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# TOPOLOGICAL ALGEBRAS WITH PSEUDOCONVEXLY BOUNDED ELEMENTS

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**Abstract.** It is shown that every commutative sequentially bornologically complete Hausdorff algebra $A$ with bounded elements is representable in the form of an (algebraic) inductive limit of an inductive system of locally bounded Fréchet algebras with continuous monomorphisms if the von Neumann bornology of $A$ is pseudoconvex. Several classes of topological algebras $A$ for which $r_A(a) \leq \beta_A(a)$ or $r_A(a) = \beta_A(a)$ for each $a \in A$ are described.

## 1. Introduction

**1.1.** Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers and $A$ a topological algebra over $\mathbb{K}$ with separately continuous multiplication (in short *topological algebra*). If the underlying linear topological space of $A$ is locally pseudoconvex, then $A$ is called a *locally pseudoconvex algebra* (in [14] a semiconvex algebra). In this case $A$ has a base $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of neighbourhoods of zero consisting of *balanced* (that is, $\mu U_\lambda \subset U_\lambda$ whenever $|\mu| \leq 1$) and *pseudoconvex* (that is, $U_\lambda + U_\lambda \subset \mu U_\lambda$ for some $\mu \geq 2$) sets. This base defines a set of numbers $\{k_\lambda : \lambda \in \Lambda\}$ in $(0, 1]$ such that $U_\lambda + U_\lambda \subset 2^{\frac{1}{k_\lambda}} U_\lambda$ and $\Gamma_{k_\lambda}(U_\lambda) \subset 2^{\frac{1}{k_\lambda}} U_\lambda$ for each $\lambda \in \Lambda$ (see [13], p. 115, or [20], p. 4 - 5). Here $\Gamma_k(U)$ denotes the *absolutely k-convex hull* of $U$ in $A$, that is, the set of elements

$$\sum_{v=1}^{n} \alpha_v u_v,$$

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where $n \in \mathbb{N}$, $u_1, \ldots, u_n \in U$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ are such that

$$\sum_{v=1}^{n} |\alpha_v|^k \leq 1.$$

In particular, $A$ is called a *locally k-convex algebra* if $k_\lambda = k$ for each $\lambda \in \Lambda$; a *locally convex algebra* if $k_\lambda = 1$ for each $\lambda \in \Lambda$; a *locally m-pseudoconvex algebra* if every $U \in \mathcal{U}$ is idempotent that is, $UU \subset U$; a *locally m-(k-convex) algebra* if $A$ is a locally $m$-pseudoconvex algebra for which $k_\lambda = k$ for each $\lambda \in \Lambda$ and a *locally bounded algebra* if the topology of $A$ contains a bounded neighbourhood of zero.

It is well known (see [20], p. 6) that the topology of every locally pseudoconvex algebra $A$ can be given by means of a family $\mathcal{P} = \{p_\lambda : \lambda \in \Lambda\}$ of $k_\lambda$-homogeneous seminorms, where $k_\lambda \in (0, 1]$ for each $\lambda \in \Lambda$ and

$$p_\lambda(a) = \inf\{|\mu|^{k_\lambda} : a \in \mu\Gamma_{k_\lambda}(U_\lambda)\}$$

for each $a \in A$ and $\lambda \in \Lambda$. Herewith, the topology of locally $m$-pseudoconvex algebra can be given by a family $\mathcal{P} = \{p_\lambda : \lambda \in \Lambda\}$ of $k_\lambda$-homogeneous submultiplicative (that is, $p_\lambda(ab) \leq p_\lambda(a)p_\lambda(b)$ for each $a, b \in A$) seminorms, and the topology of locally bounded Hausdorff algebra by a $k$-homogeneous norm with $k \in (0, 1]$. Therefore a locally bounded Hausdorff algebra is called a *k-normed algebra* and a complete locally bounded Hausdorff algebra a *k-Banach algebra*. Moreover, a locally $k$-convex algebra $A$ is *locally uniformly absorbing* if for each $a \in A$ there is a number $N(a) > 0$ (which does not depend on $\lambda$) such that $p_\lambda(ab) \leq N(a)p_\lambda(b)$ for each $b \in A$ and each $\lambda \in \Lambda$. Hence, a locally $k$-convex algebra $A$ is locally uniformly absorbing if $\sup_{\lambda \in \Lambda} p_\lambda(a)$ is finite for each $a \in A$.

**1.2.** A topological algebra $A$ is a *Q-algebra* if the set Qinv$A$ of all quasi-invertible elements (that is, of elements $a \in A$ such that[1] $a \circ b = b \circ a = \theta_A$ for an element $b \in A$) is open in $A$. It is easy to see that a unital algebra $A$ is a $Q$-algebra if and only if the set Inv$A$ (of all invertible elements in $A$) is open in $A$. Furthermore, $A$ is a *Mackey Q-algebra* if Qinv$A - a$ is a bornivore (that is, Qinv$A - a$ absorbs all bounded subsets of $A$) for each $a \in$ Qinv$A$. It is easy to see that every $Q$-algebra is a Mackey $Q$-algebra, but there is a Mackey $Q$-algebra which is not a $Q$-algebra (see [8], Example 3.9).

**1.3.** A net $(a_\lambda)_{\lambda \in \Lambda}$ in a topological algebra $A$ is said to be *advertibly convergent* in $A$ if there exists an element $a \in A$ such that $(a \circ a_\lambda)_{\lambda \in \Lambda}$ and $(a_\lambda \circ a)_{\lambda \in \Lambda}$ converge to $\theta_A$ in the topology of $A$. A topological algebra $A$ is *advertibly complete* if every advertibly convergent Cauchy net is convergent in $A$. It is known (see [16], p. 45) that all complete algebras and all $Q$-algebras are advertibly complete algebras. In particular, when only advertibly convergent Cauchy sequences are convergent in $A$, then we call $A$ a *sequentially advertibly complete algebra* (in short *sa-complete algebra*).

**1.4.** Let $X$ be a linear topological space over $\mathbb{K}$. A net $(x_\lambda)_{\lambda \in \Lambda}$ in $X$ is said to *converge bornologically to* $x_0$ if there exists a bounded subset $B$ of $X$ and for each $\epsilon > 0$ an index $\lambda_\epsilon \in \Lambda$ such that $x_\lambda - x_0 \in \epsilon B$ whenever $\lambda > \lambda_\epsilon$. Since every neighbourhood of zero absorbs all bounded sets, every bornologically convergent net is topologically convergent.

[1]Here, and later on, $\theta_A$ denotes the zero element of $A$ and $a \circ b = a + b - ab$ for each $a, b \in A$.

The converse is false in general (see [12], p. 122), but it is true in the case of metrizable linear topological space (see [12], p. 27). Sometimes (see, for example, [12]) instead of bornological convergence of sequences we talk about *Mackey convergence of sequences* because G. W. Mackey was the first who studied this notion of convergence.

A net $(x_\lambda)_{\lambda\in\Lambda}$ in $X$ is said to be a *Mackey-Cauchy net* if there exist a bounded subset $B$ of $X$ and for each $\epsilon > 0$ an index $\lambda_\epsilon \in \Lambda$ such that $x_\lambda - x_\mu \in \epsilon B$ whenever $\lambda > \mu > \lambda_\epsilon$. Herewith, a linear topological space $X$ is called a *bornologically complete space* if every Mackey-Cauchy net in $X$ is convergent. It is easy to see that every complete linear topological space is bornologically complete. In particular, when only Mackey-Cauchy sequences in $X$ are convergent, we will speak about a *sb-complete* space $X$.

**1.5.** Let $X$ be a linear topological space over $\mathbb{K}$ and $\mathcal{B}$ the *von Neumann bornology* on $X$, that is, the set of all bounded subsets in $X$. The bornology $\mathcal{B}$ on $X$ is called *k-convex* (see [11] or [15]) if $\Gamma_k(B)$ is bounded in $X$ for each bounded subset $B$ of $X$ (in this case $\mathcal{B}$ has a basis consisting of bounded absolutely $k$-convex sets) and $\mathcal{B}$ is called *pseudoconvex* if for each $B \in \mathcal{B}$ there is a number $k = k(B) \in [0,1]$ such that $\Gamma_k(B)$ is bounded in $X$. It is known (see [15], Theorems 1 and 2, [17] and [11], p. 102-103) that the von Neumann bornology $\mathcal{B}$ on a locally pseudoconvex space $X$ is pseudoconvex if $\mathcal{B}$ has a countable basis, and every metrizable linear topological space $X$ is locally $k$-convex for some $k \in (0,1]$ if $\mathcal{B}$ is pseudoconvex.

**1.6.** Let $A$ be a topological algebra over $\mathbb{K}$ and

$$S(a,\mu) = \left\{ \left(\frac{a}{\mu}\right)^n : n \in \mathbb{N} \right\}$$

for each $a \in A$ and $\mu \in \mathbb{K} \setminus \{0\}$.

An element $a \in A$ is *bounded* if there is a number $\mu \in \mathbb{K} \setminus \{0\}$ such that $S(a,\mu)$ is bounded in $A$. We will say that an element $a \in A$ is *pseudoconvexly bounded* if there are numbers $\mu \in \mathbb{K} \setminus \{0\}$ and $k \in (0,1]$ such that $\Gamma_k(S(a,\mu))$ is bounded in $A$.

It is easy to see that every pseudoconvexly bounded element is bounded in $A$ and every bounded element is pseudoconvexly bounded in $A$ if the von Neumann bornology of $A$ is pseudoconvex or $A$ is a locally $k$-convex algebra for some $k \in (0,1]$. If all elements in $A$ are bounded (pseudoconvexly bounded), then we will say that $A$ is a *topological algebra with bounded* (respectively *pseudoconvexly bounded*) *elements.*

**1.7.** Let $A$ be a topological algebra over $\mathbb{C}$, $\mathrm{sp}_A(a)$ the *spectrum* of $a \in A$, $r_A(a)$ the *spectral radius* of $a \in A$ and $\beta_A(a)$ the *radius of boundedness* of $a \in A$. If $A$ is an algebra with the unit element $e_A$, then

$$\mathrm{sp}_A(a) = \{\lambda \in \mathbb{C} : a - \lambda e_A \notin \mathrm{Inv}A\}$$

otherwise

$$\mathrm{sp}_A(a) = \{\lambda \in \mathbb{C} \setminus \{0\} : a/\lambda \notin \mathrm{Qinv}A\} \cup \{0\},$$
$$\mathrm{r}_A(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}_A(a)\}$$

and

$$\beta_A(a) = \inf\{\lambda \in \mathbb{C} \setminus \{0\} : S(a, \lambda) \text{ is bounded in } A\}$$
$$= \inf\left\{\lambda \in \mathbb{C} \setminus \{0\} : \left(\frac{a}{\lambda}\right)^n \text{ vanishes in } A\right\}.$$

We put $\mathrm{r}_A(a) = 0$ if $\mathrm{sp}_A(a)$ is empty and $\beta_A(a) = +\infty$ if there does not exist any $\mu$ such that the set $S(a, \mu)$ is bounded in $A$.

**1.8.** In the present paper it is shown that every commutative $sb$-complete Hausdorff algebra $A$ with bounded elements is representable in the form of an (algebraic) inductive limit of an inductive system of commutative locally bounded Fréchet algebras with continuous monomorphisms if the von Neumann bornology of $A$ is pseudoconvex. It is shown that $r_A(a) = \beta_A(a)$ if $A$ is a $sb$-complete Hausdorff algebra over $\mathbb{C}$ with bounded elements and pseudoconvex von Neumann bornology and $r_A(a) \leq \beta_A(a)$ if $A$ is a topological algebra in which Qinv$A$ is a bornivore or $A$ is a $sa$-complete algebra for which from $\beta_A(a) < 1$ it follows that $(\sum_{k=0}^{n} a^k)$ is a Cauchy sequence.

**2. Preliminary results.** Let $A$ be a topological algebra and $\mathcal{B}_{pc}$ the set of all closed, idempotent, bounded and absolutely pseudoconvex[2] subsets of $A$. For each $k \in (0, 1]$ let $\mathcal{B}_k$ be the set of all closed, idempotent, bounded and absolutely $k$-convex subsets of $A$.

PROPOSITION 2.1. *Let $A$ be a topological algebra and $B$ an absolutely pseudoconvex subset of $A$. Then the linear span $A_B$ generated by $B$ is expressible in the form*

$$A_B = \cup\{\lambda B : \lambda \in \mathbb{K}\}.$$

*Proof.* Let $b \in A_B$. Then there is $n \in \mathbb{N}$, $b_1, \ldots, b_n \in B$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that

$$b = \sum_{v=1}^{n} \lambda_v b_v$$

and a number $k \in (0, 1]$ such that $B = \Gamma_k(B)$. Now

$$b = (n\mu)^{\frac{1}{k}} \sum_{v=1}^{n} \frac{\lambda_v}{(n\mu)^{\frac{1}{k}}} b_v,$$

where $\mu = \max\{|\lambda_1|^k, \ldots, |\lambda_n|^k\}$ and

$$\sum_{v=1}^{n} \left|\frac{\lambda_v}{(n\mu)^{\frac{1}{k}}}\right|^k = \frac{1}{n} \sum_{v=1}^{n} \frac{|\lambda_v|^k}{\mu} \leq 1.$$

Hence $A_B \subset (n\mu)^{\frac{1}{k}} B \subset \cup\{\lambda B : \lambda \in \mathbb{K}\}$. The converse inclusion is evident. ■

It is well known (see, for example, [3] and [4]) that every closed, idempotent, bounded and absolutely convex subset $B$ in a locally convex Hausdorff algebra $A$ defines a submultiplicative norm on $A_B$. The following result describes these subsets $B$ of a topological algebra $A$ which define nonhomogeneous submultiplicative norms on $A_B$.

[2]That is, absolutely $k$-convex subsets for some $k \in (0, 1]$.

PROPOSITION 2.2. *Let $k \in (0,1]$, $A$ be a Hausdorff algebra and $B \in \mathcal{B}_k$. Then*

a) *$A_B$ is a $k$-normed algebra with respect to the $k$-homogeneous submultiplicative norm $p_B$ defined for each $a \in A_B$ by*

$$p_B(a) = \inf\{|\mu|^k : a \in \mu B\}; \tag{1}$$

b) *if $A$ is sb-complete, then $A_B$ is a $k$-Banach algebra;*

c) *the topology on $A_B$ defined by $p_B$ is not weaker than the subset topology on $A_B$.*

*Proof.* a) Let $B \in \mathcal{B}_k$. In the same way as in [20], p. 4, it is easy to show (by Proposition 2.1) that $p_B$ is a $k$-homogeneous submultiplicative seminorm on $A_B$. As $A_B$ is a Hausdorff algebra, for each non-zero $a \in A_B$ there is a neighbourhood of zero $V$ in $A$ such that $a \notin V \cap A_B$. By assumption $B$ is bounded. Therefore there is a number $\lambda > 0$ such that $B \subset \lambda V$. Hence, from $\lambda a \notin B$ (if $\lambda a \in B$, then $a \in V$) it follows that $p_B(a) > 1/|\lambda|^k > 0$. Consequently, $p_B$ is a $k$-homogeneous submultiplicative norm on $A_B$.

b) Let $B \in \mathcal{B}_k$, $\epsilon > 0$ and $\delta \in (0, \min\{\epsilon, \epsilon^{\frac{1}{k}}\})$. If $(a_n)$ is a Cauchy sequence in $A_B$ in the topology defined by $p_B$, then there exists a number $n_\epsilon \in \mathbb{N}$ such that $p_B(a_n - a_m) < \delta^k$ or $a_n - a_m \in \delta B \subset \epsilon B$ whenever $n > m > n_\epsilon$. Hence $(a_n)$ is a Mackey-Cauchy sequence in $A$. Since $A$ is $sb$-complete, $(a_n)$ converges in $A$.

Let now $a_0 \in A$ be the limit of $(a_n)$, $m_0 > n_\epsilon$ be fixed, $a_{m_0} = \alpha_0 b_0$ (here $\alpha_0 \in \mathbb{K}$ and $b_0 \in B$), $\rho = \max\{\epsilon, \alpha_0\}$ and $\phi = \rho 2^{\frac{1}{k}}$. Then

$$a_n = (a_n - a_{m_0}) + a_{m_0} \in \epsilon B + \alpha_0 B \subset \rho(B + B) \subset \phi B$$

for each $n > m_0$. Since $B$ is closed, $\phi B$ is also closed in $A$. Hence, $a_0 \in \phi B \subset A_B$.

Fix now $m$ and let $n \to \infty$. Then $(a_n - a_m)$ converges to $(a_0 - a_m)$ in the topology of $A$. Since $\delta B$ is a closed in $A$, we have $a_m - a_0 \in \delta B$ or $p_B(a_m - a_0) \le \delta^k < \epsilon$ for each $m > n_\epsilon$. Thus, $(a_n)$ converges in $A_B$. Consequently, $A_B$ is a $k$-Banach algebra.

c) Let $B \in \mathcal{B}_k$ and $O$ be a neighbourhood of zero in $A_B$ in the subspace topology. Then there exists a neighbourhood $U$ of zero in $A$ such that $O = U \cap A_B$. Since $B$ is bounded, there is a positive number $\mu$ such that $B \subset \mu U$. Now

$$O' = \{a \in A_B : p_B(a) < \mu^{-k}\}$$

is a neighborhood of zero in $A_B$ in the topology defined by the norm $p_B$, and $O' \subset \mu^{-1} B \subset U \cap A_B = O$. Therefore the topology defined on $A_B$ by the norm $p_B$ is not weaker than the subspace topology. ■

Let $A_b$ denote the set of all bounded elements of $A$ and $A_{pb}$ the set of all pseudoconvexly bounded elements in $A$. The following result describes the set $A_{pb}$.

PROPOSITION 2.3. *Let $A$ be a Hausdorff algebra. Then*

$$A_{pb} = \cup\{A_B : B \in \mathcal{B}_{pc}\}.$$

*Proof.* Let $a \in A_{pb}$. Then there exist $\mu \in \mathbb{K} \setminus \{0\}$ and $k \in (0,1]$ such that

$$S(a, \mu) \subset B = \mathrm{cl}_A(\Gamma_k(S(a, \mu))).$$

Since $B \in \mathcal{B}_{pc}$, we have $a \in \cup\{A_B : B \in \mathcal{B}_{pc}\}$.

Let now $a \in \cup\{A_B : B \in \mathcal{B}_{pc}\}$. Then $a \in A_B$ for some $B \in \mathcal{B}_{pc}$. Therefore there is a number $k \in (0,1]$ such that $B \in \mathcal{B}_k$. Now $p_B$ is a $k$-homogeneous submultiplicative norm on $A_B$ by Proposition 2.2 a). Hence

$$p_B\left(\left(\frac{a}{\lambda}\right)^n\right) \leq \left(p_B\left(\frac{a}{\lambda}\right)\right)^n = \left(\frac{p_B(a)}{|\lambda|^k}\right)^n < 1$$

for each $n \in \mathbb{N}$ whenever $|\lambda| > p_B(a)^{\frac{1}{k}}$. Since

$$S(a,\lambda) \subset \{a \in A : p_B(a) < 1\} \subset B,$$

we have $\Gamma_k(S(a,\lambda)) \subset \Gamma_k(B) = B$. This means that $a \in A_{pb}$. ∎

For each $a \in A$ let $I(a) = \{B \in \mathcal{B}_{pc} : a \in A_B\}$. Then we have

Corollary 2.4. *Let $A$ be a Hausdorff algebra. If the von Neumann bornology of $A$ is pseudoconvex,*

$$\beta_A(a) = \inf\{\beta_{A_B}(a) : B \in I(a)\} \tag{2}$$

*for each $a \in A_{pb}$.*

*Proof.* Let $a \in A_{pb}$ and $B \in I(a)$. Since $A_B \subset A$, we have $\beta_A(a) \leq \beta_{A_B}(a)$. Hence,

$$\beta_A(a) \leq \inf\{\beta_{A_B}(a) : B \in I(a)\}.$$

Let now $\lambda$ and $\mu$ be positive numbers such that $\beta_A(a) < \lambda$ and $\beta_A(a) < \mu < \lambda$. Because the von Neumann bornology on $A$ is pseudoconvex, there is a number $k \in (0,1]$ such that $B = \mathrm{cl}_A(\Gamma_k(S(a,\mu))) \in \mathcal{B}_k$. Hence,

$$p_B\left(\left(\frac{a}{\mu}\right)^n\right) \leq 1$$

for each $n \in \mathbb{N}$. Thus, $S(a,\mu)$ is bounded in $A_B$. Consequently, from $\beta_{A_B}(a) \leq \mu < \lambda$ it follows that (2) holds. ∎

Proposition 2.5. *Let $k \in (0,1]$ and $A$ be a unital locally uniformly absorbingly $k$-convex algebra with jointly continuous multiplication. Then $A_{pb} = A_b = A$.*

*Proof.* Let $\{p_\lambda : \lambda \in \Lambda\}$ be a family of $k$-homogeneous seminorms which defines the topology of $A$, $a \in A$, $N(a) > 0$ be the number such that $p_\lambda(ab) \leq N(a)p_\lambda(b)$ for each $b \in A$ and $\lambda \in \Lambda$, $\mu > N(a)^{\frac{1}{k}}$ and

$$B = \bigcap_{\lambda \in \Lambda} \{b \in A : p_\lambda(b) \leq 1\}.$$

Since $B \in \mathcal{B}_k$ and[3]

$$p_\lambda\left(\left(\frac{a}{\mu}\right)^n\right) \leq \left(\frac{N(a)}{\mu^k}\right)^n < 1$$

for each $n \in \mathbb{N}$ and $\lambda \in \Lambda$, we have $\Gamma_k(S(a,\mu)) \subset \Gamma_k(B) = B$. Hence, $a \in A_{pb}$. ∎

Proposition 2.6. *Let $A$ be a sb-complete Hausdorff algebra, $B$ a bounded subset of $A$ and $C \in \mathcal{B}_{pc}$. Then $BC$ is bounded in $A$. In particular, when $A$ is a commutative sb-complete Haustorff algebra, $B$ a bounded and idempotent subset of $A$ and the von Neumann bornology of $A$ is pseudoconvex, then there exists a set $D \in \mathcal{B}_{pc}$ such that $B \cup C \subset D$.*

[3]We can assume (see [23]) that $p_\lambda(e_A) = 1$ for each $\lambda \in \Lambda$.

*Proof.* Let $B$ be a bounded subset of $A$, $b \in B$, $C \in \mathcal{B}_{pc}$, $L_b$ a map from $A_C$ into $A$ defined by $L_b(a) = ba$ for each $a \in A_C$ and $\mathcal{L} = \{L_b : b \in B\}$. Then there is a number $k \in (0,1]$ such that $C \in \mathcal{B}_k$. Let $\mathcal{L}(A_C, A)$ be the space of all linear continuous maps from $A_C$ into $A$ endowed with the topology of simple convergence[4] and $O$ a neighbourhood of zero in $\mathcal{L}(A_C, A)$. Then there exist $n \in \mathbb{N}$, $S = \{a_1, \dots, a_n\} \subset A_C$ and neighbourhoods $U$ and $V$ of zero in $A$ such that $T(S,U) \subset O$ and $Va_v \subset U$ for each $v$. Since $B$ is bounded, there exists a positive number $\mu$ such that $B \subset \mu V$. Hence, from

$$\frac{1}{\mu}L_b(a_v) = \frac{1}{\mu}ba_v \in Va_v \subset U$$

for each $b \in B$ and each $v$ it follows that $\mathcal{L} \subset T(S, \mu U) = \mu T(S,U) \subset \mu O$. This means that $\mathcal{L}$ is a bounded subset of $\mathcal{L}(A_C, A)$. As $A$ is a *sb*-complete Hausdorff algebra, $A_C$ is a $k$-Banach algebra by Proposition 2.2 b). Hence, $A_C$ is a Baire space (see [13], p. 87, Theorem 1). Therefore (see [19], Theorem 4.2, p. 83), $\mathcal{L}$ is an equicontinuous subset of $\mathcal{L}(A_C, A)$. This means that for each neighbourhood $U$ of zero in $A$ there is a neighbourhood $V$ of zero in $A_C$ such that $BV \subset U$. Since $C$ is bounded in $A_C$, there exists a positive number $\nu$ such that $C \subset \nu V$. Thus, $BC \subset \nu BV \subset \nu U$. Consequently, $BC$ is a bounded subset of $A$.

Let now $A$ be a commutative *sb*-complete Hausdorff algebra, $B$ a bounded and idempotent subset of $A$, $C \in \mathcal{B}_{pc}$ and let the von Neumann bornology on $A$ be pseudoconvex. Then $E = B \cup C \cup BC$ is a bounded and idempotent subset of $A$ and there is a number $k \in (0,1]$ such that $\Gamma_k(E)$ is a bounded and idempotent set in $A$. Hence $D = \mathrm{cl}_A(\Gamma_k(E)) \in \mathcal{B}_k$ and $B \cup C \subset D$. ∎

Corollary 2.7. *Let $A$ be a commutative sb-complete Hausdorff algebra with bounded elements. If the von Neumann bornology on $A$ is pseudoconvex, then $A_{pb}$ is a subalgebra of $A$.*

*Proof.* Let $a, b \in A_{pb}$. Then there are $\mu_1, \mu_2 \in \mathbb{K} \setminus \{0\}$ and $k_1, k_2 \in (0,1]$ such that $B_1 = \mathrm{cl}_A(\Gamma_{k_1}(S(a,\mu_1))) \in \mathcal{B}_{k_1}$ and $B_2 = \mathrm{cl}_A(\Gamma_{k_2}(S(a,\mu_2))) \in \mathcal{B}_{k_2}$. As $B_1B_2$ is bounded in $A$ by Proposition 2.6 and the von Neumann bornology on $A$ is pseudoconvex, then there is a number $k \in (0,1]$ such that $D = \mathrm{cl}_A(\Gamma_k(B_1 \cup B_2 \cup B_1B_2)) \in \mathcal{B}_k$. Now from

$$\left(\frac{ab}{\mu_i\mu_2}\right)^n = \left(\frac{a}{\mu_i}\right)^n \left(\frac{b}{\mu_2}\right)^n \in B_1B_2 \subset D$$

for each $n \in \mathbb{N}$ it follows that $\Gamma_k(S(ab, \mu_1\mu_2)) \subset \Gamma_k(D) = D$. Hence, $ab \in A_{pb}$.

Since $a = \mu_1 b_1$ and $b = \mu_2 b_2$ for some $b_1 \in B_1$ and $b_2 \in B_1$, we have

$$a + b \in \mu_1 B_1 + \mu_2 B_2 \subset \mu(D + D) \subset \mu 2^{\frac{1}{k}} D \subset A_D,$$

where $\mu = \max\{\mu_1, \mu_2\}$. Hence, $a + b \in A_{pb}$. It is easy to see that $\mu a \in A_{pb}$ for each $\mu \in \mathbb{K}$ and $a \in A_{pb}$. Consequently, $A_{pb}$ is a subalgebra of $A$. ∎

[4] A base of neighbourhoods of zero in the topology of simple convergence consists of sets $T(S,U) = \{L \in \mathcal{L}(A_C, A) : L(S) \subset U\}$, where $S$ is a finite subset of $A_C$ and $U$ is a neighbourhood of zero of $A$.

PROPOSITION 2.8. *Let $A$ be a commutative sb-complete Hausdorff algebra over $\mathbb{C}$ with bounded elements. If the von Neumann bornology on $A$ is pseudoconvex, then*[5] *for each $a \in A$*

(a) $\mathrm{sp}_A(a) = \bigcap\{\mathrm{sp}_{A_B}(a) : B \in I(a)\}$;
(b) $\mathrm{r}_A(a) = \inf\{r_{A_B}(a) : B \in I(a)\}$;
(c) $\mathrm{sp}_A(a)$ *is a closed subset of* $\mathbb{C}$.

*Proof.* Let $a \in A$. Since $A_B \subset A$ for each $B \in \mathcal{B}_{pc}$, we have

$$\mathrm{sp}_A(a) \subset \bigcap\{\mathrm{sp}_{A_B}(a) : B \in I(a)\}. \tag{3}$$

To prove the converse inclusion, let first $A$ be a unital algebra and $\lambda \notin \mathrm{sp}_A(a)$. Then $(a - \lambda e_A)^{-1}$ exists in $A$. Since $A = \cup\{A_B : B \in \mathcal{B}_{pc}\}$ by Proposition 2.3, there is a set $B_0 \in \mathcal{B}_{pc}$ such that $e_A \in A_{B_0}$. Thus, by Proposition 2.6, for each $B \in I(a)$ there is a set $D \in \mathcal{B}_{pc}$ such that $B \cup B_0 \subset D$. Therefore we can assume that $e_A, a \in A_{B_1}$ for some $B_1 \in \mathcal{B}_{pc}$. Now there is a set $B_2 \in \mathcal{B}_{pc}$ such that $(a - \lambda e_A)^{-1} \in A_{B_2}$ and a set $B_3 \in \mathcal{B}_{pc}$ such that $B_1 \cup B_2 \subset B_3$ by Proposition 2.6. Hence, $e_A$, $a$, $a - \lambda e_A$ and $(a - \lambda e_A)^{-1}$ belong to $A_{B_3}$. This means that $\lambda \notin \mathrm{sp}_{A_{B_3}}(a)$, which proves the statement (a). The proof for non-unital algebra $A$ is similar.

b) It is clear by (3) that

$$\mathrm{r}_A(a) \leq \inf\{\mathrm{r}_{A_B}(a) : B \in I(a)\}$$

for each $a \in A$. Let now $A$ be a unital algebra and $\lambda > \mathrm{r}_A(a)$. Then $\lambda \notin \mathrm{sp}_A(a)$. Therefore $(a - \lambda e_A)^{-1} \in A_B$ for some $B \in \mathcal{B}_{pc}$. As above we can show that there is a set $B \in I(a)$ such that $a - \lambda e_A \in \mathrm{Inv}A_B$. This means that $\lambda > \mathrm{r}_{A_B}(a)$. Consequently, the statement (b) holds for unital algebra $A$. The proof for non-unital algebra $A$ is similar.

c) As every $A_B$ with $B \in \mathcal{B}_{pc}$ is a commutative $k$-Banach algebra for some $k \in (0, 1]$ by Proposition 2.2 b), all $A_B$ are $Q$-algebras (see [5], Proposition 3.6.23; for unital case see [21], p. 10, and [ 22], Lemma 3.6). Therefore $\mathrm{sp}_{A_B}(a)$ is a closed subset in $\mathbb{C}$ for each $a \in A_B$ with $B \in I(a)$ (see [16], p. 60). Hence, $\mathrm{sp}_A(a)$ is a closed subset of $\mathbb{C}$. ∎

**3. Main result.** It is known (see [4], Proposition 1.2) that every pseudo-Banach algebra with respect to some bound structure is representable in the form of the inductive limit of an (algebraic) inductive system of unital Banach algebras with continuous unital monomorphisms. Next we prove a similar result for topological algebras with pseudoconvexly bounded elements.

THEOREM 3.1. *Let $A$ be a commutative sb-complete Hausdorff algebra with bounded elements. If the von Neumann bornology of $A$ is pseudoconvex, $A$ is representable in the form of an (algebraic) inductive limit of an inductive system of locally bounded Fréchet algebras with continuous monomorphisms.*

*Proof.* By assumptions every $B \in \mathcal{B}_{pc}$ defines a number $k \in (0, 1]$ such that $A_B$ is a commutative $k$-Banach algebra by Proposition 2.2 b). We define the ordering in $\mathcal{B}_{pc}$ in

[5] In case of unital algebra $A$ let $I(a) = \{B \in \mathcal{B}_{pc} : a, e_A \in A_B\}$, otherwise let $I(a) = \{B \in \mathcal{B}_{pc} : a \in A_B\}$.

the following way: for each $B, C \in \mathcal{B}_{pc}$ (then there are numbers $k, k' \in (0,1]$ such that $B \in \mathcal{B}_k$ and $C \in \mathcal{B}_{k'}$) we shall say that $B < C$ if and only if $B \subset \mu C$ for some positive number $\mu$. It is easy to see that the set $\mathcal{B}_{pc}$ is upward directed (by Proposition 2.6) and preordered. Now $A_B \subset A_C$ by Proposition 2.1 whenever $B < C$. For any $B, C \in \mathcal{B}_{pc}$ with $B < C$ let $i_{CB}$ be the inclusion map from $A_B$ into $A_C$. Then

$$p_C(i_{CB}(a)) = p_C(a) \leq \mu^{k'} p_B(a)^{\frac{k'}{k}}$$

for each $a \in A_B$. Therefore $i_{CB}$ is continuous by Theorem III.2.10 from [5] whenever $B < C$. Hence, $(A_B; i_{CB}; \mathcal{B}_{pc})$ is an inductive system of locally bounded Fréchet algebras $A_B$ with continuous monomorphisms.

Let now $O$ be a neighbourhood of zero in $A$, $\gamma$ a positive number such that $B \subset \gamma O$ and $O' = \{a \in A_B : p_B(a) < \gamma^{-k}\}$. Then $O'$ is a neighbourhood of zero in $A_B$ in the topology defined by $p_B$ and $O' \subset \gamma^{-1} B \subset O$. Therefore the inclusion map $i_B$ from $A_B$ into $A$ is continuous. Since $A = \cup\{A_B : B \in \mathcal{B}_{pc}\}$ by Proposition 2.3, $A$ is the inductive limit of the system $(A_B; i_{CB}; \mathcal{B}_{pc})$. ■

Corollary 3.2. *If $k \in [0,1]$, then every commutative sb-complete locally $k$-convex Hausdorff algebra with bounded elements is representable in the form of an (algebraic) inductive limit of an inductive system of $k$-Banach algebras with continuous monomorphisms.*

**4. Relations between the spectral radius and the radius of boundedness.** It is well known (see, for example, [3], [6], [7], [8], [9] and [10]) that there are locally pseudoconvex algebras $A$ for which $\mathrm{r}_A(a) = \beta_A(a)$, $\mathrm{r}_A(a) < \beta_A(a)$ or $\mathrm{r}_A(a) > \beta_A(a)$ for all $a \in A$ as well as there are topological algebras $A$ for which $\mathrm{r}_A(a) = 0$ and $\beta_A(a) = +\infty$ or $\mathrm{r}_A(a) = +\infty$ and $\beta_A(a)$ is finite for some $a \in A$. To describe these classes of topological (not necessarily locally pseudoconvex) algebras $A$ for which $\mathrm{r}_A(a) \leq \beta_A(a)$ and $\mathrm{r}_A(a) = \beta_A(a)$ for each $a \in A$ we first prove

Proposition 4.1. *Let $A$ be a commutative complete locally m-pseudoconvex Hausdorff algebra over $\mathbb{C}$. Then $\mathrm{r}_A(a) = \beta_A(a)$ for each $a \in A_b$.*

*Proof.* Let $\{p_\lambda : \lambda \in \Lambda\}$ be a saturated family of $k_\lambda$-homogeneous submultiplicative seminorms (with $k_\lambda \in (0,1]$ for each $\lambda \in \Lambda$) which defines the topology of $A$. First we assume that $A$ is a unital algebra. Let $a \in A_b$ and $\nu$ be a positive number such that $\beta_A(a) < \nu$. Then $S(a,\nu)$ is bounded in $A$. Therefore for each $\lambda \in \Lambda$ there is a positive number $\mu_\lambda$ such that

$$p_\lambda\left(\frac{1}{\mu_\lambda}\left(\frac{a}{\nu}\right)^n\right) < 1$$

for each $n \in \mathbb{N}$. Hence $\sqrt[k_\lambda n]{p_\lambda(a^n)} < \mu_\lambda^{\frac{1}{n}} \nu$ for each $\lambda \in \Lambda$ and $n \in \mathbb{N}$. Thus

$$\mathrm{r}_A(a) = \sup_{\lambda \in \Lambda} \lim_{n \to \infty} \sqrt[k_\lambda n]{p_\lambda(a^n)} \leq \nu$$

by Proposition 12 from [2]. Consequently, $\mathrm{r}_A(a) \leq \beta_A(a)$.

If $\mathrm{r}_A(a) = \infty$ for some $a \in A$, then $\beta_A(a) \leq \mathrm{r}_A(a)$. Let now $a \in A$ and $\nu$ be a positive number such that $\mathrm{r}_A(a) < \nu$. Then $\mathrm{r}_A((a/\nu)^n) < 1$ for each $n \in \mathbb{N}$, because

$r_A(a^n) = r_A(a)^n$ for[6] each $a \in A$ and $n \in \mathbb{N}$. Therefore

$$T_\lambda = \lim_{n\to\infty} \sqrt[k_\lambda n]{p_\lambda\left(\left(\frac{a}{\nu}\right)^n\right)} < 1$$

for each $\lambda \in \Lambda$. Now for each $\lambda \in \Lambda$ there is a number $\eta_\lambda$ such that $T_\lambda < \eta_\lambda < 1$. Then for each $\lambda \in \Lambda$ there is an $n_\lambda \in \mathbb{N}$ such that

$$p_\lambda\left(\left(\frac{a}{\nu}\right)^n\right) < \eta_\lambda^{nk_\lambda}$$

whenever $n > n_\lambda$. Hence, $(\sum (a/\nu)^k)$ converges in $A$. Thus, the set $S(a,\nu)$ is bounded in $A$. Now from $\beta_A(a) < \nu$ it follows that $\beta_A(a) \le r_A(a)$.

Let now $A$ be an algebra without unit element. Then the unitization $A \times \mathbb{C}$ of $A$ (in the topology defined by the family $\{q_\lambda : \lambda \in \Lambda\}$ of $k_\lambda$-homogeneous seminorms, where

$$q_\lambda((a,\mu)) = p_\lambda(a) + |\mu|^{k_\lambda}$$

for each $\lambda \in \Lambda$ and $(a,\mu) \in A \times \mathbb{C}$) is a unital commutative complete locally $m$-pseudoconvex Hausdorff algebra. Therefore $r_A(a) = r_{A\times\mathbb{C}}((a,0)) = \beta_{A\times\mathbb{C}}((a,0)) = \beta_A(a)$ for each $a \in A$. ■

THEOREM 4.2. *If either*

(a) *$A$ is a sa-complete topological algebra which has the property*

$$\textit{if } \beta_A(a) < 1, \textit{ then } \left(\sum_{k=1}^{n} a^n\right) \textit{ is a Cauchy sequence in } A; \tag{4}$$

*or*

(b) *$A$ is a topological algebra in which QinvA is a bornivore,*

*then $r_A(a) \le \beta_A(a)$ for each $a \in A$. Moreover, if*

(c) *$A$ is a sb-complete Hausdorff algebra over $\mathbb{C}$ with bounded elements and the von Neumann bornology of $A$ is pseudoconvex,*

*then* $r_A(a) = \beta_A(a)$ *for each $a \in A$.*

*Proof.* a) Let $a \in A$. If $\beta_A(a) = \infty$, then $r_A(a) \le \beta_A(a)$. Let now $\rho > 0$ be a number such that $\beta_A(a) < \rho$. Then $\beta_A(a/\rho) < 1$. Therefore the sequence $((a/\rho)^n)$ vanishes in $A$ and $(S_n)$, where

$$S_n = -\sum_{k=1}^{n} \left(\frac{a}{\rho}\right)^k$$

for each $n \in \mathbb{N}$, is a Cauchy sequence in $A$ by (4). Since

$$S_n \circ \frac{a}{\rho} = \frac{a}{\rho} \circ S_n = \left(\frac{a}{\rho}\right)^{n+1}, \tag{5}$$

[6] The proof of this statement is similar to the proof in case of complete locally $m$-convex algebras using here Arens-Michael teorem for complete locally $m$-pseudoconvex algebras (see [5], Theorem 4.5.3, or [1], Theorem 5) and the fact that every locally bounded Fréchet algebra has the functional spectrum (see [2], Theorem 12).

$(S_n)$ is an advertibly convergent Cauchy sequence in $A$. Hence, $(S_n)$ converges in $A$. This means that $a/\rho \in \mathrm{Qinv}A$ or $\rho \notin \mathrm{sp}_A(a)$ by (5). Consequently, $r_A(a) < \rho$. Thus, $r_A(a) \le \beta_A(a)$.

Next let $A$ be a topological algebra in which $\mathrm{Qinv}A$ is a bornivore and let $a \in A$ be such that $\beta_A(a) < 1$. Then there is a number $\mu$ such that $\beta_A(a) < \mu < 1$. Since $\beta(\frac{a}{\mu}) < 1$, the set $S(a, \mu)$ is bounded in $A$. Let $B'$ be a balanced bounded subset of $A$ such that $S(a, \mu) \subset B'$ and for each $\varepsilon > 0$ let $n_\varepsilon \in \mathbb{N}$ be such that $\mu^n < \varepsilon$ whenever $n > n_\varepsilon$. Then $a^n \in \varepsilon B'$ whenever $n > n_\varepsilon$. Thus, $(a^n)$ Mackey converges to $\theta_A$. Let $S_n = \sum_{k=1}^n a^k$ for each $n \in \mathbb{N}$. Since $S_n \circ a = a \circ S_n = a^{n+1}$, it follows that $(S_n \circ a)$ and $(a \circ S_n)$ Mackey converge to $\theta_A$. Therefore for any $\varepsilon > 0$ there is a $m_\varepsilon \in \mathbb{N}$ such that $S_n \circ a \in \varepsilon B'$ and $a \circ S_n \in \varepsilon B'$ whenever $n > m_\varepsilon$. As $\mathrm{Qinv}A$ is a bornivore in $A$, there is a positive number $\rho$ such that $B' \subset \rho \mathrm{Qinv}A$. If now $n_0 > m_\varepsilon$ and $\varepsilon < \frac{1}{\rho}$, then $S_{n_0} \circ a \in \mathrm{Qinv}A$ and $a \circ S_{n_0} \in \mathrm{Qinv}A$. Hence $a \in \mathrm{Qinv}A$ or $1 \notin \mathrm{sp}_A(a)$. Consequently, $r_A(a) < 1$. This means that $r_A(a) \le \beta(a)$ for each $a \in A$.

Let now $A$ be a commutative topological algebra which has the property (c) and let $a \in A$. Then $A = \cup\{A_B : B \in \mathcal{B}_{pc}\}$ by Proposition 2.3, where every $A_B$ is a commutative $k$-Banach algebra for some $k \in (0, 1]$. Therefore

$$\mathrm{r}_A(a) = \inf\{\mathrm{r}_{A_B} : B \in I(a)\} = \inf\{\beta_{A_B}(a) : B \in I(a)\} = \beta_A(a)$$

by Corollary 2.4 and Propositions 2.8 and 4.1.

If $A$ is not commutative, then let $C$ be a maximal commutative subalgebra of $A$. Then $C$ has the property (c) and $\mathrm{r}_A(a) = \mathrm{r}_C(a) = \beta_C(a) = \beta_A(a)$, because $\mathrm{sp}_A(a) = \mathrm{sp}_C(a)$ for each $a \in C$. ∎

Corollary 4.3. *If either*

(a) *$A$ is a sa-complete locally pseudoconvex algebra or*
(b) *$A$ is a Mackey $Q$-algebra,*

*then $r_A(a) \le \beta_A(a)$ for each $a \in A$.*

*Proof.* Let $\{p_\lambda : \lambda \in \Lambda\}$ be a saturated family of $k_\lambda$-homogeneous seminorms (with $k_\lambda \in (0, 1]$ for each $\lambda \in \Lambda$) which defines the topology of $A$. If $a \in A$ and $\beta_A(a) < 1$, then there is a number $\rho$ such that $\beta_A(a) < \rho < 1$. Since

$$\beta_A(a) = \sup_{\lambda \in \Lambda} \limsup_{n \to \infty} \sqrt[k_\lambda n]{p_\lambda(a^n)}$$

for locally pseudoconvex algebras (see [7]), $p_\lambda(a^n) < (\rho^{k_\lambda})^n$ for each $n \in \mathbb{N}$ and $\lambda \in \Lambda$. Therefore $(\sum_{k=1}^n a^n)$ is a Cauchy sequence in $A$. Hence, the condition (4) has been satisfied. If now $A$ is a Mackey $Q$-algebra, then $\mathrm{Qinv}A$ is a bornivore in $A$. Consequently, in both cases $r_A(a) \le \beta_A(a)$ for each $a \in A$ by Theorem 4.2. ∎

Corollary 4.4. *Let $k \in (0, 1]$ and $A$ be a sb-complete locally $k$-convex Hausdorff algebra over $\mathbb{C}$ with bounded elements. Then $\mathrm{r}_A(a) = \beta_A(a)$ for each $a \in A$.*

Remark 4.5. Part (a) of Corollary 4.3 for unital *sa*-complete locally pseudoconvex algebras and part (b) of Corollary 4.3 for unital locally $k$-convex Mackey $Q$-algebras for some $k \in (0, 1]$ have been proved in [8], Propositions 2.1 and 2.4; part (b) of Corollary

4.3 has been proved in [7], Proposition 10; Corollary 4.4 for $k = 1$ has been proved in [9], Proposition III.4 and similar results to parts (a) and (c) of Theorem 4.2 have been proved in [18], Proposition 8. It is easy to show by Corollary 6 from [2] that Proposition 4.1 holds also for commutative advertibly complete locally $m$-pseudoconvex Hausdorff algebras over $\mathbb{C}$.

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