{8\pi}\right)\|v\|_E^2 + \frac{\lambda_1}{8\pi} I_{8\pi}(v) + \frac{\lambda_2}{8\pi} I_{8\pi}(-v),$$

and therefore,

$$\inf_{v\in E} J_{\lambda_1,\lambda_2}(v) > -\infty \quad \text{if } 1 - \frac{\lambda_1}{8\pi} - \frac{\lambda_2}{8\pi} \geq 0.$$

In our previous work [34] we improve this trivial inequality to the following optimal one:

Theorem 1.1.

$$\inf_{v\in E} J_{\lambda_1,\lambda_2}(v) > -\infty \quad \textit{if } (\lambda_1, \lambda_2) \in [0, 8\pi] \times [0, 8\pi], \tag{2}$$

and in particular J_{λ_1,λ_2} *has a global minimizer on* E *if* $0 \leq \lambda_1, \lambda_2 < 8\pi$ *and*

$$\inf_{v\in E} J_{\lambda_1,\lambda_2}(v) = -\infty \quad \textit{if } \lambda_1 > 8\pi, \textit{ or } \lambda_2 > 8\pi.$$

We note that similar results for $H_0^1(\Omega)$ on a bounded domain $\Omega \subset \mathbb{R}^2$ follow from the above theorem by a simple extension argument, see [34].

Although Shafrir and Wolansky [37] obtained a related result that leads to (2), we proved the above result by a completely different method. We developed blow-up analysis for the solution sequence to (1), and apply the argument of Jost and Wang [18] concerning $SU(3)$ Toda system. The purpose of this paper is to develop further the blow-up analysis and clarify the possible singular limits of the solution sequence to (1) to some extent.

First, we recall the following result on the blow-up analysis from [34]:

Theorem 1.2. *Let* $\{\lambda_{1,n}\}$ *and* $\{\lambda_{2,n}\}$ *be sequences of nonnegative constants satisfying*

$$\lambda_{i,n} \to \lambda_i (\geq 0) \quad \textit{as } n \to \infty \textit{ for } i = 1, 2, \tag{3}$$

and $\{v_n\} \subset E$ *be a sequence of solutions to (1) corresponding to* $(\lambda_{1,n}, \lambda_{2,n})$*. Then, up to a subsequence, the following alternatives hold:*

(1) *(compactness) There exist $v \in E$ and a subsequence of $\{v_n\}$ (denoted by the same symbol, also hereafter) such that*

$$v_n \to v \quad \text{in } E,$$

where this v is a solution to (1) for those λ_1 and λ_2.

(2) *(one-sided concentration) Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the blow-up set of (this subsequence of) $\{v_n\}$ and $\{-v_n\}$, respectively, that is,*

$$\mathcal{S}_1 = \{x \in M | \exists x_n \to x \text{ s.t. } v_n(x_n) \to +\infty\},$$
$$\mathcal{S}_2 = \{x \in M | \exists x_n \to x \text{ s.t. } v_n(x_n) \to -\infty\}.$$

Then there exists $i \in \{1,2\}$ such that $\mathcal{S}_i \neq \emptyset$ and $\mathcal{S}_j = \emptyset$ for $j \in \{1,2\}\backslash\{i\}$. Moreover, put

$$\mu_{1,n} \equiv \lambda_{1,n} \frac{e^{v_n}}{\int_M e^{v_n}\, dv_g}, \qquad \mu_{2,n} \equiv \lambda_{2,n} \frac{e^{-v_n}}{\int_M e^{-v_n}\, dv_g},$$

and identify them with $\mu_{k,n}\, dv_g$ $(k = 1,2)$ in the space of measures $\mathcal{M}(M) = C(M)^*$. Then

$$\mu_{i,n} \to \mu_i = \sum_{x_0 \in \mathcal{S}_i} 8\pi\delta_{x_0} \quad \text{weakly } * \text{ in } \mathcal{M}(M),$$

and

$$\mu_{i,n} \to 0 \quad \text{in } L^\infty(\omega)$$

for every $\omega \Subset M\backslash\mathcal{S}_i$. On the other hand, there exists $u_j \in E$ and a subsequence of $\{u_{j,n}\}$ such that

$$u_{j,n} \to u_j \qquad \text{in } E,$$

where this u_j is a solution to

$$-\Delta_g v = \lambda\left(\frac{K(x)e^v}{\int_M K(x)e^v\, dv_g} - \frac{1}{|M|}\right), \quad \int_M v\, dv_g = 0, \tag{4}$$

with $K(x) = e^{-\sum_{x_0 \in \mathcal{S}_i} 8\pi G(x,x_0)}$. Here $G = G(x,y)$ indicates the Green function of $-\Delta_g$, that is,

$$-\Delta_g G(\cdot, y) = \delta_x - \frac{1}{|M|} \quad \text{in } M, \qquad \int_M G(\cdot,y)\, dv_g = 0.$$

(3) *(concentration) For each $i = 1,2$, we have $\mathcal{S}_i \neq \emptyset$ and there exists a positive constant*

$$m_i(x_0) \geq 4\pi \quad \text{for each } x_0 \in \mathcal{S}_i. \tag{5}$$

We have, furthermore, a nonnegative function

$$r_i(x) \in L^1(M) \cap L^\infty_{\mathrm{loc}}(M\backslash\mathcal{S}_i)$$

such that

$$\mu_{i,n} \to r_n + \sum_{x_0 \in \mathcal{S}_i} m_i(x_0)\delta_{x_0} \quad \text{weakly } * \text{ in } \mathcal{M}(M),$$

and

$$\mu_{i,n} \to r_i \quad \text{in } L^p(\omega)$$

for every $p \in [1,\infty)$ and every $\omega \Subset M\backslash\mathcal{S}_i$. Finally, the following facts hold:

3-i) *If there exists* $x_0 \in \mathcal{S}_i \backslash \mathcal{S}_j$ *for* $i \neq j$*, then we have* $m_i(x_0) = 8\pi$ *and* $r_i \equiv 0$*.*

3-ii) *For every* $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$*, we have*

$$(m_1(x_0) - m_2(x_0))^2 = 8\pi(m_1(x_0) + m_2(x_0)). \tag{6}$$

Moreover, if $\mathcal{S}_i \subset \mathcal{S}_j$ *and there exists* $x_0 \in \mathcal{S}_i$ *satisfying*

$$m_i(x_0) - m_j(x_0) > 4\pi,$$

then $r_i \equiv 0$*, see Figure 1.*

In this paper, we improve the minimum mass (5) as follows:

THEOREM 1.3 (Main Result). *In the conclusion of Theorem 1.2, (5) is improved as follows, see Figure 1:*

$$m_i(x_0) \geq 8\pi \quad \textit{for each } x_0 \in \mathcal{S}_i.$$

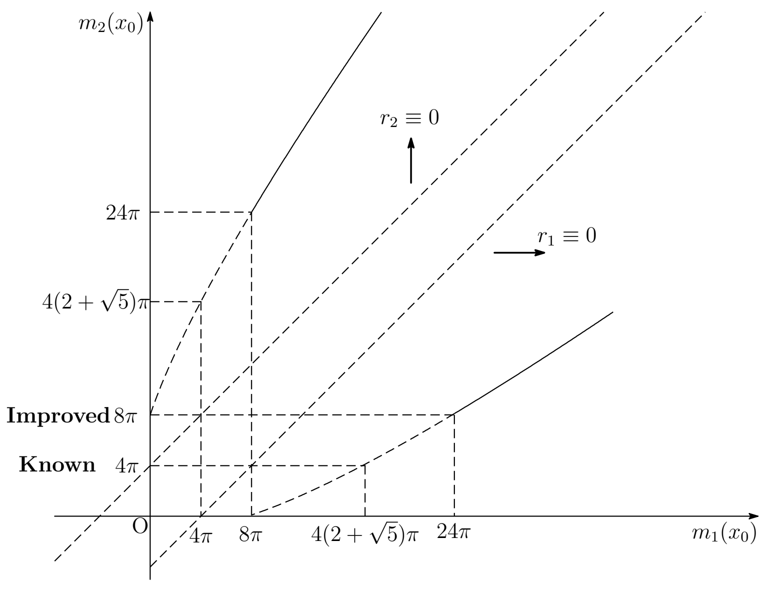

Fig. 1. The mass of concentration at $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$

The conclusion follows from Theorem 1.2 3-i) when $x_0 \in \mathcal{S}_i \backslash \mathcal{S}_j$ for some $i \neq j$. Thus we only consider the case $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$ to prove Theorem 1.3.

The above result guarantees the following compactness result for solution sequences to (1):

COROLLARY 1.4. *Let* $\{\lambda_{1,n}\}$ *and* $\{\lambda_{2,n}\}$ *be sequences of nonnegative constants satisfying (3) for some*

$$(\lambda_1, \lambda_2) \in \{[0, 24\pi) \backslash 8\pi\mathbb{N}\} \times \{[0, 24\pi) \backslash 8\pi\mathbb{N}\},$$

and $\{v_n\} \subset E$ *be a sequence of solutions to (1) corresponding to* $(\lambda_{1,n}, \lambda_{2,n})$*. Then* $\{v_n\}$ *is relatively compact in* E*.*

The possible values of $(m_1(x_0), m_2(x_0))$ for $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$ will be more restrictive and we expect that

$$(m_1(x_0), m_2(x_0)) = 8\pi\left(\frac{(\ell-1)\ell}{2}, \frac{\ell(\ell+1)}{2}\right), \quad 8\pi\left(\frac{\ell(\ell+1)}{2}, \frac{(\ell-1)\ell}{2}\right)$$

for $\ell = 1, 2, 3, \cdots$, see [34]. To describe the background of this conjecture, let us define $(u_1, u_2) \in E \times E$ by

$$u_i(x) \equiv \int_M G(x,y)\mu_i(y)\, dv_g \quad \text{for } i = 1, 2. \tag{7}$$

Then, the function $v = u_1 - u_2$ satisfies (1). A basic idea is obtained by regarding these u_1 and u_2 as the positive and the negative parts of v, respectively, and in this case (1) becomes the Liouville system

$$\begin{aligned} -\Delta_g u_1 &= \lambda_1\left(\frac{e^{a_{11}u_1 + a_{12}u_2}}{\int_M e^{a_{11}u_1 + a_{12}u_2}\, dv_g} - \frac{1}{|M|}\right), \\ -\Delta_g u_2 &= \lambda_2\left(\frac{e^{a_{21}u_1 + a_{22}u_2}}{\int_M e^{a_{21}u_1 + a_{22}u_2}\, dv_g} - \frac{1}{|M|}\right), \\ \int_M u_1\, dv_g &= 0, \qquad \int_M u_2\, dv_g = 0, \end{aligned} \tag{8}$$

with a_{ij} $(i, j = 1, 2)$ constituting

$$A = (a_{ij}) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

When this matrix is given by

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

in (8), it comprises the $SU(3)$ Toda system (in the simplest form without the vortex term) arising in nonabelian relativistic self-dual gauge theory [22, 15, 41] studied by several authors mathematically [18, 19, 26, 7, 32, 17].

Each equation of the general Liouville system (8) is regarded as (4) by putting

$$v = a_{ii}u_i, \quad \lambda = a_{ii}\lambda_i, \quad K = e^{a_{ij}u_j} \quad (j \neq i),$$

for $i = 1, 2$ and, especially, to

$$-\Delta_g v = \lambda\left(\frac{e^v}{\int_M e^v\, dv_g} - \frac{1}{|M|}\right), \qquad \int_M v\, dv_g = 0, \tag{9}$$

if λ_1 or $\lambda_2 = 0$. Here, the equation (9) and its generalization (4) with the inhomogeneous coefficient $K(x) > 0$ appear also in the self-dual gauge field theory [41], stationary system of chemotaxis or self-interacting particles [40], and the prescribing Gaussian curvature problem [1]. It has been studied in recent years [29, 39, 4, 24, 23, 5, 6, 21, 38, 14, 36, 31, 33, 2, 13, 8, 9], and especially, we have the quantization phenomenon [23] of

$$\lambda \in 8\pi\mathbb{N}$$

for the noncompact sequence of solutions (v_n, λ_n) with $\lambda_n \to \lambda$ (based on [4, 24], see also [29, 39, 36] for another method) and the classification of the singular limit using the Green function [29, 27, 33]. We note that these results are provided with fundamental tools or motivations for the variational method [38, 14], the singular perturbation of the solution (see [2] for bounded domain), and the calculation of the topological degree [23, 9].

Similar problems are also considered for $SU(3)$ Toda systems. Putting

$$\mu_{1,n} \equiv \lambda_{1,n} \frac{e^{2u_{1,n}-u_{2,n}}}{\int_M e^{2u_{1,n}-u_{2,n}}\, dv_g}, \qquad \mu_{2,n} \equiv \lambda_{2,n} \frac{e^{-u_{1,n}+2u_{2,n}}}{\int_M e^{-u_{1,n}+2u_{2,n}}\, dv_g},$$

we obtain a result like the above Theorem 1.2 ([18, 26, 7]). In this case, (since $a_{ii} = 2$) the estimate corresponding to (5) is

$$m_i(x_0) \geq 2\pi \quad \text{for each } x_0 \in \mathcal{S}_i. \tag{10}$$

Furthermore, if $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$, then $m_1(x_0)$ and $m_2(x_0)$ satisfy the relation describing an ellipse

$$m_1(x_0)^2 - m_1(x_0)m_2(x_0) + m_2(x_0)^2 = 4\pi(m_1(x_0) + m_2(x_0)),$$

instead of (6). In fact, in the general form of (8), it holds that

$$\begin{aligned} &a_{11}a_{21}m_1(x_0)^2 + 2a_{12}a_{21}m_1(x_0)m_2(x_0) + a_{22}a_{12}m_2(x_0)^2 \\ &\quad = 8\pi(a_{21}m_1(x_0) + a_{12}m_2(x_0)). \end{aligned}$$

For the $SU(3)$ Toda case, the improvement of the estimate (10) to

$$m_i(x_0) \geq 4\pi \quad \text{for each } x_0 \in \mathcal{S}_i$$

was obtained in [26], see also [32, 17]. In this case it is expected that

$$(m_1(x_0), m_2(x_0)) \in \{(4\pi, 8\pi), (8\pi, 4\pi), (8\pi, 8\pi)\}$$

for any $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$.

2. Preliminaries. In this section, we describe several results obtained in [34] to be used in the proof of the main theorem of this paper.

First, given $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$, we take an isothermal chart (Ψ, U) satisfying

$$\Psi(x_0) = 0, \quad \Psi(x) = X \in \mathbb{R}^2, \quad g = e^{\xi(X)}(dX_1^2 + dX_2^2),$$

and $\overline{U} \cap (\mathcal{S}_1 \cup \mathcal{S}_2) = \{x_0\}$. Then, $v_n(X) = v_n \circ \Psi^{-1}(X)$ is a solution of

$$-\Delta v_n = \lambda_{1,n}\left(\frac{e^{v_n}}{\int_M e^{v_n}} - \frac{1}{|M|}\right)e^{\xi} - \lambda_{2,n}\left(\frac{e^{-v_n}}{\int_M e^{-v_n}} - \frac{1}{|M|}\right)e^{\xi} \quad \text{in } \Omega.$$

Let us define the functions h_ξ by

$$\Delta h_\xi = e^{\xi} \quad \text{in } \Omega, \qquad h_\xi = 0 \quad \text{on } \partial\Omega,$$

where $\Omega = \Psi(U) \subset \mathbb{R}^2$. Without loss of generality, we may assume that $\partial\Omega$ is smooth.

Putting

$$\begin{aligned} w_{1,n}(X) &= v_n(\Phi^{-1}(X)) - \log \int_M e^{v_n} - \frac{\lambda_{1,n} - \lambda_{2,n}}{|M|} h_\xi, \\ w_{2,n}(X) &= -v_n(\Phi^{-1}(X)) - \log \int_M e^{-v_n} - \frac{-\lambda_{1,n} + \lambda_{2,n}}{|M|} h_\xi, \end{aligned}$$

we obtain

$$-\Delta w_{1,n} = V_{1,n}(x)e^{w_{1,n}} - V_{2,n}e^{w_{2,n}},$$
$$-\Delta w_{2,n} = -V_{1,n}(x)e^{w_{1,n}} + V_{2,n}(x)e^{w_{2,n}}, \tag{11}$$

in Ω for

$$V_{1,n} = \lambda_{1,n}e^{\xi+\frac{\lambda_{1,n}-\lambda_{2,n}}{|M|}h_\xi},$$
$$V_{2,n} = \lambda_{2,n}e^{\xi+\frac{-\lambda_{1,n}+\lambda_{2,n}}{|M|}h_\xi},$$

satisfying

$$0 \le V_{1,n}(X) \le b, \quad 0 \le V_{2,n}(X) \le b \qquad (\forall X \in \Omega)$$
$$\int_\Omega e^{w_{1,n}} \le c, \quad \int_\Omega e^{w_{2,n}} \le c, \tag{12}$$

with some constants $b, c > 0$ independent of n, and

$$V_{1,n} \quad \to \quad V_1 = \lambda_1 e^{\xi+(\lambda_1-\lambda_2)h_\xi},$$
$$V_{2,n} \quad \to \quad V_2 = \lambda_2 e^{\xi+(-\lambda_1+\lambda_2)h_\xi}, \tag{13}$$

uniformly on $\overline{\Omega}$. By (5) we have only to consider the case $\min(\lambda_1, \lambda_2) > 0$, that is, $V_1, V_2 > 0$. We have $x_{i,n} \to x_0$ such that $v_n(x_{1,n}), -v_n(x_{2,n}) \to +\infty$. This implies $X_{i,n} = \Psi(x_{i,n}) \to 0$ and also

$$v_n(x_n) - \log \int_M e^{v_n}, \quad -v_n(x_n) - \log \int_M e^{-v_n} \to +\infty$$

from the proof of [34, Lemma 2.2], or equivalently, $w_{i,n} \to +\infty$ for each $i = 1, 2$. This means $0 \in \mathcal{S}_i^0$, where

$$\mathcal{S}_i^0 = \{X_0 \in \Omega \mid \text{there exists } X_n \to X_0 \text{ such that } w_{i,n}(X_n) \to +\infty\}.$$

We also obtain $\mathcal{S}_i^0 = \Psi(U \cap \mathcal{S}_i) = \{0\}$ similarly to the proof of [34, Lemma 2.2].

Next, by Theorem 1.2 we have

$$V_{1,n}e^{w_{1,n}} \quad \to \quad m_1\delta_0 + r_1,$$
$$V_{2,n}e^{w_{2,n}} \quad \to \quad m_2\delta_0 + r_2,$$

in $\mathcal{M}(\overline{\Omega})$ with $\min(m_1, m_2) \ge 4\pi$, $r_1, r_2 \in L^1(\Omega) \cap L^\infty_{loc}(\overline{\Omega} \setminus \{0\})$, and

$$V_{i,n}e^{w_{i,n}} \to r_i \qquad \text{in } L^p_{loc}(\overline{\Omega} \setminus \{0\})$$

for any $1 \le p < \infty$. These m_i coincide with $m_i(x_0)$ $(i = 1, 2)$. By Theorem 1.2 we have $r_1 = 0$ and $r_2 = 0$ in the cases of $m_1 - m_2 > 4\pi$ and $-m_1 + m_2 > 4\pi$, respectively, and

$$(m_1 - m_2)^2 = 8\pi(m_1 + m_2).$$

Thus, we obtain (11), (12), and (13) in a bounded domain $\Omega \subset \mathbb{R}^2$, taking $x = (x_1, x_2)$ to indicate the standard coordinates in $\mathbb{R}^2$. We have to show $m_i \ge 8\pi$ $(i = 1, 2)$ to prove the main theorem. Here, we recall that Brezis-Merle [4] type theorem for (11) holds by a similar argument discussed for the $SU(3)$ Toda system [26, Theorem 4.2].

LEMMA 2.1. *If $\{(w_{1,n}, w_{2,n})\}_n$ is a solution sequence to (11) and (12), then there is a subsequence (denoted by the same symbol) satisfying the following alternatives.*

(1) *Both* $\{w_{1,n}\}_n$ *and* $\{w_{2,n}\}_n$ *are locally uniformly bounded in* Ω.
(2) *There is* $i \in \{1,2\}$ *such that* $\{w_{i,n}\}_n$ *is uniformly bounded in* Ω *and* $w_{j,n} \to -\infty$ *locally uniformly in* Ω *for* $j \neq i$.
(3) *Both* $w_{1,n} \to -\infty$ *and* $w_{2,n} \to -\infty$ *locally uniformly in* Ω.
(4) *For the blow-up sets* $\mathcal{S}_1^0$, $\mathcal{S}_2^0$ *defined for this subsequence, we have* $\mathcal{S}_1^0 \cup \mathcal{S}_2^0 \neq \emptyset$ *and* $\sharp(\mathcal{S}_1^0 \cup \mathcal{S}_2^0) < +\infty$. *Furthermore, for each* $i \in \{1,2\}$, *either* $\{w_{i,n}\}_n$ *is locally uniformly bounded in* $\Omega \setminus (\mathcal{S}_1^0 \cup \mathcal{S}_2^0)$ *or* $w_{i,n} \to -\infty$ *locally uniformly in* $\Omega \setminus (\mathcal{S}_1^0 \cup \mathcal{S}_2^0)$. *Here, if* $\mathcal{S}_i^0 \setminus (\mathcal{S}_1^0 \cap \mathcal{S}_2^0) \neq \emptyset$, *then* $w_{i,n} \to -\infty$ *locally uniformly in* $\Omega \setminus (\mathcal{S}_1^0 \cup \mathcal{S}_2^0)$, *and each* $x_0 \in \mathcal{S}_i^0$ *takes* $m_i(x_0) \geq 4\pi$ *such that*
$$V_{i,n}(x) e^{w_{i,n}} \rightharpoonup \sum_{x_0 \in \mathcal{S}_i^0} m_i(x_0) \delta_{x_0} \qquad *\text{-weakly in } \mathcal{M}(\Omega).$$

Finally, performing the rescaling argument using the above lemma, we arrive at one of the following:

(1) (Liouville equation in $\mathbb{R}^2$)
$$-\Delta w = e^w \qquad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^w < +\infty, \tag{14}$$
(2) (singular Liouville equation in $\mathbb{R}^2$)
$$-\Delta w = e^w - \sum_{x_0 \in \mathcal{S}} m(x_0) \delta_{x_0}, \qquad \int_{\mathbb{R}^2} e^w < +\infty, \tag{15}$$
where $\mathcal{S} \subset \mathbb{R}^2$ is a finite set and $m(x_0) \geq 4\pi$ for any $x_0 \in \mathcal{S}$.

Lemma 2.2 ([10, 11]). *We have the following:*

(1) *For the solution* w *to (14) we have* $\int_{\mathbb{R}^2} e^w = 8\pi$.
(2) *For the solution* w *to (15) we have* $\int_{\mathbb{R}^2} e^w > 4\pi + \sum_{x_0 \in \mathcal{S}} m(x_0)$.

Remark 2.3. Lemma 2.2 (2) follows from [11, Theorem 2.3], but the statement there assumes $0 > m(x_0) > -4\pi$ for each $x_0 \in \mathcal{S}$, which seems not to cover our cases $m(x_0) \geq 4\pi$. Nevertheless Lemma 2.2 (2) holds because the proof of [11, Theorem 2.3] is applicable to our cases. Indeed, it is necessary to show
$$v(x) := w(x) + \frac{m(x_0)}{2\pi} \log |x - x_0|^{-1} \leqq C + C_1 \log(|x| + 1) \tag{16}$$
for some constants C and C_1 in each sufficiently small neighbourhood of $x_0 \in \mathcal{S}$, say $B_\varepsilon(x_0)$, in the course of the proof. Here we note that $v(x)$ satisfies
$$-\Delta v = V(x) e^v \quad \text{in } B_\varepsilon(x_0)$$
with
$$V(x) = e^{-\frac{m(x_0)}{2\pi} \log |x - x_0|^{-1}} = |x - x_0|^{\frac{m(x_0)}{2\pi}},$$
which belongs to $L^\infty(B_\varepsilon(x_0))$ if $m(x_0) \geq 0$ (and to $L^p(B_\varepsilon(x_0))$ for some $p \in (1, \infty)$ if $0 > m(x_0) > -4\pi$). Now taking smaller $\varepsilon > 0$ if necessary, we get (16) with $C_1 = 0$ from [4, Corollary 4].

3. Proof of Theorem 1.3. We have $\mathcal{S}_1^0 = \mathcal{S}_2^0 = \{0\}$, and there are $x_{1,n} \to 0$ and $x_{2,n} \to 0$ such that

$$w_{1,n}(x_{1,n}) = \sup_\Omega w_{1,n} \to +\infty \qquad \text{and} \qquad w_{2,n}(x_{2,n}) = \sup_\Omega w_{2,n} \to +\infty.$$

We take the rescaling of $w_{i,n}$ around $x_{k,n}$ by

$$w_{i,n}^k(x) = w_{i,n}(x_{k,n} + \varepsilon_{k,n} x) - w_{k,n}(x_{k,n}),$$

where $i, k = 1, 2$ and $\varepsilon_{k,n} = e^{-w_{k,n}(x_{k,n})/2}$. Then

$$\begin{aligned} -\Delta w_{1,n}^k &= V_{1,n}(x_{k,n} + \varepsilon_{k,n} x) e^{w_{1,n}^k} - V_{2,n}(x_{k,n} + \varepsilon_{k,n} x) e^{w_{2,n}^k}, \\ -\Delta w_{2,n}^k &= -V_{1,n}(x_{k,n} + \varepsilon_{k,n} x) e^{w_{1,n}^k} + V_{2,n}(x_{k,n} + \varepsilon_{k,n} x) e^{w_{2,n}^k} \end{aligned}$$

in $\Omega_n^k = \{x \in \mathbb{R}^2 \mid \frac{x - x_{k,n}}{\varepsilon_{k,n}} \in \Omega\}$ with $\int_{\Omega_{i,n}^k} e^{w_{i,n}^k} = \int_\Omega e^{w_{i,n}} \leq b$. Without loss of generality, we may suppose

$$\varepsilon_{1,n} \leq \varepsilon_{2,n}$$

for $n = 1, 2, \cdots$, i.e., $w_{1,n}(x_{1,n}) \geq w_{2,n}(x_{2,n})$. Then, we take the rescaled solution around $x_{1,n}$, i.e., $(w_{1,n}^1, w_{2,n}^1)$. Since

$$\begin{aligned} & w_{1,n}^1(x) \leq w_{1,n}^1(0) = 0, \\ & w_{2,n}^1(x) \leq w_{2,n}^1\left(\frac{x_{2,n} - x_{1,n}}{\varepsilon_{1,n}}\right) \leq w_{2,n}(x_{2,n}) - w_{1,n}(x_{1,n}) \leq 0 \end{aligned}$$

on Ω_n^1, Lemma 2.1 assures the following alternatives:

(1) Both $\{w_{1,n}^1\}$ and $\{w_{2,n}^1\}$ are locally uniformly bounded in $\mathbb{R}^2$.
(2) $\{w_{1,n}^1\}$ is locally uniformly bounded in $\mathbb{R}^2$, while $w_{2,n}^1 \to -\infty$ locally uniformly in $\mathbb{R}^2$.

The first alternative, however, never occurs. Indeed, we have

$$\begin{aligned} w_{2,n}^1(x) &= w_{2,n}(x_{1,n} + \varepsilon_{1,n} x) - w_{1,n}(x_{1,n}) \\ &= w_{1,n}(x_{1,n} + \varepsilon_{1,n} x) + w_{2,n}(x_{1,n} + \varepsilon_{1,n} x) \\ &\quad - (w_{1,n}(x_{1,n} + \varepsilon_{1,n} x) - w_{1,n}(x_{1,n})) - 2w_{1,n}(x_{1,n}) \\ &= w_{1,n}(x_{1,n} + \varepsilon_{1,n} x) + w_{2,n}(x_{1,n} + \varepsilon_{1,n} x) - w_{1,n}^1(x) - 2w_{1,n}(x_{1,n}) \end{aligned}$$

and, from the definition of $w_{i,n}$, we have also

$$w_{1,n}(x) + w_{2,n}(x) = -\log \int_M e^{v_n} - \log \int_M e^{-v_n}.$$

Here it follows from the Jensen inequality that

$$\log \int_M e^{v_n} \geq \log |M|, \quad \log \int_M e^{-v_n} \geq \log |M|$$

and consequently we have

$$w_{1,n}(x) + w_{2,n}(x) \leq -2 \log |M|.$$

From these we obtain

$$w_{2,n}^1(x) \leq -2 \log |M| - w_{1,n}^1(x) - 2w_{1,n}(x_{1,n}) \to -\infty$$

for every $x \in \mathbb{R}^2$ if $\{w_{1,n}^1(x)\}$ is locally uniformly bounded in $\mathbb{R}^2$.

Henceforth, we consider the second alternative concerning this rescaling around $x_{1,n}$. Then we have a subsequence (denoted by the same symbol) such that $w_{1,n}^1 \to w_1^1$ in $C_{loc}^{1,\alpha}(\mathbb{R}^2)$ and this w_1^1 satisfies

$$-\Delta w_1^1 = V_1(0)e^{w_1^1}, \qquad \int_{\mathbb{R}^2} e^{w_1^1} < +\infty.$$

Therefore, from the first case of Lemma 2.2 we have

$$m_1 \geq \int_{\mathbb{R}^2} V_1(0)e^{w_1^1} = 8\pi.$$

Henceforth, we put $w_2^1 = -\infty$ for simplicity, and therefore, this alternative is referred to as $w_1^1 \in C_{loc}^{1,\alpha}(\mathbb{R}^2)$ and $w_2^1 = -\infty$. Furthermore, we have $(m_1, m_2) \geq (8\pi, 4\pi)$, namely, $m_1 \geq 8\pi$ and $m_2 \geq 4\pi$.

Now, we use the rescaled solution $(w_{1,n}^2, w_{2,n}^2)$ around $x_{2,n}$. In this case, we have

$$\begin{aligned} &w_{2,n}^2(x) \leq w_{2,n}^2(0) = 0, \\ &w_{1,n}^2(x) \leq w_{1,n}^2\left(\frac{x_{1,n} - x_{2,n}}{\varepsilon_{2,n}}\right) = w_{1,n}(x_{1,n}) - w_{2,n}(x_{2,n}), \end{aligned}$$

in Ω_n^2. In spite of $w_{1,n}(x_{1,n}) - w_{2,n}(x_{2,n}) \geq 0$, again by Lemma 2.1 we have the following alternatives.

(1) Both $\{w_{1,n}^2\}$ and $\{w_{2,n}^2\}$ are locally uniformly bounded in $\mathbb{R}^2$.
(2) $\{w_{2,n}^2\}$ is locally uniformly bounded, while $w_{1,n}^2 \to -\infty$ locally uniformly in $\mathbb{R}^2$.
(3) There is a finite blow-up set $\mathcal{S}_1^2$ of $\{w_{1,n}^2\}$ such that $V_{1,n}(x_{2,n} + \varepsilon_{2,n}x)e^{w_{1,n}^2} \to \sum_{x_0 \in \mathcal{S}_1^2} m_1^2(x_0)\delta_{x_0}$ in $\mathcal{M}(\mathbb{R}^2)$ with $m_1^2(x_0) \geq 4\pi$ for any $x_0 \in \mathcal{S}_1^2$ and $w_{1,n}^2 \to -\infty$ locally uniformly in $\mathbb{R}^2 \setminus \mathcal{S}_1^2$. Moreover, either
   3-i) $\{w_{2,n}^2\}$ is locally uniformly bounded in $\mathbb{R}^2 \setminus \mathcal{S}_1^2$, or
   3-ii) $w_{2,n}^2 \to -\infty$ locally uniformly in $\mathbb{R}^2 \setminus \mathcal{S}_1^2$.

Here the first alternative is impossible by the preceding argument of the rescaling around $x_{1,n}$ and we proceed to the other cases.

The second alternative is indicated by $w_2^2 \in C_{loc}^{1,\alpha}(\mathbb{R}^2)$ and $w_1^2 = -\infty$. The former function satisfies the Liouville equation on $\mathbb{R}^2$, and this implies $m_2 \geq 8\pi$. On the other hand, we have already $m_1 \geq 8\pi$ from the former rescaling. Therefore, $(m_1, m_2) \geq (8\pi, 8\pi)$.

In the first case of the third alternative, passing to a subsequence, we have $w_{2,n}^2 \to w_2^2$ in $C_{loc}^{1,\alpha}(\mathbb{R}^2 \setminus \mathcal{S}_1^2)$ with w_2^2 satisfying

$$-\Delta w_2^2 = V_2(0)e^{w_2^2} \qquad \text{in } \mathbb{R}^2 \backslash \mathcal{S}_1^2.$$

Here we note that $w_2^2 \leq 0$ since $w_{2,n}^2(x) \leq 0$, which guarantees $\int_{\mathbb{R}^2} e^{w_2^2} < \infty$. Moreover, it follows that there exist $\varphi \in L_{\mathrm{loc}}^1(\mathbb{R}^2)$ and constants $\alpha(x_0) > 0$ for each $x_0 \in \mathcal{S}_1^2$ such that

$$-\Delta w_2^2 = -\sum_{x_0 \in \mathcal{S}_1^2} \alpha(x_0)\delta_{x_0} + \varphi \qquad \text{in } \mathbb{R}^2,$$

see [3, Theorem 1]. It is easy to see that $\varphi = V_2(0)e^{w_2^2}$ and $\alpha(x_0) = m_1^2(x_0)$. Consequently w_2^2 satisfies the following

$$-\Delta w_2^2 = -\sum_{x_0 \in \mathcal{S}_1^2} m_1^2(x_0)\delta_{x_0} + V_2(0)e^{w_2^2} \qquad \text{in } \mathbb{R}^2,$$

$$\int_{\mathbb{R}^2} e^{w_2^2} < +\infty,$$

where $m_1^2(x_0) \geq 4\pi$ for each $x_0 \in \mathcal{S}_1^2$. In particular,

$$\int_{\mathbb{R}^2} V_2(0)e^{w_2^2} > 4\pi + \sum_{x_0 \in \mathcal{S}_1^2} m_1^2(x_0)$$

by the second case of Lemma 2.2, and therefore,

$$m_2 > 4\pi + \sum_{x_0 \in \mathcal{S}_1^2} m_1^2(x_0) > 8\pi.$$

Finally, the second case of the third alternative does not occur. In fact, we have $w_{2,n}^2(0) = 0$, and therefore, $0 \in \mathcal{S}_1^2$. We can choose $R > 0$ satisfying $\overline{B_R(0)} \cap \mathcal{S}_1^2 = \{0\}$, and define $h_{i,n}$ $(i = 1, 2)$ by

$$\begin{aligned} -\Delta h_{i,n} &= V_{i,n}(x_{2,n} + \varepsilon_{2,n}x)e^{w_{i,n}^2} \qquad \text{in } B_R(0), \\ h_{i,n} &= 0 \qquad \text{on } \partial B_R(0). \end{aligned}$$

Then,

$$h_{0,n} = w_{2,n}^2 - (h_{2,n} - h_{1,n})$$

is a harmonic function satisfying

$$\sup_{B_R(0)} h_{0,n} \leq \sup_{\partial B_R(0)} h_{0,n} \to -\infty.$$

On the other hand, we have $0 \leq e^{w_{2,n}^2(x)} \leq e^0 = 1$ and $e^{w_{2,n}^2}(x) \to 0$ locally uniformly in $\mathbb{R}^2 \setminus \mathcal{S}_1^2$, and therefore, $e^{w_{2,n}^2(x)} \to 0$ in $L^p(B_R(0))$ for every $p \in [1, \infty)$. This implies

$$h_{2,n} \to 0 \qquad \text{in } C^{1,\alpha}(B_R(0)),$$

while $h_{1,n}$ is a nonnegative function. Thus, we obtain

$$\begin{aligned} 0 = w_{2,n}^2(0) &= h_{0,n}(0) + h_{2,n}(0) - h_{1,n}(0) \leq h_{0,n}(0) + h_{2,n}(0) \\ &\leq \sup_{B_R(0)} h_{0,n} + \|h_{2,n}\|_{L^\infty(B_R(0))} \to -\infty, \end{aligned}$$

a contradiction, and the proof is complete. ∎

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