SELF-SIMILAR SOLUTIONS OF NONLINEAR PDE
BANACH CENTER PUBLICATIONS, VOLUME 74
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2006

CONCENTRATION POINTS OF LEAST ENERGY SOLUTIONS TO THE BREZIS-NIRENBERG EQUATION WITH VARIABLE COEFFICIENTS

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1. Introduction. In this paper we consider the extensively studied problem

$$(P_{\varepsilon,k})\begin{cases} -\Delta u = u^p + \varepsilon k(x)u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$, $N \geq 4$, $p = (N+2)/(N-2)$ is the critical Sobolev exponent, $\varepsilon > 0$ and $k \in C^2(\overline{\Omega})$ is a given function.

We are interested in the asymptotic behavior of blowing up solutions to $(P_{\varepsilon,k})$ as $\varepsilon \to 0$.

Note that when Ω is star-shaped (with respect to 0), the Pohozaev identity yields the nonexistence of solutions to $(P_{\varepsilon,k})$ if $k(x) + \frac{1}{2}x \cdot \nabla k(x) \leq 0$ for any $x \in \Omega$.

On the other hand, solutions to $(P_{\varepsilon,k})$ on a general domain can be obtained by solving the constrained minimization problem

$$S_{\varepsilon,k} = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^{p+1}(\Omega)}=1}} \left\{ \int_\Omega |\nabla u|^2 dx - \varepsilon \int_\Omega k(x)u^2 dx \right\}. \tag{1.1}$$

Let S denote the best Sobolev constant. Let $\varepsilon > 0$ be sufficiently small so that the operator $-\Delta - \varepsilon k(x)$ is coercive on $H_0^1(\Omega)$ (for example, it would be enough that $0 < \varepsilon < \frac{(1-C)\lambda_1(\Omega)}{\|k\|_{L^\infty(\Omega)}}$ for some constant $0 < C < 1$, here $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ acting on $H_0^1(\Omega)$). Then, Brezis and Nirenberg proved that the conditions

(1) $k(x) > 0$ somewhere on Ω,
(2) $S_{\varepsilon,k} < S$,

2000 *Mathematics Subject Classification*: 35B40, 35J20.

The paper is in final form and no version of it will be published elsewhere.

(3) $S_{\varepsilon,k}$ is achieved

are equivalent ([2], see also [1]).

From now on, we assume the condition (1), thus the existence of a minimizer u_ε^0 of $S_{\varepsilon,k}$ is assured. We may assume $u_\varepsilon^0 > 0$ by considering $|u_\varepsilon^0|$ if necessary. $S_{\varepsilon,k}$ is positive when $\varepsilon > 0$ small, thus the Lagrange multiplier rule and elliptic regularity assure that

$$\overline{u}_\varepsilon = (S_{\varepsilon,k})^{\frac{N-2}{4}} u_\varepsilon^0 \tag{1.2}$$

is a smooth solution to $(P_{\varepsilon,k})$. We call $(\overline{u}_\varepsilon)$ the *least energy solutions* to the problem $(P_{\varepsilon,k})$.

In the following we consider only least energy solutions obtained by the method of Brezis and Nirenberg.

Note that the least energy solutions $(\overline{u}_\varepsilon)$ is also a minimizing sequence for the Sobolev best constant S, by the fact that $S_{\varepsilon,k} = S + o(1)$ as $\varepsilon \to 0$. Thus, by [3] Lemma I.1 and [7], it is known that $(\overline{u}_\varepsilon)$ concentrate at one point of $\overline{\Omega}$: There exist $\lambda_\varepsilon > 0$ with $\lambda_\varepsilon \to 0$ $(\varepsilon \to 0)$ and $a_\varepsilon \in \Omega$ with $\lambda_\varepsilon/\mathrm{dist}(a_\varepsilon, \partial\Omega) \to 0$ $(\varepsilon \to 0)$ such that, by choosing a subsequence if necessary, $a_\varepsilon \to a_\infty \in \overline{\Omega}$ and

$$\|\nabla(\overline{u}_\varepsilon - \alpha_N PU_{\lambda_\varepsilon, a_\varepsilon})\|_{L^2(\Omega)} \to 0, \quad \text{where } \alpha_N = (N(N-2))^{\frac{N-2}{4}}, \tag{1.3}$$

$$|\nabla \overline{u}_\varepsilon|^2 \overset{*}{\rightharpoonup} S^{\frac{N}{2}} \delta_{a_\infty} \tag{1.4}$$

in the sense of Radon measures of $\overline{\Omega}$ as $\varepsilon \to 0$, where δ_{a_∞} is a Dirac mass at $a_\infty \in \overline{\Omega}$. Here for $\lambda > 0$ and $a \in \Omega$, $PU_{\lambda,a}(x)$ denotes the projection of $U_{\lambda,a}$ to $H_0^1(\Omega)$ defined by $PU_{\lambda,a} = U_{\lambda,a} - \varphi_{\lambda,a} \in H_0^1(\Omega)$ where $\varphi_{\lambda,a}$ is the harmonic extension of $U_{\lambda,a}|_{\partial\Omega}$ to Ω,

$$U_{\lambda,a}(x) = \left(\frac{\lambda}{\lambda^2 + |x-a|^2}\right)^{\frac{N-2}{2}}, \quad x \in \mathbf{R}^N,$$

is the unique (up to translation and dilation) positive solution of $-\Delta U = N(N-2)U^p$ in $\mathbf{R}^N$.

Concentration phenomena in elliptic problems involving critical Sobolev exponents like $(P_{\varepsilon,k})$ are now widely studied. For the special case of $k \equiv 1$, see [4], [7], [8] and [9]. The Robin function of the domain plays an important role in these studies. Han and Rey showed that if (u_ε) is a family of solutions of $(P_{\varepsilon,1})$ which concentrate at a point $a_\infty \in \overline{\Omega}$ in the sense of (1.4), then a_∞ is interior of Ω and is a critical point of the (positive) Robin function

$$R(a) = H(a,a), \quad a \in \Omega,$$

where $H(x,a)$ is the regular part of the Green's function $G(x,a)$,

$$H(x,a) = \frac{1}{(N-2)\omega_N}|x-a|^{2-N} - G(x,a),$$

here ω_N is the $(N-1)$ dimensional volume of S^{N-1}. By the maximum principle and the Harnack inequality, we have

$$C_1 \le R(a)(\mathrm{dist}(a, \partial\Omega))^{N-2} \le C_2 \tag{1.5}$$

for some $C_1, C_2 > 0$ independent of $a \in \Omega$. Thus $R(a) \to \infty$ as $a \to \partial\Omega$. More precisely, we know

$$R(a_n) = \frac{1}{(N-2)\omega_N}\left(\frac{1}{2d_n}\right)^{N-2} + o\left(\frac{1}{d_n^{N-2}}\right)$$

as $d_n = \mathrm{dist}(a_n, \partial\Omega) \to 0$ by ([8, (2.8)]).

Later in [9], it is proved that any blow up point of least energy solutions is a minimum point of the Robin function on general bounded domains in $\mathbf{R}^{\mathbf{N}}, N \geq 4$.

Recently, Molle and Pistoia [6] studied a more general problem,

$$(P^q_{\varepsilon,k}) \begin{cases} -\Delta u = u^p + \varepsilon k(x) u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $q \geq 1$ if $N \geq 5$, $q > 1$ if $N = 4$ and $q \neq p$.

They showed that if (u_ε) is a family of solutions of $(P^q_{\varepsilon,k})$ which concentrate at a point $a_\infty \in \Omega$ (interior point) in the sense of (1.3), then a_∞ is a critical point of the function

$$\psi_q(a) = k(a)\{R(a)\}^{\frac{q-p}{2}}, \quad a \in \Omega.$$

In addition,

- $k(a_\infty) > 0$ if $q < p$ (subcritical perturbation), and
- $k(a_\infty) < 0$ if $q > p$ (supercritical perturbation).

Furthermore, they showed the existence of a family of solutions of $(P^q_{\varepsilon,k})$ which concentrates at some point in Ω as $\varepsilon \to 0$. Especially, if $q < p$ and $\max_{\overline{\Omega}} k > 0$, then there exists a family of solutions which concentrates at a maximum point of the function ψ_q.

Now, our main result in this note is the following

THEOREM 1.1. *Let $N \geq 4$. Assume that $\Omega_+ := \{a \in \Omega : k(a) > 0\} \neq \phi$. Let $(\overline{u}_\varepsilon)$ be a family of least energy solutions obtained by the method of Brezis-Nirenberg and $a_\infty \in \overline{\Omega}$ be a blow-up point of $(\overline{u}_\varepsilon)$ in the sense of (1.3). Then*

(1) $a_\infty \in \Omega_+$, and

(2) a_∞ maximizes the function $\psi_1(a) = k(a)\{R(a)\}^{\frac{-2}{N-2}}, a \in \Omega_+$:

$$\psi_1(a_\infty) = \max_{a \in \Omega_+} \psi_1(a).$$

Mainly, our proof is almost the same as in [9] in which we treated the case when $k \equiv 1$, and the argument there originates from [5]. But there is also some improvement compared to the former calculations in [9].

2. Asymptotic behavior of $S_{\varepsilon,k}$. In this section, we will obtain an asymptotic formula of $S_{\varepsilon,k}$ as $\varepsilon \to 0$.

For a given sequence $\varepsilon_n \to 0$, let $u^0_{\varepsilon_n}$ be a positive minimizer for (1.1) and define

$$v_n = S^{\frac{N-2}{4}} u^0_{\varepsilon_n}.$$

Then we see that v_n and $\overline{u}_{\varepsilon_n}$ have the same concentration point a_∞ and $|\nabla v_n|^2 dx \overset{*}{\rightharpoonup} S^{\frac{N}{2}} \delta_{a_\infty}$ in the sense of measures on $\overline{\Omega}$.

Now, by a result of Rey ([8, Proposition 2]), we know there exists $(\alpha_n, \lambda_n, a_n) \in \mathbf{R}_+ \times \mathbf{R}_+ \times \Omega$ such that

$$v_n = \alpha_n PU_{\lambda_n, a_n} + w_n \tag{2.1}$$

holds true for n large, where

$$\begin{aligned} \alpha_n &\to \alpha_N = (N(N-2))^{\frac{N-2}{4}}, \\ a_n &\to a_\infty, \\ \frac{\lambda_n}{d_n} &\to 0 \quad \text{where } d_n = \mathrm{dist}(a_n, \partial\Omega), \\ w_n &\in E_{\lambda_n, a_n}, \\ w_n &\to 0 \text{ in } H_0^1(\Omega) \end{aligned}$$

as $n \to \infty$. Here for $\lambda > 0$ and $a \in \Omega$, we set $PU_{\lambda,a} = U_{\lambda,a} - \varphi_{\lambda,a} \in H_0^1(\Omega)$ where $\varphi_{\lambda,a}$ is the harmonic extension of $U_{\lambda,a}|_{\partial\Omega}$ to Ω, and

$$\begin{aligned} E_{\lambda,a} = \Big\{ w \in H_0^1(\Omega) : 0 &= \int_\Omega \nabla w \cdot \nabla PU_{\lambda,a}\, dx \\ &= \int_\Omega \nabla w \cdot \nabla\left(\frac{\partial}{\partial a_i} PU_{\lambda,a}\right) dx \quad (i = 1, \cdots, N) \\ &= \int_\Omega \nabla w \cdot \nabla\left(\frac{\partial}{\partial \lambda} PU_{\lambda,a}\right) dx \Big\}. \end{aligned}$$

Note that the estimate

$$\|\varphi_{\lambda_n, a_n}\|_{L^\infty(\Omega)} = O\left(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^{N-2}}\right) \tag{2.2}$$

holds by the maximum principle for harmonic functions and

$$\varphi_{\lambda_n, a_n}(a_n) = (N-2)\omega_N \lambda_n^{\frac{N-2}{2}} R(a_n) + O\left(\frac{\lambda_n^{\frac{N+2}{2}}}{d_n^N}\right) \tag{2.3}$$

by [8, Proposition 1].

Let

$$J_{n,k} = \int_\Omega |\nabla v_n|^2 dx - \varepsilon_n \int_\Omega k(x) v_n^2 dx. \tag{2.4}$$

Then $S_{\varepsilon_n, k} = S^{1-\frac{N}{2}} J_{n,k}$, so in the following we calculate $J_{n,k}$ by using the expression (2.1).

The first lemma concerns the H_0^1 norm of the main part and is well known, see [9].

LEMMA 2.1. *Let $N \geq 4$. We have*

$$\int_\Omega |\nabla PU_{\lambda_n, a_n}|^2\, dx = N(N-2)A - (N-2)^2 \omega_N^2 R(a_n) \lambda_n^{N-2} + O\left(\frac{\lambda_n^N}{d_n^N} \left|\log\left(\frac{\lambda_n}{d_n}\right)\right|\right)$$

as $n \to \infty$, where

$$A = \int_{\mathbf{R}^N} U_{1,0}^{p+1} dx = \frac{\Gamma(N/2)}{\Gamma(N)} \pi^{N/2}.$$

Next we will prove

LEMMA 2.2 (Asymptotic behavior of L^2 norm of the main part). *When $N \geq 5$, we have*

$$\int_\Omega k(x) PU_{\lambda_n,a_n}^2 dx = k(a_n)\omega_N C_N \lambda_n^2 + o(\lambda_n^2) \quad \text{as } n \to \infty,$$

where

$$C_N = \int_0^\infty \frac{s^{N-1}}{(1+s^2)^{N-2}} ds = \frac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{2\Gamma(N-2)}.$$

When $N = 4$, we have

$$\int_\Omega k(x) PU_{\lambda_n,a_n}^2 dx = k(a_n)\omega_4 \lambda_n^2 |\log \lambda_n| + O\left(\frac{\lambda_n^2}{d_n} |\log \lambda_n|^{1/2}\right) + O\left(\frac{\lambda_n^2}{d_n^2}\right)$$

as $n \to \infty$.

Proof. We extend PU_{λ_n,a_n} and φ_{λ_n,a_n} to $\mathbf{R}^N$ by setting $PU_{\lambda_n,a_n} = 0$ and $\varphi_{\lambda_n,a_n} = U_{\lambda_n,a_n}$, respectively, in $\mathbf{R}^N \setminus \Omega$.

First we treat the case $N \geq 5$. We have

$$\begin{aligned}\int_\Omega k(x) PU_{\lambda_n,a_n}^2 dx &= \int_\Omega k(x) U_{\lambda_n,a_n}^2 dx + \int_\Omega k(x)\varphi_{\lambda_n,a_n}^2 dx \\ &\quad + O\biggl(\biggl(\int_\Omega U_{\lambda_n,a_n}^2\, dx\biggr)^{1/2} \biggl(\int_\Omega \varphi_{\lambda_n,a_n}^2\, dx\biggr)^{1/2}\biggr). \end{aligned} \tag{2.5}$$

We easily see

$$\int_\Omega U_{\lambda_n,a_n}^2\, dx = O\biggl(\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^2\, dx\biggr) = O(\lambda_n^2). \tag{2.6}$$

When $N \geq 5$, we can check that

$$\int_\Omega \varphi_{\lambda_n,a_n}^2 dx = O\biggl(\frac{\lambda_n^{N-2}}{d_n^{N-4}}\biggr). \tag{2.7}$$

Indeed, we represent the integral as

$$\int_\Omega \varphi_{\lambda_n,a_n}^2 dx = \int_{B_{d_n}(a_n)} \varphi_{\lambda_n,a_n}^2\, dx + \int_{\Omega\setminus B_{d_n}(a_n)} \varphi_{\lambda_n,a_n}^2 dx.$$

Now,

$$\begin{aligned}\int_{B_{d_n}(a_n)} \varphi_{\lambda_n,a_n}^2 dx &= O(\|\varphi_{\lambda_n,a_n}\|_{L^\infty(\Omega)}^2 \cdot \mathrm{vol}(B_{d_n}(a_n))) \\ &= O\biggl(\biggl(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^{N-2}}\biggr)^2 \cdot d_n^N\biggr) = O\biggl(\frac{\lambda_n^{N-2}}{d_n^{N-4}}\biggr)\end{aligned}$$

by (2.2), and

$$\begin{aligned}\int_{\Omega\setminus B_{d_n}(a_n)} \varphi_{\lambda_n,a_n}^2\, dx &= O\biggl(\int_{\mathbf{R}^N\setminus B_{d_n}(a_n)} U_{\lambda_n,a_n}^2\, dx\biggr) \\ &= O\biggl(\int_{d_n}^\infty \biggl(\frac{\lambda_n}{\lambda_n^2 + r^2}\biggr)^{N-2} r^{N-1}\, dr\biggr) = O\biggl(\frac{\lambda_n^{N-2}}{d_n^{N-4}}\biggr),\end{aligned}$$

since $0 < \varphi_{\lambda_n,a_n} < U_{\lambda_n,a_n}$ in Ω and $\varphi_{\lambda_n,a_n} = U_{\lambda_n,a_n}$ on $\mathbf{R}^N \setminus \Omega$. Thus we obtain (2.7).

By (2.6) and (2.7), we have

$$\int_\Omega k(x)\varphi_{\lambda_n,a_n}^2\,dx + O\biggl(\Bigl(\int_\Omega U_{\lambda_n,a_n}^2\,dx\Bigr)^{1/2}\Bigl(\int_\Omega \varphi_{\lambda_n,a_n}^2\,dx\Bigr)^{1/2}\biggr) = o(\lambda_n^2).$$

Now, we estimate the term $\int_\Omega k(x)U_{\lambda_n,a_n}^2\,dx$. We split the integral as

$$\int_\Omega k(x)U_{\lambda_n,a_n}^2\,dx = \int_{B_{d_n}(a_n)} k(x)U_{\lambda_n,a_n}^2\,dx + \int_{\Omega\setminus B_{d_n}(a_n)} k(x)U_{\lambda_n,a_n}^2\,dx.$$

Making a Taylor expansion of $k(x)$ on $B_{d_n}(a_n)$, we have

$$\begin{aligned}\int_{B_{d_n}(a_n)} k(x)U_{\lambda_n,a_n}^2\,dx &= k(a_n)\int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^2\,dx + \nabla k(a_n)\cdot\int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^2(x-a_n)\,dx\\ &\quad + \int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^2 O(\|\nabla^2 k\|_{L^\infty(B_{d_n}(a_n))}|x-a_n|^2)\,dx.\end{aligned}$$

A calculation shows

$$\begin{aligned}\int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^2\,dx &= \omega_N\int_0^{d_n}\Bigl(\frac{\lambda_n}{\lambda_n^2+r^2}\Bigr)^{N-2} r^{N-1}\,dr\\ &= \omega_N\lambda_n^2\int_0^{d_n/\lambda_n}\frac{s^{N-1}}{(1+s^2)^{N-2}}\,ds = \omega_N\lambda_n^2\Bigl(\int_0^\infty - \int_{d_n/\lambda_n}^\infty\Bigr)\\ &= \omega_N\lambda_n^2\biggl(C_N + O\biggl(\biggl|\int_{d_n/\lambda_n}^\infty \frac{s^{N-1}}{(1+s^2)^{N-2}}\,ds\biggr|\biggr)\biggr)\\ &= \omega_N C_N\lambda_n^2 + O\Bigl(\frac{\lambda_n^{N-2}}{d_n^{N-4}}\Bigr),\end{aligned}$$

here we have used the assumption $N \ge 5$. Since the integrand is odd, we also have

$$\int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^2(x-a_n)\,dx = \vec{0}.$$

Now, a direct calculation shows

$$\begin{aligned}\int_{B_{d_n}(a_n)} |x-a_n|^2 U_{\lambda_n,a_n}^2\,dx &= \omega_N\int_0^{d_n}\Bigl(\frac{\lambda_n}{\lambda_n^2+r^2}\Bigr)^{N-2} r^{N+1}\,dr\\ &= \omega_N\lambda_n^4\int_0^{d_n/\lambda_n}\frac{s^{N+1}}{(1+s^2)^{N-2}}\,ds\end{aligned}$$

and

$$\int_0^{d_n/\lambda_n}\frac{s^{N+1}}{(1+s^2)^{N-2}}ds = \begin{cases} O(1), & N > 7,\\ O(|\log(\lambda_n/d_n)|), & N = 6,\\ O(\lambda_n^{N-6}/d_n^{N-6}), & N < 6.\end{cases}$$

Thus we obtain

$$\int_{B_{d_n}(a_n)} k(x)U_{\lambda_n,a_n}^2\,dx = k(a_n)\omega_N C_N\lambda_n^2 + o(\lambda_n^2)$$

as $n \to \infty$. On the other hand, we estimate

$$\left| \int_{\Omega \setminus B_{d_n}(a_n)} k(x) U^2_{\lambda_n, a_n}\, dx \right| = O\Big(\|k\|_{L^\infty} \int_{\mathbf{R}^N \setminus B_{d_n}(a_n)} U^2_{\lambda_n, a_n}\, dx \Big), \\ = O\Big(\int_{d_n}^{\infty} \Big(\frac{\lambda_n}{\lambda_n^2 + r^2} \Big)^{N-2} r^{N-1}\, dr \Big) = O\Big(\frac{\lambda_n^{N-2}}{d_n^{N-4}} \Big)$$

as before when $N \geq 5$. Thus returning to (2.5), we have

$$\int_\Omega k(x) P U^2_{\lambda_n, a_n}\, dx = k(a_n) \omega_N C_N \lambda_n^2 + o(\lambda_n^2)$$

as $n \to \infty$.

When $N = 4$, we argue as follows.

We fix a bounded domain $\tilde{\Omega} \supset\supset \Omega$. Note that $\exists C_1 \geq \tilde{d}_n := \mathrm{dist}(a_n, \partial \tilde{\Omega}) \geq C_2 > 0$ for all n since $a_n \in \Omega$. Denote $\tilde{R} = \mathrm{diam}(\tilde{\Omega})$. We extend $k \in C^2(\overline{\Omega})$ to $\tilde{\Omega}$ (which we also denote by k) so that $k \in C^2(\tilde{\Omega})$.

A calculation shows

$$\begin{aligned} \int_{B_L(a_n)} U^2_{\lambda_n, a_n}\, dx &= \omega_4 \int_{r=0}^{r=L} \frac{\lambda_n^2}{(\lambda_n^2 + r^2)^2} r^3\, dr = \omega_4 \lambda_n^2 \int_{s=0}^{s=L/\lambda_n} \frac{s^3}{(1+s^2)^2}\, ds \\ &= \omega_4 \lambda_n^2 \Big[\frac{1}{2} \log(1+s^2) + \frac{1}{2} (1+s^2)^{-1} \Big]_{s=0}^{s=L/\lambda_n} \\ &= \omega_4 \lambda_n^2 \Big[\frac{1}{2} \log(\lambda_n^2 + L^2) + |\log \lambda_n| + \frac{1}{2} \Big(\frac{\lambda_n^2}{\lambda_n^2 + L^2} \Big) - \frac{1}{2} \Big] \\ &= \omega_4 \lambda_n^2 |\log \lambda_n| + O(\lambda_n^2) \end{aligned}$$

for n sufficiently large. Thus

$$\int_{B_{\tilde{d}_n}(a_n)} U^2_{\lambda_n, a_n}\, dx = \omega_4 \lambda_n^2 |\log \lambda_n| + O(\lambda_n^2) \tag{2.8}$$

since $\log(\lambda_n^2 + \tilde{d}_n^2) = O(1)$ as $n \to \infty$ and

$$\int_{\tilde{\Omega}} U^2_{\lambda_n, a_n}\, dx = \Big(\int_{B_{\tilde{R}}(a_n)} U^2_{\lambda_n, a_n}\, dx \Big) = O\left(\lambda_n^2 |\log \lambda_n| \right). \tag{2.9}$$

Next, splitting the integral

$$\int_{\tilde{\Omega}} \varphi^2_{\lambda_n, a_n}\, dx = \int_{B_{\tilde{d}_n}(a_n)} \varphi^2_{\lambda_n, a_n}\, dx + \int_{\tilde{\Omega} \setminus B_{\tilde{d}_n}(a_n)} \varphi^2_{\lambda_n, a_n}\, dx,$$

and estimating

$$\begin{aligned} \int_{B_{\tilde{d}_n}(a_n)} \varphi^2_{\lambda_n, a_n}\, dx &= O\Big(\|\varphi_{\lambda_n, a_n}\|_{L^\infty(\tilde{\Omega})} \cdot \int_{B_{\tilde{d}_n}(a_n)} \varphi_{\lambda_n, a_n}\, dx \Big) \\ &= O\Big(\Big(\frac{\lambda_n}{d_n^2} \Big) \int_0^{\tilde{d}_n} \frac{\lambda_n}{\lambda_n^2 + r^2} r^3\, dr \Big) = O\Big(\frac{\lambda_n^2}{d_n^2} \Big), \end{aligned}$$

$$\int_{\tilde\Omega\setminus B_{\tilde d_n}(a_n)} \varphi_{\lambda_n,a_n}^2\,dx = O\biggl(\int_{B_{\tilde R}(a_n)\setminus B_{\tilde d_n}(a_n)} U_{\lambda_n,a_n}^2\,dx\biggr)$$
$$= O\biggl(\int_{\tilde d_n}^{\tilde R}\Bigl(\frac{\lambda_n}{\lambda_n^2+r^2}\Bigr)^2 r^3\,dr\biggr) = O(\lambda_n^2\,[\log r]_{r=\tilde d_n}^{r=\tilde R}) = O(\lambda_n^2),$$

we have

$$\int_{\tilde\Omega} \varphi_{\lambda_n,a_n}^2\,dx = O\biggl(\frac{\lambda_n^2}{d_n^2}\biggr). \tag{2.10}$$

Now, by (2.9) and (2.10), we have

$$\begin{aligned}\int_\Omega k(x)PU_{\lambda_n,a_n}^2\,dx &= \int_{\tilde\Omega} k(x)PU_{\lambda_n,a_n}^2\,dx \qquad (2.11)\\ &= \int_{\tilde\Omega} k(x)U_{\lambda_n,a_n}^2\,dx + O\biggl(\Bigl(\int_{\tilde\Omega} U_{\lambda_n,a_n}^2\,dx\Bigr)^{1/2}\Bigl(\int_{\tilde\Omega}\varphi_{\lambda_n,a_n}^2\,dx\Bigr)^{1/2}\biggr)\\ &\quad+\int_{\tilde\Omega} k(x)\varphi_{\lambda_n,a_n}^2\,dx\\ &= \int_{\tilde\Omega} k(x)U_{\lambda_n,a_n}^2\,dx + O\biggl(\frac{\lambda_n^2}{d_n^2}\biggr) + O\biggl(\frac{\lambda_n^2}{d_n}|\log\lambda_n|^{1/2}\biggr).\end{aligned}$$

Finally, as before we split the integral

$$\int_{\tilde\Omega} k(x)U_{\lambda_n,a_n}^2\,dx = \int_{B_{\tilde d_n}(a_n)} k(x)U_{\lambda_n,a_n}^2\,dx + \int_{\tilde\Omega\setminus B_{\tilde d_n}(a_n)} k(x)U_{\lambda_n,a_n}^2\,dx.$$

Taylor expansion of $k(x)$ leads to

$$\begin{aligned}\int_{B_{\tilde d_n}(a_n)} k(x)U_{\lambda_n,a_n}^2\,dx &= k(a_n)\int_{B_{\tilde d_n}(a_n)} U_{\lambda_n,a_n}^2\,dx + \nabla k(a_n)\cdot\int_{B_{\tilde d_n}(a_n)} U_{\lambda_n,a_n}^2(x-a_n)\,dx\\ &\quad+ O\biggl(\int_{B_{\tilde d_n}(a_n)}|x-a_n|^2U_{\lambda_n,a_n}^2\,dx\biggr)\\ &= k(a_n)\int_{B_{\tilde d_n}(a_n)} U_{\lambda_n,a_n}^2\,dx + 0 + O\biggl(\frac{\lambda_n^2}{d_n^2}\biggr)\\ &= k(a_n)\omega_4\lambda_n^2|\log\lambda_n| + O\biggl(\frac{\lambda_n^2}{d_n^2}\biggr),\end{aligned}$$

since

$$\int_{B_{\tilde d_n}(a_n)}|x-a_n|^2U_{\lambda_n,a_n}^2\,dx = \omega_4\lambda_n^4\int_0^{\tilde d_n/\lambda_n}\frac{s^5}{(1+s^2)^2}ds = O\biggl(\lambda_n^4\cdot\Bigl(\frac{\tilde d_n}{\lambda_n}\Bigr)^2\biggr)$$

and (2.8). On the other hand, we see that

$$\int_{\tilde\Omega\setminus B_{\tilde d_n}(a_n)} k(x)U_{\lambda_n,a_n}^2\,dx = O\biggl(\int_{\tilde d_n}^{\tilde R}\Bigl(\frac{\lambda_n}{\lambda_n^2+r^2}\Bigr)^2 r^3\,dr\biggr) = O(\lambda_n^2) = O\biggl(\frac{\lambda_n^2}{d_n^2}\biggr)$$

since $C_1 \ge \tilde d_n \ge C_2$. Going back to (2.11), we obtain Lemma 2.2 when $N = 4$. ∎

Since $w_n \in E_{\lambda_n,a_n}$ (see (2.2)), we have

$$\int_\Omega|\nabla v_n|^2\,dx = \alpha_n^2\int_\Omega|\nabla PU_{\lambda_n,a_n}|^2\,dx + \int_\Omega|\nabla w_n|^2\,dx.$$

Also we can estimate

$$\int_\Omega k(x)v_n^2\,dx = \alpha_n^2 \int_\Omega k(x)PU_{\lambda_n,a_n}^2\,dx + O(\|PU_{\lambda_n,a_n}\|_{L^2}\|w_n\|_{L^2}) + \int_\Omega k(x)w_n^2\,dx$$

and

$$\begin{aligned}
\varepsilon_n O(\|w_n\|_{L^2(\Omega)}^2) &= o(\|\nabla w_n\|_{L^2(\Omega)}^2),\\
O\left(\frac{\lambda_n^N}{d_n^N}\left|\log\left(\frac{\lambda_n}{d_n}\right)\right|\right) &= o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right),\\
\varepsilon_n O(\|PU_{\lambda_n,a_n}\|_{L^2}\|w_n\|_{L^2}) &= O(\varepsilon_n^{3/2}\|PU_{\lambda_n,a_n}\|_{L^2}^2) + O(\varepsilon_n^{1/2}\|\nabla w_n\|_{L^2}^2)\\
&= o(\varepsilon_n\|PU_{\lambda_n,a_n}\|_{L^2}^2) + o(\|\nabla w_n\|_{L^2}^2)
\end{aligned}$$

by the Poincaré inequality. Combining these with Lemma 2.1 and Lemma 2.2, we have the following lemma concerning $J_{n,k}$ defined by (2.4):

LEMMA 2.3 (Asymptotic behavior of $J_{n,k}$). *We have*

$$\begin{aligned}
J_{n,k} &= \int_\Omega |\nabla v_n|^2\,dx - \varepsilon_n \int_\Omega k(x)v_n^2\,dx\\
&= \alpha_n^2\left\{N(N-2)A - (N-2)^2\omega_N^2 R(a_n)\lambda_n^{N-2} - \varepsilon_n k(a_n)\omega_N C_N\lambda_n^2\right\}\\
&+ \|\nabla w_n\|_{L^2(\Omega)}^2\\
&+ o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o(\varepsilon_n\lambda_n^2) \quad \text{as } n\to\infty
\end{aligned}$$

when $N \geq 5$, and

$$\begin{aligned}
J_{n,k} &= \alpha_n^2\left\{8A - 4\omega_4^2 R(a_n)\lambda_n^2 - \varepsilon_n k(a_n)\omega_4\lambda_n^2|\log\lambda_n|\right\}\\
&+ \|\nabla w_n\|_{L^2(\Omega)}^2\\
&+ o\left(\frac{\lambda_n^2}{d_n^2}\right) + o(\|\nabla w_n\|_{L^2(\Omega)}^2) + o(\varepsilon_n\lambda_n^2|\log\lambda_n|) \quad \text{as } n\to\infty
\end{aligned}$$

when $N = 4$.

To proceed further, we need to know the precise asymptotic behavior of α_n as $n \to \infty$. This is the subject of the next lemma.

LEMMA 2.4 (Asymptotic behavior of α_n). *As $n \to \infty$, we have*

$$\begin{aligned}
\alpha_n^2 = \alpha_N^2 + 2\alpha_N^2\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right)R(a_n)\lambda_n^{N-2} - \frac{N+2}{A(N-2)}\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1}w_n^2\,dx\\
+o(\|\nabla w_n\|_{L^2(\Omega)}^2) + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right)
\end{aligned}$$

for $N \geq 4$, where $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$.

Proof. After extending v_n, PU_{λ_n,a_n}, and w_n by 0 outside Ω, we have

$$S^{N/2} = \int_\Omega v_n^{p+1}\,dx = \int_{\mathbf{R}^N} |\alpha_n PU_{\lambda_n,a_n} + w_n|^{p+1}\,dx. \tag{2.12}$$

We set $W_n := -\alpha_n\varphi_{\lambda_n,a_n} + w_n$, here we extend φ_{λ_n,a_n} to $\mathbf{R}^N$ as U_{λ_n,a_n} on $\mathbf{R}^N \setminus \Omega$.

By expanding the right hand side of (2.12), we have

$$\begin{aligned} S^{N/2} &= \int_{\mathbf{R}^N} (\alpha_n U_{\lambda_n,a_n} + W_n)^{p+1}\,dx \\ &= \alpha_n^{p+1}\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p+1}\,dx + (p+1)\alpha_n^p \int_{\mathbf{R}^N} U_{\lambda_n,a_n}^p W_n\,dx \\ &+ \frac{(p+1)p}{2}\alpha_n^{p-1}\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} W_n^2\,dx \\ &+ \begin{cases} O\left(\int_{\mathbf{R}^N} |W_n|^{p+1}\,dx\right) & (N \ge 6). \\ O\left(\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-2}|W_n|^3\,dx + \int_{\mathbf{R}^N} |W_n|^{p+1}\,dx\right) & (N = 4,5). \end{cases} \end{aligned} \tag{2.13}$$

First, we know

$$\alpha_n^{p+1}\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p+1}\,dx = \alpha_n^{p+1}A. \tag{2.14}$$

Next, by using the equation $-\Delta U_{\lambda_n,a_n} = N(N-2)U_{\lambda_n,a_n}^p$ in $\mathbf{R}^N$, we calculate

$$\begin{aligned} &(p+1)\alpha_n^p\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^p W_n\,dx = \frac{2\alpha_n^p}{(N-2)^2}\int_{\mathbf{R}^N}(-\Delta U_{\lambda_n,a_n})W_n\,dx \\ &= \frac{2\alpha_n^p}{(N-2)^2}\int_{\mathbf{R}^N}\nabla U_{\lambda_n,a_n}\cdot\nabla W_n\,dx \\ &= \frac{2\alpha_n^p}{(N-2)^2}\int_{\mathbf{R}^N}(\nabla PU_{\lambda_n,a_n} + \nabla\varphi_{\lambda_n,a_n})\cdot(-\alpha_n\nabla\varphi_{\lambda_n,a_n} + \nabla w_n)\,dx \\ &= \frac{-2\alpha_n^{p+1}}{(N-2)^2}\int_{\mathbf{R}^N}|\nabla\varphi_{\lambda_n,a_n}|^2\,dx \\ &= -2\alpha_n^{p+1}\omega_N^2 R(a_n)\lambda_n^{N-2} + O\left(\frac{\lambda_n^N}{d_n^N}\left|\log\left(\frac{\lambda_n}{d_n}\right)\right|\right). \end{aligned} \tag{2.15}$$

Here we have used the fact that φ_{λ_n,a_n} is a harmonic function on Ω, $w_n \in E_{\lambda_n,a_n}$ and

$$\begin{aligned} &\int_{\mathbf{R}^N}|\nabla\varphi_{\lambda_n,a_n}|^2\,dx = \int_{\mathbf{R}^N}|\nabla U_{\lambda_n,a_n}|^2\,dx - \int_{\mathbf{R}^N}|\nabla PU_{\lambda_n,a_n}|^2\,dx \\ &= (N-2)^2\omega_N^2 R(a_n)\lambda_n^{N-2} + O\left(\frac{\lambda_n^N}{d_n^N}\left|\log\left(\frac{\lambda_n}{d_n}\right)\right|\right) \end{aligned} \tag{2.16}$$

by Lemma 2.1.

The third term of (2.13) is calculated as follows: we split

$$\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1}W_n^2\,dx = \int_{\mathbf{R}^N\setminus\Omega} U_{\lambda_n,a_n}^{p-1}W_n^2\,dx + \int_\Omega U_{\lambda_n,a_n}^{p-1}W_n^2\,dx := I_1 + I_2.$$

Since $W_n = -\alpha_n U_{\lambda_n,a_n}$ on $\mathbf{R}^N\setminus\Omega$, the first term is estimated as

$$I_1 = \int_{\mathbf{R}^N\setminus\Omega} U_{\lambda_n,a_n}^{p-1}W_n^2\,dx = \alpha_n^2\int_{\mathbf{R}^N\setminus\Omega} U_{\lambda_n,a_n}^{p+1}\,dx = O\left(\int_{\mathbf{R}^N\setminus B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p+1}\,dx\right).$$

Now we compute

$$\int_{\mathbf{R}^N\setminus B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p+1}\,dx = \omega_N\int_{d_n}^\infty \left(\frac{\lambda_n}{\lambda_n^2 + r^2}\right)^N r^{N-1}\,dr = O\left(\frac{\lambda_n^N}{d_n^N}\right), \tag{2.17}$$

so we have

$$I_1 = \int_{\mathbf{R}^N\setminus\Omega} U_{\lambda_n,a_n}^{p-1} W_n^2\,dx = O\left(\frac{\lambda_n^N}{d_n^N}\right). \tag{2.18}$$

Next, substituting W_n by $-\alpha_n\varphi_{\lambda_n,a_n} + w_n$ in I_2, we have

$$\begin{aligned} I_2 = \int_\Omega U_{\lambda_n,a_n}^{p-1} W_n^2\,dx &= \alpha_n^2 \int_\Omega U_{\lambda_n,a_n}^{p-1}\varphi_{\lambda_n,a_n}^2\,dx + \int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} w_n^2\,dx \\ &+ O\left(\left(\int_\Omega U_{\lambda_n,a_n}^{p-1} w_n^2\,dx\right)^{1/2}\left(\int_\Omega U_{\lambda_n,a_n}^{p-1}\varphi_{\lambda_n,a_n}^2\,dx\right)^{1/2}\right). \end{aligned} \tag{2.19}$$

The Hölder and the Sobolev inequalities imply

$$\begin{aligned} \int_\Omega U_{\lambda_n,a_n}^{p-1} w_n^2\,dx &= O\left(\left(\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p+1}\,dx\right)^{\frac{p-1}{p+1}}\left(\int_\Omega w_n^{p+1}\,dx\right)^{\frac{2}{p+1}}\right) \\ &= O(\|\nabla w_n\|_{L^2(\Omega)}^2). \end{aligned} \tag{2.20}$$

When we estimate the first term on the right hand side of (2.19), we consider the cases according to the dimension. First we assume $N \geq 5$. We split the integral

$$\int_\Omega U_{\lambda_n,a_n}^{p-1}\varphi_{\lambda_n,a_n}^2\,dx = \int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1}\varphi_{\lambda_n,a_n}^2\,dx + \int_{\Omega\setminus B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1}\varphi_{\lambda_n,a_n}^2\,dx. \tag{2.21}$$

Then,

$$\begin{aligned} \int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1}\varphi_{\lambda_n,a_n}^2\,dx &= O\left(\|\varphi_{\lambda_n,a_n}\|_{L^\infty(\Omega)}^2 \cdot \int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1}\,dx\right) \\ &= O\left(\left(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^{N-2}}\right)^2 \cdot \lambda_n^2 d_n^{N-4}\right) = O\left(\frac{\lambda_n^N}{d_n^N}\right) \end{aligned} \tag{2.22}$$

since

$$\int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1}\,dx = \omega_N \int_0^{d_n}\left(\frac{\lambda_n}{\lambda_n^2 + r^2}\right)^2 r^{N-1}\,dr = O(\lambda_n^2 d_n^{N-4})$$

for $N \geq 5$. On the other hand, by (2.17),

$$\int_{\Omega\setminus B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1}\varphi_{\lambda_n,a_n}^2\,dx = O\left(\int_{\mathbf{R}^N\setminus B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p+1}\,dx\right) = O\left(\frac{\lambda_n^N}{d_n^N}\right).$$

Thus we have

$$\int_\Omega U_{\lambda_n,a_n}^{p-1}\varphi_{\lambda_n,a_n}^2\,dx = O\left(\frac{\lambda_n^N}{d_n^N}\right) \tag{2.23}$$

when $N \geq 5$.

When $N = 4$, we have

$$\begin{aligned} \int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1}\varphi_{\lambda_n,a_n}^2\,dx &= O\left(\|\varphi_{\lambda_n,a_n}\|_{L^\infty(\Omega)}^2 \cdot \int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1}\,dx\right) \\ &= O\left(\left(\frac{\lambda_n}{d_n^2}\right)^2 \cdot \lambda_n^2\left(\log\left(1 + \left(\frac{d_n}{\lambda_n}\right)^2\right) + O(1)\right)\right) = o\left(\frac{\lambda_n^3}{d_n^3}\right) \end{aligned}$$

since

$$\int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1}\,dx = \omega_4\lambda_n^2\left[\frac{1}{2}\log(1+s^2) + \frac{1}{2}(1+s^2)^{-1}\right]_{s=0}^{s=\frac{d_n}{\lambda_n}}.$$

So arguing as above, we have

$$\int_\Omega U_{\lambda_n,a_n}^{p-1}\varphi_{\lambda_n,a_n}^2\,dx = o\left(\frac{\lambda_n^3}{d_n^3}\right) \tag{2.24}$$

when $N=4$. Combining (2.23) and (2.24), we conclude

$$\int_\Omega U_{\lambda_n,a_n}^{p-1}\varphi_{\lambda_n,a_n}^2\,dx = o\left(\frac{\lambda_n^{N-1}}{d_n^{N-1}}\right) \tag{2.25}$$

when $N\geq 4$.

Returning to (2.19) with (2.20) and (2.25), and using the Young inequality

$$\left(\frac{\lambda_n}{d_n}\right)^{\frac{N-1}{2}}\|\nabla w\|_{L^2} = O\left(\left(\frac{\lambda_n}{d_n}\right)^{\frac{2N-3}{2}}\right) + O\left(\left(\frac{\lambda_n}{d_n}\right)^{\frac{1}{2}}\|\nabla w\|_{L^2}^2\right),$$

we have

$$\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1}W_n^2\,dx = \int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1}w_n^2\,dx + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o(\|\nabla w\|_{L^2}^2). \tag{2.26}$$

Finally, by the Sobolev inequality and the inequality $(a+b)^t\leq C(a^t+b^t)$ for $C=2^{t-1}$ $(a,b>0,t>1)$, we have

$$\begin{aligned}
\int_{\mathbf{R}^N}|W_n|^{p+1}\,dx &= O\left(\left(\int_{\mathbf{R}^N}|\nabla W_n|^2\,dx\right)^{\frac{p+1}{2}}\right)\\
&= O\left(\left(\int_{\mathbf{R}^N}|\nabla\varphi_{\lambda_n,a_n}|^2\,dx + \int_{\mathbf{R}^N}|\nabla w_n|^2\,dx\right)^{\frac{p+1}{2}}\right)\\
&= O\left(\left(\int_{\mathbf{R}^N}|\nabla\varphi_{\lambda_n,a_n}|^2\,dx\right)^{\frac{p+1}{2}}\right) + O\left(\left(\int_{\mathbf{R}^N}|\nabla w_n|^2\,dx\right)^{\frac{p+1}{2}}\right).
\end{aligned}$$

So by (2.16) and the estimate of the Robin function (1.5),

$$\begin{aligned}
\int_{\mathbf{R}^N}|W_n|^{p+1}\,dx &= O\left(\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right)^{\frac{N}{N-2}}\right) + O(\|\nabla w_n\|_{L^2(\Omega)}^{\frac{2N}{N-2}})\\
&= O\left(\frac{\lambda_n^N}{d_n^N}\right) + o(\|\nabla w_n\|_{L^2(\Omega)}^2).
\end{aligned} \tag{2.27}$$

When $N=4,5$, we also need to estimate the term $\int_{\mathbf{R}^N}U_{\lambda_n,a_n}^{p-2}W_n^3\,dx$. But the calculation is almost the same. Indeed, by the Hölder inequality and (2.27), we have

$$\begin{aligned}
\int_{\mathbf{R}^N}U_{\lambda_n,a_n}^{p-2}|W_n|^3\,dx &= O\left(\left(\int_{\mathbf{R}^N}|W_n|^{p+1}\,dx\right)^{\frac{3}{p+1}}\right)\times O\left(\left(\int_{\mathbf{R}^N}U_{\lambda_n,a_n}^{p+1}\,dx\right)^{\frac{p-2}{p+1}}\right)\\
&= \left(O\left(\frac{\lambda_n^N}{d_n^N}\right) + O(\|\nabla w_n\|_{L^2(\Omega)}^{p+1})\right)^{\frac{3}{p+1}}\times O(1)\\
&= O\left(\left(\frac{\lambda_n^N}{d_n^N}\right)\right)^{\frac{3(N-2)}{2N}} + O(\|\nabla w_n\|_{L^2(\Omega)}^{\frac{2N}{N-2}})^{\frac{3(N-2)}{2N}}\\
&= o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o(\|\nabla w_n\|_{L^2(\Omega)}^2).
\end{aligned} \tag{2.28}$$

Here we have used the inequality $(a+b)^t\leq(a^t+b^t)$ for $a,b\geq$ and $0<t<1$.

Inserting (2.14), (2.15), (2.26), (2.27) (and (2.28) when $N = 4, 5$) to (2.13), we obtain

$$S^{N/2} = \alpha_n^{p+1} A - 2\alpha_n^{p+1} \cdot \omega_N^2 R(a_n)\lambda_n^{N-2} + \frac{(p+1)p}{2}\alpha_n^{p-1} \int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} w_n^2\,dx + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o(\|\nabla w\|_{L^2}^2).$$

Dividing both sides by A and noting that $\frac{S^{N/2}}{A} = \alpha_N^{p+1}$, we have

$$\alpha_N^{p+1} = \alpha_n^{p+1} - \alpha_n^{p+1}\left(\frac{2\omega_N^2}{A}\right) R(a_n)\lambda_n^{N-2} + \frac{p(p+1)}{2}\frac{\alpha_n^{p-1}}{A} \int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} w_n^2\,dx + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o(\|\nabla w\|_{L^2}^2). \tag{2.29}$$

Since $\alpha_n^{p+1} = \alpha_N^{p+1} + o(1)$ and $R(a_n) = O(\frac{1}{d_n^{N-2}})$, we know

$$\alpha_n^{p+1} R(a_n)\lambda_n^{N-2} = \alpha_N^{p+1} R(a_n)\lambda_n^{N-2} + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right).$$

Similarly, we have

$$\alpha_n^{p-1} \int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} w_n^2\,dx = \alpha_N^{p-1} \int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} w_n^2\,dx + o(\|\nabla w\|_{L^2}^2).$$

Substituting these in (2.29), we have

$$\alpha_n^{p+1} = \alpha_N^{p+1} + \alpha_N^{p+1}\left(\frac{2\omega_N^2}{A}\right) R(a_n)\lambda_n^{N-2} - \frac{p(p+1)}{2}\frac{\alpha_N^{p-1}}{A} \int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} w_n^2\,dx + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o(\|\nabla w\|_{L^2}^2),$$

which implies

$$\alpha_n^{p+1} = \alpha_N^{p+1}\left\{1 + \left(\frac{2\omega_N^2}{A}\right) R(a_n)\lambda_n^{N-2} - \frac{(p+1)p}{2A\alpha_N^2} \int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} w_n^2\,dx\right\} + o(\|\nabla w_n\|_{L^2(\Omega)}^2) + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right).$$

By Taylor expansion $(1+x)^{\frac{2}{p+1}} = 1 + \frac{2}{p+1}x + o(x)$ as $x \to 0$, we conclude that

$$\alpha_n^2 = \alpha_N^2\left\{1 + \left(\frac{2}{p+1}\right)\left(\frac{2\omega_N^2}{A}\right) R(a_n)\lambda_n^{N-2} - \frac{p}{A\alpha_N^2} \int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} w_n^2\,dx\right\} + o(\|\nabla w_n\|_{L^2(\Omega)}^2) + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right)$$

as $n \to \infty$.

This completes the proof of Lemma 2.4. ■

Combining Lemma 2.3 and Lemma 2.4, and noting $\alpha_N^2 N(N-2)A = S^{N/2}$, we obtain:

PROPOSITION 2.5 (Asymptotic behavior of $S_{\varepsilon_n,k}$). *As $n \to \infty$, we have*

$$\begin{aligned} S_{\varepsilon_n,k} = &\inf_{\substack{v\in H_0^1(\Omega)\\ \|v\|_{L^{p+1}(\Omega)}=1}} \left\{ \int_\Omega |\nabla v|^2\,dx - \varepsilon_n \int_\Omega k(x)v^2\,dx \right\} \\ &= S\cdot S^{-\frac{N}{2}} J_{n,k} = S + S\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right) R(a_n)\lambda_n^{N-2} \\ &- \varepsilon_n k(a_n)\left(\frac{S\omega_N C_N}{N(N-2)A}\right)\lambda_n^2 \\ &+ S^{(2-N)/2}\left\{\|\nabla w_n\|_{L^2}^2 - N(N+2)\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} w_n^2\,dx\right\} \\ &+ o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o(\|\nabla w_n\|_{L^2(\Omega)}^2) + o(\varepsilon_n\lambda_n^2) \end{aligned}$$

when $N \ge 5$, and

$$\begin{aligned} S_{\varepsilon_n,k} = S &+ \frac{S}{2}\left(\frac{\omega_4^2}{A}\right)R(a_n)\lambda_n^2 - \varepsilon_n k(a_n)\left(\frac{S\omega_4}{8A}\right)\lambda_n^2|\log\lambda_n| \\ &+ S^{-1}\left\{\|\nabla w_n\|_{L^2}^2 - 24\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^2 w_n^2\,dx\right\} \\ &+ o\left(\frac{\lambda_n^2}{d_n^2}\right) + o(\|\nabla w_n\|_{L^2(\Omega)}^2) + o(\varepsilon_n\lambda_n^2|\log\lambda_n|) \end{aligned}$$

when $N = 4$.

To proceed further, we need the nondegeneracy result first shown by Rey ([8, Appendix D]).

LEMMA 2.6 (Nondegeneracy inequality). *There exists a constant $C > 0$ which depends only on the dimension N such that for any $w_n \in E_{\lambda_n,a_n}$,*

$$\int_{\mathbf{R}^N} |\nabla w_n|^2\,dx - N(N+2)\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} w_n^2\,dx \ge C\int_{\mathbf{R}^N} |\nabla w_n|^2\,dx.$$

Furthermore, we need the appropriate bound of the value $S_{\varepsilon_n,k}$ from the above. The following Lemma is proved by the same argument of Lemma 2.7 in [9], so we omit the proof.

LEMMA 2.7 (Upper bound of $S_{\varepsilon,k}$). *For any $a \in \Omega_+ = \{a \in \Omega : k(a) > 0\}$ and $\rho > 0$, there exists an $\varepsilon_0 = \varepsilon_0(a,\rho)$ such that if $\varepsilon \in (0,\varepsilon_0)$, then*

$$S_{\varepsilon,k} \le S - \left(\frac{N-4}{N-2}\right)\varepsilon k(a)\left\{\frac{S\omega_N C_N}{N(N-2)A} - \rho\right\}\left[\frac{2C_N\varepsilon k(a)}{(N-2)^3\omega_N R(a)}\right]^{\frac{2}{N-4}}$$

when $N \ge 5$, and

$$S_{\varepsilon,k} \le S - \frac{S\varepsilon k(a)\omega_4}{16Ae}\exp\left(-\frac{8\omega_4 R(a) + \varepsilon k(a)/e + 2\rho}{\varepsilon k(a)}\right)$$

when $N = 4$.

3. Proof of Theorem. In this section, we prove Theorem 1.1.

The following elementary facts are important in the argument: for constants $C_A, C_B > 0$, the function

$$f_N(\lambda) = S + C_A\lambda^{N-2} - C_B\lambda^2$$

has the unique global minimum value

$$\min_{\lambda>0} f_N(\lambda) = S - \left(\frac{N-4}{N-2}\right) C_B \left(\frac{2C_B}{(N-2)C_A}\right)^{\frac{2}{N-4}} \tag{3.1}$$

when $N \geq 5$, and

$$f_4(\lambda) = S + C_A\lambda^2 - C_B\lambda^2|\log\lambda|, \quad 0 < \lambda < 1,$$

has the unique global minimum value

$$\min_{\lambda>0} f_4(\lambda) = S - \left(\frac{C_B}{2e}\right)\exp\left(-\frac{2C_A}{C_B}\right) \tag{3.2}$$

when $N = 4$.

Now, we treat the case $N \geq 5$. Set

$$K_1 = S\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right), \quad K_2 = \frac{S\omega_N C_N}{N(N-2)A}.$$

First, we prove that $k(a_n) > 0$ for n sufficiently large. Assume the contrary that there exists a subsequence such that $k(a_n) \leq 0$. In addition if $k(a_n) \leq -C < 0$ for some $C > 0$ independent of n, then Proposition 2.5 and Lemma 2.6 yield the inequality $S_{\varepsilon_n,k} \geq S$. This is a contradiction to the fact $S > S_{\varepsilon_n,k}$ by Brezis and Nirenberg, see Introduction. Thus $k(a_n) \to 0$ for every sequence with $k(a_n) \leq 0$.

On the other hand, by the result of Brezis and Nirenberg, Proposition 2.5, Lemma 2.6 and (1.5), we have $C_1 > 0$ independent of n such that

$$S > S_{\varepsilon_n,k} \geq S + C_1\lambda_n^{N-2} - (k(a_n)K_2 + p_n)\varepsilon_n\lambda_n^2$$

for some $p_n > 0, p_n \to 0$. Therefore we have

$$C_B(n) := K_2 k(a_n) + p_n > 0$$

for n large. Thus by (3.1), we obtain

$$S_{\varepsilon_n,k} \geq S - \left(\frac{N-4}{N-2}\right) C_B(n)\varepsilon_n \left(\frac{2C_B(n)\varepsilon_n}{(N-2)C_1}\right)^{\frac{2}{N-4}}.$$

Connecting this with the upper bound

$$S_{\varepsilon_n,k} \leq S - C_2\varepsilon_n^{1+\frac{2}{N-4}}$$

for some $C_2 > 0$, which is assured by Lemma 2.7, we have a contradiction since we have seen that $C_B(n) \to 0$ as $n \to 0$. Thus we have proved $k(a_n) > 0$ for n sufficiently large.

The same argument shows that when $k(a_n) > 0$ for n sufficiently large, it cannot happen that $k(a_\infty) = \lim_{n\to\infty} k(a_n) = 0$.

Next, we will show that the blow up point a_∞ is in the interior of Ω. Suppose the contrary. Then $a_\infty \in \partial\Omega$ and $d_n = d(a_n, \partial\Omega) \to 0$ as $n \to \infty$. Then by Proposition 2.5, Lemma 2.6, (1.5) and the fact that $k(a_n) \geq C > 0$ for large n, we can find constants $C_1, C_2, C_3 > 0$ such that when $N \geq 5$,

$$\begin{aligned}
S_{\varepsilon_n,k} &= S + S\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right)R(a_n)\lambda_n^{N-2} \\
&\quad - \varepsilon_n k(a_n)\left(\frac{S\omega_N C_N}{N(N-2)A}\right)\lambda_n^2 \\
&\quad + S^{(2-N)/2}\left\{\|\nabla w_n\|_{L^2}^2 - N(N+2)\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} w_n^2\,dx\right\} \\
&\quad + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o(\|\nabla w_n\|_{L^2(\Omega)}^2) + o(\varepsilon_n\lambda_n^2) \\
&\geq S + C_1\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) - C_2\varepsilon_n\lambda_n^2 \\
&\geq S - \left(\frac{N-4}{N-2}\right)C_2\varepsilon_n\left\{\frac{2C_2\varepsilon_n}{(N-2)C_1(\frac{1}{d_n^{N-2}})}\right\}^{\frac{2}{N-4}} \\
&= S - C_3\varepsilon_n^{\frac{N-2}{N-4}} d_n^{\frac{2(N-2)}{N-4}} = S + o(\varepsilon_n^{\frac{N-2}{N-4}}),
\end{aligned}$$

since we assumed $d_n \to 0$. Here we used (3.1) in deriving the second inequality.

On the other hand, we know that $S_{\varepsilon_n,k} \leq S - C\varepsilon_n^{\frac{N-2}{N-4}}$ for some $C > 0$ by Lemma 2.7. This contradicts the above estimate, so we conclude that a_∞ is in the interior of Ω.

Now, since we have proved that $d_n \geq C$ for some constant $C > 0$ uniformly in n, we may drop d_n in the asymptotic formula of Proposition 2.5. Then we can find $p_n, q_n > 0$, $p_n, q_n \to 0$ such that

$$\begin{aligned}
S_{\varepsilon_n,k} &= S + K_1 R(a_n)\lambda_n^{N-2} - \varepsilon_n k(a_n)K_2\lambda_n^2 \\
&\quad + S^{(2-N)/2}\left\{\|\nabla w_n\|_{L^2}^2 - N(N+2)\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} w_n^2\,dx\right\} \\
&\quad + o(\lambda_n^{N-2}) + o(\|\nabla w_n\|_{L^2(\Omega)}^2) + o(\varepsilon_n\lambda_n^2) \\
&\geq S + (K_1R(a_n) - p_n)\lambda_n^{N-2} - (K_2k(a_n) + q_n)\varepsilon_n\lambda_n^2 \\
&\geq S - \left(\frac{N-4}{N-2}\right)\varepsilon_n(K_2k(a_n)+q_n)\left[\frac{2\varepsilon_n(K_2k(a_n)+q_n)}{(N-2)(K_1R(a_n)-p_n)}\right]^{\frac{2}{N-4}}. \qquad (3.3)
\end{aligned}$$

The last inequality of (3.3) follows again by (3.1) and the fact that $K_2k(a_n) + q_n > 0$ for n large.

On the other hand, Lemma 2.7 gives an upper bound

$$S_{\varepsilon_n,k} \leq S - (K_2 - \rho)\varepsilon_n k(a)\left[\frac{2K_2\varepsilon_n k(a)}{(N-2)K_1R(a)}\right]^{\frac{2}{N-4}}$$

for any $a \in \Omega_+$ and $\rho > 0$ sufficiently small. Therefore by combining these, we have

$$\begin{aligned}
&(K_2k(a_n) + q_n)\varepsilon_n\left[\frac{2\varepsilon_n(K_2k(a_n)+q_n)}{(N-2)(K_1R(a_n)-p_n)}\right]^{\frac{2}{N-4}} \\
&\geq (K_2-\rho)k(a)\varepsilon_n\left[\frac{2K_2\varepsilon_n k(a)}{(N-2)K_1R(a)}\right]^{\frac{2}{N-4}}.
\end{aligned}$$

Dividing both sides by $\varepsilon_n^{\frac{N-2}{N-4}}$, letting $n \to \infty$ and $\rho \to 0$, we check that a_∞ will maximize

$$k(a)\left(\frac{k(a)}{R(a)}\right)^{\frac{2}{N-4}} = (\psi_1(a))^{\frac{N-2}{N-4}}.$$

When $N = 4$, the fact that $k(a_n) > 0$ for n large and $k(a_\infty) \neq 0$ is proved similarly as in the proof when $N \geq 5$.

The proof of the fact that the blow up point belongs to the interior of Ω is also the same. We give a proof for the reader's convenience. Since $k(a_n) \geq C > 0$ uniformly in n large, we have some $C_1, C_2 > 0$ such that

$$\begin{aligned}
S_{\varepsilon_n,k} &= S + \left(\frac{S\omega_4^2}{2A}\right)R(a_n)\lambda_n^2 - \varepsilon_n k(a_n)\left(\frac{S\omega_4}{8A}\right)\lambda_n^2|\log\lambda_n| \\
&\quad + S^{-1}\left\{\|\nabla w_n\|_{L^2}^2 - 24\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p-1} w_n^2\,dx\right\} \\
&\quad + o\left(\frac{\lambda_n^2}{d_n^2}\right) + o(\varepsilon_n\lambda_n^2|\log\lambda_n|) + o(\|\nabla w_n\|_{L^2(\Omega)}^2) \\
&\geq S + C_1\left(\frac{\lambda_n^2}{d_n^2}\right) - C_2 k(a_n)\varepsilon_n\lambda_n^2|\log\lambda_n| \\
&\geq S - \frac{\varepsilon_n k(a_n)C_2}{2e}\exp\left(-\frac{2C_1(\frac{1}{d_n^2})}{\varepsilon_n k(a_n)C_2}\right),
\end{aligned}$$

again we used (3.2).

On the other hand, Lemma 2.7 yields constants $C_3, C_4 > 0$ such that

$$S_{\varepsilon_n,k} \leq S - C_3\varepsilon_n\exp\left(-\frac{C_4}{\varepsilon_n}\right).$$

Combining these, we obtain

$$C_5 k(a_n)\exp\left(-\frac{C_6(\frac{1}{d_n^2})}{\varepsilon_n k(a_n)}\right) \geq 3\exp\left(-\frac{C_4}{\varepsilon_n}\right)$$

for some $C_5, C_6 > 0$. Taking logarithms of both sides and multiplying by ε_n, we have

$$\varepsilon_n\log C_5 + \varepsilon_n\log k(a_n) - \frac{C_6(\frac{1}{d_n^2})}{k(a_n)} \geq \varepsilon_n\log C_3 - C_4.$$

Note that $\varepsilon_n \log k(a_n) \to 0$ as $n \to \infty$. Then the above inequality leads to the contradiction if $d_n \to 0$. Thus, $d_n \not\to 0$.

Now, as before, we may drop d_n in the expansion of $S_{\varepsilon_n,k}$ and

$$\begin{aligned}
S_{\varepsilon_n,k} &\geq S + K_3R(a_n)\lambda_n^2 - \varepsilon_n k(a_n)K_4\lambda_n^2|\log\lambda_n| \\
&\quad + o(\lambda_n^2) + o(\varepsilon_n\lambda_n^2|\log\lambda_n|) \\
&\geq S + (K_3R(a_n) - p_n)\lambda_n^2 - \varepsilon_n(K_4k(a_n) + q_n)\lambda_n^2|\log\lambda_n| \\
&\geq S - \left(\frac{K_4k(a_n) + q_n}{2e}\right)\varepsilon_n\exp\left(-\frac{2(K_3R(a_n) - p_n)}{(K_4k(a_n) + q_n)\varepsilon_n}\right) \qquad (3.4)
\end{aligned}$$

where $p_n, q_n > 0, p_n, q_n = o(1)$ and

$$K_3 = \frac{S}{2}\left(\frac{\omega_4^2}{A}\right), \quad K_4 = \frac{S\omega_4}{8A}.$$

The last inequality of (3.4) follows again from (3.2).

Combining (3.4) with the upper bound Lemma 2.7 as before, we have

$$\begin{aligned} & S - \left(\frac{K_4 k(a_n) + q_n}{2e}\right)\varepsilon_n \exp\left(- \frac{2(K_3 R(a_n) - p_n)}{(K_4 k(a_n) + q_n)\varepsilon_n}\right) \\ & \le S_{\varepsilon_n, k} \le \\ & S - \left(\frac{K_4 k(a)}{2e}\right)\varepsilon_n \exp\left(- \frac{8\omega_4 R(a) + \varepsilon_n/e + 2\rho}{\varepsilon_n k(a)}\right) \end{aligned}$$

for any $a \in \Omega_+$ and $\rho > 0$. This leads to

$$\varepsilon_n \log(K_4 k(a_n) + q_n) - \frac{2(K_3 R(a_n) - p_n)}{(K_4 k(a_n) + q_n)} \ge \varepsilon_n \log(K_4 k(a)) - (8\omega_4 R(a) + \varepsilon_n/e + 2\rho)/k(a).$$

Finally, letting $n \to \infty$ and then $\rho \to 0$, we obtain

$$-8\omega_4 \frac{R(a_\infty)}{k(a_\infty)} \ge -8\omega_4 \frac{R(a)}{k(a)}.$$

This completes the proof of Theorem.

Acknowledgements. Preparation of this work was partially supported by JSPS Grant-in-Aid for Scientific Research, No. 17540186.

References

[1] H. Brezis, *Elliptic equations with limiting Sobolev exponents. The impact of topology,* Comm. Pure and Appl. Math. 39 (1986), 17–39.

[2] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents,* Comm. Pure and Appl. Math. 36 (1983), 437–477.

[3] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. Part 1,* Rev. Mat. Iberoamericana 1 (1985), 145–201.

[4] Z. C. Han, *Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent,* Ann. Inst. Henri Poincaré 8 (1991), 159–174.

[5] T. Isobe, *On the asymptotic analysis of H-systems, I: asymptotic behavior of large solutions,* Adv. Diff. Eq. 6 (2001), 513–546.

[6] R. Molle and A. Pistoia, *Concentration phenomena in elliptic problem with critical and supercritical growth,* Adv. Diff. Eq. 8 (2003), 547–570.

[7] O. Rey, *Proof of two conjectures of H. Brezis and L. A. Peletier,* Manuscripta Math. 65 (1989), 19–37.

[8] O. Rey, *The role of the Green's function in a non-linear elliptic equation involving the critical Sobolev exponent,* J. Funct. Anal. 89 (1990), 1–52.

[9] F. Takahashi, *On location of blow up points of least energy solutions to the Brezis-Nirenberg equations,* Funkcialaj Ekvacioj 47 (2004), 145–166.