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SPECTRAL WELL-BEHAVED $*$-REPRESENTATIONS

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Abstract. In this brief account we present the way of obtaining unbounded $*$-representations in terms of the so-called "unbounded" C^*-seminorms. Among such $*$-representations we pick up a special class with "good behaviour" and characterize them through some properties of the Pták function.

1. Introduction. Our physical world consists mostly of unbounded operators, like e.g., position, momentum, energy and angular momentum operators, as well as Schröndiger operator. Of course, in some cases unbounded operators turn out to be bounded; for instance any symmetric operator defined on the entirety of a Hilbert space $\mathcal{H}$ is

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bounded (Hellinger-Toeplitz theorem [17; p. 195]). Nevertheless, it was only in 1962, when H. J. Borchers [4] and A. Uhlmann [21] considered unbounded $*$-representations in their reformulation of Wightman's axioms of quantum field theory. Borchers and Uhlmann noticed that the delay of such a consideration was mainly due to the lack of an adequate development of the theory of non-normed topological $*$-algebras. Later on, in 1970, a systematic study of $*$-algebras of unbounded operators started, first by R. T. Powers [16] in United States, and then by A. Uhlmann and G. Lassner in Leipzig. In particular, in 1972, G. Lassner invokes the reasons of Borchers and Uhlmann to point out in [13] the delay of the development of the unbounded $*$-representation theory, adding that this is a little bit surprising, since in the sixties the theory of Banach $*$-algebras and C^*-algebras (founded in 1943 by I. M. Gel'fand and M. A. Naimark [11]) was already well established. Gradually, a school of unbounded operator theory was created by G. Lassner in Leipzig, where K. Schmüdgen, W. Timmermann, K.-D. Kürsten and others stemmed from.

Thus in 1990, the first textbook on "Unbounded operator algebras and representation theory" appeared by K. Schmüdgen [19]. In the same year, a second book by D. A. Dubin and M. A. Hennings [9] is added in the same area of research, under the title "Quantum mechanics, algebras and distributions".

Going back to the beginning of 70's, far more to the East, one of us, A. Inoue in Fukuoka, started working on the generalization, to algebras of unbounded operators, of the celebrated modular Tomita-Takesaki theory (the so-called "non-commutative integration theory"). The results of this research are included in a monograph by him [12] published in 1998.

Recently a new book by J.-P. Antoine, A. Inoue and C. Trapani [2] appeared, dealing with "Partial $*$-algebras and their operator realizations", which enriches even more our knowledge on unbounded operators and their physical applications.

The unbounded operators may sometimes show pathological behaviour, so naturally one tries to find ways to pick up among all unbounded $*$-representations of a given $*$-algebra, the best possible ones. In this way the "well-behaved" $*$-representations appeared on the scene some years ago by S. J. Bhatt, A. Inoue and H. Ogi [8] and independently by K. Schmüdgen [20]. A major rôle for the construction of such $*$-representations is played by the so-called "unbounded C^*-seminorms". The last terminology, being rather surprising, is justified by the fact that this kind of C^*-seminorms are defined not on the whole, but on a $*$-subalgebra of a given $*$-algebra and they give rise to unbounded $*$-representations (i.e., to $*$-representations consisting of linear operators acting on a (dense) subspace of a Hilbert space). Because of the existence of spectral (unbounded) C^*-seminorms, A. Inoue and his collaborators S. J. Bhatt, H. Ogi and K.-D. Kürsten were led naturally to the investigation of spectral $*$-representations and spectral well-behaved $*$-representations in [7] and [6] respectively. In this note we shall announce some results relating the existence of spectral well-behaved $*$-representations on (locally convex) $*$-algebras to the Pták function (square root of the spectral radius on elements of the form x^*x), whose C^*-property on a Banach involutive algebra is characterized by hermiticity, according to Pták's celebrated theory for hermitian Banach algebras [18].

2. Preliminaries. The algebras we deal with are complex and associative.

Let $\mathcal{A}$ be a $*$-algebra, $\mathcal{H}$ a Hilbert space and $\mathfrak{D}$ a dense subspace of $\mathcal{H}$. $\mathcal{L}(\mathfrak{D})$ denotes all linear operators from $\mathfrak{D}$ into $\mathfrak{D}$. If $X \in \mathcal{L}(\mathfrak{D})$, $\mathfrak{D}(X^*)$ stands for the domain of the linear operator X^*, adjoint of X. The set

$$\mathcal{L}^+(\mathfrak{D}) := \{X \in \mathcal{L}(\mathfrak{D}) : \mathfrak{D}(X^*) \supseteq \mathfrak{D} \ \text{ and } \ X^*\mathfrak{D} \subseteq \mathfrak{D}\},$$

is a $*$-algebra under the usual algebraic operations and the involution $X \mapsto X^*|_{\mathfrak{D}}$. A $*$-homomorphism

$$\pi : \mathcal{A} \to \mathcal{L}^+(\mathfrak{D}),$$

is called (unbounded) *$*$-representation* of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$, with domain $\mathfrak{D}$. If $\mathcal{A}$ is unital with unit $\mathbf{1}$, we suppose that $\pi(\mathbf{1}) = I$, with I the identity operator in $\mathcal{L}^+(\mathfrak{D})$. In the sequel, $\mathfrak{D}(\pi)$ stands for the domain of the operators $\pi(a)$, $a \in \mathcal{A}$ and $\mathcal{H}_\pi$ for the corresponding Hilbert space. We say that π is *closed* if $\mathfrak{D}(\pi)[t_\pi]$ is complete, where t_π is the graph topology on $\mathfrak{D}(\pi)$ (cf. [12, 19]).

Let $\mathcal{A}$ be a $*$-algebra. An *unbounded C^*-seminorm* of $\mathcal{A}$ is a C^*-seminorm p defined on a $*$-subalgebra $\mathfrak{D}(p)$ of $\mathcal{A}$. Given an unbounded C^*-seminorm p, the set $\ker p \equiv \{x \in \mathfrak{D}(p) : p(x) = 0\}$ is a $*$-ideal of $\mathfrak{D}(p)$ and $\mathcal{I}_p \equiv \{x \in \mathfrak{D}(p) : ax \in \mathfrak{D}(p), \ \forall\, a \in \mathcal{A}\}$ is the largest left ideal of $\mathcal{A}$ contained in $\mathfrak{D}(p)$. The position of this left ideal in relation to the $*$-ideal $\ker p$ determines the existence of an unbounded $*$-representation derived by the unbounded C^*-seminorm p.

The Hausdorff completion $\mathcal{A}_p$ of $(\mathfrak{D}(p), p)$ is clearly a C^*-algebra, and in fact, it is the enveloping C^*-algebra of $(\mathfrak{D}(p), p)$; i.e., $\mathcal{A}_p$ is the completion of the normed $*$-algebra $\mathfrak{D}(p)/\ker(p)$, under the C^*-norm $\|\cdot\|_p$ induced by p. More precisely, if $x_p \equiv x + \ker p$, $x \in \mathcal{A}$, then $\|x_p\|_p := p(x)$. It is clear now that the C^*-seminormed algebra $\mathfrak{D}(p)$ can be endowed with a bounded $*$-representation, say π_p. In Section 3 we shall see that whenever $\mathcal{I}_p \not\subseteq \ker p$, the preceding bounded $*$-representation extends to an unbounded $*$-representation of $\mathcal{A}$. This is the way that an unbounded C^*-seminorm of $\mathcal{A}$ is related to an unbounded $*$-representation of $\mathcal{A}$. From these unbounded $*$-representations, one picks up well-behaved ones by using suitable unbounded C^*-seminorms (cf. [5, 6]).

3. Representable unbounded C^*-seminorms. We first give some simple examples of unbounded C^*-seminorms p with the property $\mathcal{I}_p \not\subseteq \ker p$ (see also [5; Section 3]).

3.1. Let $\mathcal{A}[\tau] := \mathcal{C}[0,1]$ be the $*$-algebra of all continuous functions on $[0,1]$, endowed with the topology τ determined by the $*$-seminorms

$$p_n(f) := \left[\int_0^1 |\, f(x) \,|^n \, dx\right]^{1/n}, \ f \in \mathcal{C}[0,1], \ n = 1, 2, \ldots$$

Let $\mathfrak{D}(p_\infty) := \{f \in \mathcal{A}[\tau] : \sup_n p_n(f) < \infty\}$ with $p_\infty(f) := \sup_n p_n(f)$, $f \in \mathfrak{D}(p_\infty)$. Then $\mathfrak{D}(p_\infty)$ is a $*$-subalgebra of $\mathcal{A}[\tau]$ and $p_\infty(f) = \|f\|_\infty$, for each $f \in \mathcal{A}[\tau]$. In particular, p_∞ is an unbounded C^*-(semi)norm of $\mathcal{A}[\tau]$ and $\mathfrak{D}(p_\infty) = (\mathcal{C}[0,1], \|\cdot\|_\infty) = \mathcal{I}_{p_\infty}$; so that $\mathcal{I}_{p_\infty} \neq \{0\} = \ker p_\infty$.

3.2. Let $\mathcal{A}[\tau] := \mathcal{C}(\mathbb{R})$ be the $*$-algebra of all continuous functions on $\mathbb{R}$ with the topology of compact convergence; i.e., τ is defined by the C^*-seminorms p_n such that

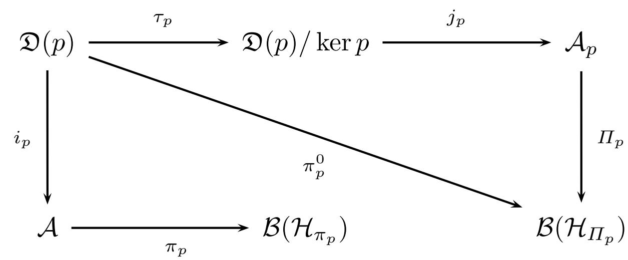
$$p_n(f) := \sup\{| f(t) |: t \in [-n, n]\},\ n \in \mathbb{N},\ f \in \mathcal{C}(\mathbb{R}).$$

Take $\mathfrak{D}(p_\infty) := \{f \in \mathcal{C}(\mathbb{R}) : \sup_n p_n(f) < \infty\}$ with $p_\infty(f) := \sup_n p_n(f)$, $f \in \mathfrak{D}(p_\infty)$. Then $\mathfrak{D}(p_\infty) = \mathcal{C}_b(\mathbb{R})$, the C^*-algebra of all continuous bounded functions on $\mathbb{R}$, $p_\infty(f) = \|f\|_\infty$, $f \in \mathcal{C}_b(\mathbb{R})$ and $\mathcal{I}_{p_\infty} = \mathcal{C}_c(\mathbb{R})$, the algebra of all continuous functions on $\mathbb{R}$ with compact support. So again p_∞ is an unbounded C^*-(semi)norm of $\mathcal{A}[\tau]$ and $\mathcal{I}_{p_\infty} \not\subseteq \{0\} = \ker p_\infty$.

For further examples, see e.g., [8, 6, 3].

3.3. Construction of an unbounded $*$-representation from an unbounded C^*-seminorm p with $\mathcal{I}_p \not\subseteq \ker p$ (see [8]).

Let $\mathcal{A}$ be a $*$-algebra and p an unbounded C^*-seminorm of $\mathcal{A}$ such that $\mathcal{I}_p \not\subseteq \ker p$. The Hausdorff completion $\mathcal{A}_p$ of $(\mathfrak{D}(p), p)$, as a C^*-algebra attains a faithful nondegenerate $*$-representation Π_p on a Hilbert space $\mathcal{H}_{\Pi_p}$. If $\mathcal{B}(\mathcal{H}_{\Pi_p})$ is the C^*-algebra of all bounded linear operators on $\mathcal{H}_{\Pi_p}$, consider the following diagram:



where τ_p is the natural quotient map, j_p the canonical injection, i_p the restriction of the identity map of $\mathcal{A}$ on $\mathfrak{D}(p)$ and $\pi_p^0 := \Pi_p \circ j_p \circ \tau_p$, i.e., $\pi_p^0(x) := \Pi_p(x_p)$, for each $x \in \mathfrak{D}(p)$. Thus π_p^0 is a bounded $*$-representation of $\mathfrak{D}(p)$ on the Hilbert space $\mathcal{H}_{\Pi_p}$.

Let now $\mathfrak{D}(\pi_p)$:=linear span of $\{\Pi_p(x + \ker p)\xi : x \in \mathcal{I}_p, \xi \in \mathcal{H}_{\Pi_p}\} \subseteq \mathcal{H}_{\Pi_p}$. Let $a \in \mathcal{A}$ and $\sum_{\text{finite}} \Pi_p(x_\kappa + \ker p)\xi_\kappa \in \mathfrak{D}(\pi_p)$. Then the map

$$\pi_p(a)\Big(\sum_\kappa \Pi_p(x_\kappa + \ker p)\xi_\kappa\Big) := \sum_\kappa \Pi_p(ax_\kappa + \ker p)\xi_\kappa$$

is a well defined linear operator on $\mathfrak{D}(\pi_p)$. Taking the Hilbert space $\mathcal{H}_{\pi_p}$ given by the norm-closure of $\mathfrak{D}(\pi_p)$ in $\mathcal{H}_{\Pi_p}$, π_p is a non-trivial unbounded $*$-representation of $\mathcal{A}$ on $\mathcal{H}_{\pi_p}$ (clearly $\mathcal{H}_{\pi_p} \neq \{0\} \Leftrightarrow \mathcal{I}_p \not\subseteq \ker p$).

3.4. Selection of well-behaved $*$-representations among all unbounded $*$-representations derived by unbounded C^*-seminorms.

Let $\mathcal{A}$ be a $*$-algebra and p an unbounded C^*-seminorm of $\mathcal{A}$. Let $Rep\mathcal{A}_p$ be the set of all faithful nondegenerate $*$-representations of the C^*-algebra $\mathcal{A}_p$ and $Rep(\mathcal{A}, p) \equiv \{\pi_p : \Pi_p \in Rep\mathcal{A}_p\}$. An unbounded C^*-seminorm p of $\mathcal{A}$ is said to be *finite* resp. *semifinite*, if $\mathcal{I}_p = \mathfrak{D}(p)$, resp. $\mathcal{I}_p$ is dense in the C^*-seminormed algebra $(\mathfrak{D}(p), p)$; p is called *weakly semifinite* (abbreviated to w-semifinite) if there exists $\Pi_p \in Rep\mathcal{A}_p$ such that $\mathcal{H}_{\Pi_p} = \overline{\mathfrak{D}(\pi_p)}^{\|\cdot\|} \Leftrightarrow \mathcal{H}_{\Pi_p} = \mathcal{H}_{\pi_p}$. Evidently, each finite unbounded C^*-seminorm is semifinite and each semifinite is w-semifinite (see, for instance, [8; Proposition 2.6, (4)]). Now, an unbounded $*$-representation $\pi_p \in Rep(\mathcal{A}, p)$ is called *well-behaved* whenever $\mathcal{H}_{\pi_p} = \mathcal{H}_{\Pi_p}$.

In Example 3.1, the unbounded C^*-(semi)norm p_∞ is clearly finite, therefore the corresponding unbounded $*$-representation π_{p_∞} of $\mathcal{A}[\tau]$ is well-behaved. The $*$-algebra $\mathcal{A}[\tau]$ is a metrizable locally convex $*$-algebra, the completion $\tilde{\mathcal{A}}[\tau]$ of which is the Arens algebra $L^w[0,1] := \bigcap_{1 \leq p < \infty} L^p[0,1]$. In this case the corresponding $\mathcal{I}_{\tilde{p}_\infty}$ is trivial, so that we cannot have an unbounded well-behaved $*$-representation of $\tilde{\mathcal{A}}[\tau]$.

Looking at the diagram of 3.3 and taking $p = p_\infty$, with p_∞ as in Example 3.2, we conclude that the corresponding bounded $*$-representation $\pi^0_{p_\infty}$ of $\mathfrak{D}(p_\infty)$ is nondegenerate with the property $\mathcal{I}_{p_\infty} \not\subseteq ker\pi^0_{p_\infty}$. This implies (see proof of [5; Theorem 3.5 (ii)$\Rightarrow$(iii)]) that a w-semifinite unbounded C^*-seminorm r of $\mathcal{A}[\tau]$ is defined with $\mathfrak{D}(r) = \mathfrak{D}(p_\infty)$ and $r \leq p_\infty$; therefore the respective unbounded $*$-representation π_r of $\mathcal{A}[\tau] = \mathcal{C}[0,1]$ will be a well-behaved one.

4. Spectral $*$-representations. In 1992, T. W. Palmer [14] introduced the concept of a *spectral algebra*, that is, of an algebra $\mathcal{A}$ which can be equipped with an *m-seminorm* ($\Leftrightarrow$ submultiplicative seminorm) p such that

$$r_{\mathcal{A}}(x) \leq p(x),\ \forall\ x \in \mathcal{A},$$

where $r_{\mathcal{A}}$ is the spectral radius of $\mathcal{A}$. This equivalently means that the group $G^q_{\mathcal{A}}$ of quasi-invertible elements of $\mathcal{A}$ is open, which is further characterized by a realization of the spectral radius in terms of a limit, as in the Banach algebra case; i.e.,

$$r_{\mathcal{A}}(x) = \lim_n p(x^n)^{1/n},\ \forall\ x \in \mathcal{A}. \tag{4.1}$$

It is clear that spectral algebras need not be normed algebras, nevertheless they enjoy a lot of the important properties of Banach algebras (see, e.g., (4.1) and for many more [14, 15]).

A $*$-algebra $\mathcal{A}$ that can be endowed with a C^*-seminorm p such that $r_{\mathcal{A}}(x) \leq p(x)$, for each $x \in \mathcal{A}$, is called by Bhatt-Inoue-Ogi in [7] a *C^*-spectral algebra.* If such a p exists, it is unique and coincides with the *Pták function* $p_{\mathcal{A}}$, defined by

$$p_{\mathcal{A}}(x) := r_{\mathcal{A}}(x^*x)^{1/2},\ \forall\ x \in \mathcal{A}.$$

All hermitian Banach algebras (with not necessarily continuous involution) are C^*-spectral algebras. The same is true for several important algebras from distribution theory, like for instance, the algebra $\mathcal{C}^\infty[0,1]$ of smooth functions on $[0,1]$, the algebra $\mathcal{S}(\mathbb{R})$ of fast decreasing smooth functions on $\mathbb{R}$, the algebra $\mathfrak{D}(\mathbb{R})$ of compactly supported smooth functions on $\mathbb{R}$, etc.

Let now $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}$ is a C^*-spectral algebra that admits a faithful $*$-representation π on a Hilbert space $\mathcal{H}_\pi$ and

$$sp_{\mathcal{A}}(x) = sp_{\pi(\mathcal{A})}(\pi(x)) \cup \{0\} = sp_{\mathcal{B}(\mathcal{H}_\pi)}(\pi(x)) \cup \{0\},\ \forall\ x \in \mathcal{A}.$$

With this as a motive, Bhatt-Inoue-Ogi in their first investigation of C^*-spectral algebras (cf. [7]) introduced the notion of a spectral $*$-representation. That is, if $\mathcal{A}$ is a $*$-algebra and π a (bounded) $*$-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\pi$, then π is called a *spectral $*$-representation* if

$$sp_{\mathcal{A}}(x) = sp_{\mathcal{C}^*(\pi)}(\pi(x)) \cup \{0\},\ \forall\, x \in \mathcal{A}, \tag{4.2}$$

where $\mathcal{C}^*(\pi)$ is the C^*-subalgebra of $\mathcal{B}(\mathcal{H}_\pi)$ given by the norm-closure $\overline{\pi(\mathcal{A})}^{\|\cdot\|}$ of the image of π in $\mathcal{B}(\mathcal{H}_\pi)$. In fact, what one demands in (4.2) is that $sp_{\mathcal{A}}(x) \subseteq sp_{\mathcal{C}^*(\pi)}(\pi(x)) \cup \{0\}$, $\forall\, x \in \mathcal{A}$, since the inverse inclusion is always true. In the same paper (see, for example, Theorem 1.6 in [7]), Bhatt-Inoue-Ogi proved that *a $*$-algebra $\mathcal{A}$ is C^*-spectral iff $\mathcal{A}$ admits a spectral $*$-representation.* After the introduction and investigation of the unbounded C^*-seminorms, the above authors naturally were led to the concepts of an unbounded spectral C^*-seminorm resp. unbounded spectral $*$-representation (cf. [8]) and the consequences of their relationship. Note that both of the unbounded C^*-seminorms of the Examples 3.1, 3.2 are spectral. Now if $\mathcal{A}$ is a $*$-algebra and π an unbounded $*$-representation of $\mathcal{A}$, let $\mathcal{A}_b^\pi := \{x \in \mathcal{A} : \overline{\pi(x)} \in \mathcal{B}(\mathcal{H}_\pi)\}$ and let $\mathcal{C}^*(\pi)$ be the C^*-algebra generated by the norm-closure of the $*$-subalgebra $\{\overline{\pi(x)} : x \in \mathcal{A}_b^\pi\}$ of $\mathcal{B}(\mathcal{H}_\pi)$. Then π is called *spectral* if

$$sp_{\mathcal{A}_b^\pi}(x) = sp_{\mathcal{C}^*(\pi)}(\overline{\pi(x)}) \cup \{0\},\ \forall\, x \in \mathcal{A}_b^\pi.$$

Let $\pi_b := \pi\lceil_{\mathcal{A}_b^\pi}$. In [8; Theorem 6.8] it was proved that: *$\mathcal{A}$ admits a strongly nondegenerate $*$-representation π such that π_b is spectral iff there is a maximal w-semifinite, spectral unbounded C^*-seminorm of $\mathcal{A}$.* We remark that π is called *strongly nondegenerate* if there is a left ideal $\mathcal{I}$ of $\mathcal{A}$ contained in $\mathcal{A}_b^\pi$ such that the linear span of $\overline{\pi(\mathcal{I})}\mathcal{H}_\pi$ is dense in $\mathcal{H}_\pi$.

In more or less the same period Bhatt-Inoue-Kürsten considered in [6] spectral $*$-representations of locally convex $*$-algebras and characterized spectral well-behaved $*$-representations of this sort of algebras. More precisely, if $\mathcal{A}$ is a unital locally convex $*$-algebra and $\mathcal{A}_0 = \{x \in \mathcal{A} : x \text{ is bounded}\}$ is the bounded part of $\mathcal{A}$ in the sense of G. R. Allan [1], then since $\mathcal{A}_0$ is not even a subspace, in general, one considers the $*$-subalgebra $\mathcal{A}_b$ of $\mathcal{A}$ generated by the elements $h \in \mathcal{A}_0$ with $h^* = h$. In this case, $\mathcal{A}_b \subseteq \mathcal{A}_b^\pi$ with π an unbounded $*$-representation of $\mathcal{A}$ (cf. [6; Lemma 3.1]). Thus if $\mathcal{C}_u^*(\pi)$ is the C^*-algebra generated by the norm-closure of $\{\overline{\pi(x)} : x \in \mathcal{A}_b\}$ in $\mathcal{B}(\mathcal{H}_\pi)$, π is called spectral, if

$$sp_{\mathcal{A}_b}(x) = sp_{\mathcal{C}_u^*(\pi)}(\overline{\pi(x)}),\ \forall\, x \in \mathcal{A}_b.$$

In this respect, *if $\mathcal{A}$ is a pseudo-complete unital locally convex $*$-algebra, a spectral well-behaved $*$-representation of $\mathcal{A}$ exists iff a spectral tw-semifinite unbounded C^*-seminorm of $\mathcal{A}$ exists, whose domain contains $\mathcal{A}_b$* (cf. [6; Theorem 4.8]).

An unbounded C^*-seminorm p of a locally convex $*$-algebra $\mathcal{A}$ with $\mathfrak{D}(p) := \mathcal{A}_b$ is called *tw-semifinite* (topologically w-semifinite) if there exists a faithful nondegenerate $*$-representation Π_p of $\mathcal{A}_p$ such that $\Pi_p(\mathcal{I}_b + \ker p)\mathcal{H}_{\Pi_p}$ is dense in $\mathcal{H}_{\Pi_p}$.

5. Characterization of spectral well-behaved $*$-representations in terms of Pták function. In this section we present some new results on spectral well-behaved $*$-representations of $*$-algebras and locally convex $*$-algebras.

Let $\mathcal{A}$ be a $*$-algebra, π a $*$-representation of $\mathcal{A}$ and p an *unbounded m^*-seminorm* of $\mathcal{A}$ (i.e., p is an unbounded m-seminorm of $\mathcal{A}$ (see beginning of Section 4) that preserves involution) such that $\mathfrak{D}(p) \subseteq \mathcal{A}_b^\pi$. Then π is said to be *p-spectral* if

$$sp_{\mathfrak{D}(p)}(x) = sp_{\mathcal{C}_p^*(\pi)}(\overline{\pi(x)}) \cup \{0\}, \ \forall\, x \in \mathfrak{D}(p),$$

where $\mathcal{C}_p^*(\pi)$ is the C^*-algebra corresponding to the norm-closure of $\{\overline{\pi(x)} : x \in \mathfrak{D}(p)\}$ in $\mathcal{B}(\mathcal{H}_\pi)$.

A $*$-algebra $\mathcal{A}$ is called *hermitian* if every element $h \in \mathcal{A}$ with $h^* = h$ has real spectrum. For the next lemma see [10; Theorems 4.1 and 4.2, as well as Proposition 4.6].

LEMMA 5.1. *Let $\mathcal{A}$ be a $*$-algebra and p an unbounded spectral m^*-seminorm of $\mathcal{A}$ such that $(\mathfrak{D}(p), p)$ is complete. Then the following are equivalent:*

(i) $(\mathfrak{D}(p), p)$ is hermitian.

(ii) The Pták function $p_{\mathfrak{D}(p)}$ is a C^*-seminorm on $\mathfrak{D}(p)$.

(iii) $r_{\mathfrak{D}(p)}(x) \le p_{\mathfrak{D}(p)}(x)$, $\forall\, x \in \mathfrak{D}(p)$ (Pták inequality). ■

THEOREM 5.2. *Let $\mathcal{A}$ be a unital $*$-algebra and p an unbounded m^*-seminorm of $\mathcal{A}$. The following are equivalent:*

(i) There exists a p-spectral well-behaved $*$-representation π_r of $\mathcal{A}$ induced by a w-semifinite C^*-seminorm r on $\mathfrak{D}(p)$.

(ii) There exists a p-spectral $*$-representation π of $\mathcal{A}$ such that $[\pi(\mathcal{I}_p)\mathcal{H}_\pi] = \mathcal{H}_\pi$, where $[\cdot]$ means closed linear span.

(iii) The Pták function $p_{\mathfrak{D}(p)}$ is a w-semifinite (spectral) C^*-seminorm on $\mathfrak{D}(p)$. ■

COROLLARY 5.3. *Let $\mathcal{A}$ be a unital $*$-algebra and p an unbounded semifinite m^*-seminorm of $\mathcal{A}$ such that $(\mathfrak{D}(p), p)$ is complete. Consider the following statements:*

(i) There exists a p-spectral, well-behaved $*$-representation of $\mathcal{A}$.

(ii) There exists a p-spectral $*$-representation π of $\mathcal{A}$ such that $\pi\restriction_{\mathfrak{D}(p)}$ is nondegenerate.

(iii) The Pták function $p_{\mathfrak{D}(p)}$ is a (spectral) C^*-seminorm on $\mathfrak{D}(p)$ with $p_{\mathfrak{D}(p)} \le p$.

(iv) The unbounded m^*-seminorm p is spectral and $(\mathfrak{D}(p), p)$ is hermitian.

Then (iv) $\Leftrightarrow$ (iii) $\Rightarrow$ (i) $\Leftrightarrow$ (ii). ■

COROLLARY 5.4. *Let $\mathcal{A}[\tau]$ be a unital Fréchet locally convex $*$-algebra; i.e., $\mathcal{A}[\tau]$ is a complete locally convex $*$-algebra, whose topology τ is determined by an increasing sequence $\{p_n\}_{n\in\mathbb{N}}$ of $*$-seminorms such that $p_n(xy) \le p_{n+1}(x)p_{n+1}(y)$, for any $x, y \in \mathcal{A}$ and $n \in \mathbb{N}$. Then (according to the notation of 3.1) the following are equivalent:*

(i) There exists a p_∞-spectral, well-behaved $*$-representation π_r of $\mathcal{A}$, induced by a w-semifinite unbounded C^*-seminorm r of $\mathcal{A}$ with $\mathfrak{D}(r) = \mathfrak{D}(p_\infty)$.

(ii) The Pták function $p_{\mathfrak{D}(p_\infty)}$ is a w-semifinite (spectral) C^*-seminorm on $\mathfrak{D}(p_\infty)$.

(iii) The Banach algebra $\mathfrak{D}(p_\infty)$ is hermitian and $p_{\mathfrak{D}(p_\infty)}$ is w-semifinite. ■

THEOREM 5.5. *Let $\mathcal{A}$ be a unital pseudo-complete locally convex $*$-algebra. The following are equivalent:*

(i) There exist a well-behaved $*$-representation π_r of $\mathcal{A}$ induced by a tw-semifinite unbounded C^*-seminorm r of $\mathcal{A}$, with $\mathfrak{D}(r) = \mathcal{A}_b$ and a spectral unbounded C^*-seminorm r' of $\mathcal{A}$ such that $\mathfrak{D}(r') = \mathcal{A}_b$ and $\|\overline{\pi_r(x)}\| \le r'(x)$, for each $x \in \mathcal{A}_b$.

(ii) There exist a tw-semifinite unbounded C^*-seminorm r of $\mathcal{A}$ with $\mathfrak{D}(r) = \mathcal{A}_b$ and a spectral unbounded C^*-seminorm r' of $\mathcal{A}$ such that $\mathfrak{D}(r') = \mathcal{A}_b$ and $r(x) \leq r'(x)$, $\forall x \in \mathcal{A}_b$.

(iii) The Pták function $p_{\mathcal{A}_b}$ is a C^*-seminorm on $\mathcal{A}_b$ with $p_{\mathcal{A}_b}(\mathcal{I}_b) \neq \{0\}$. ∎

EXAMPLE 5.6. Let $(\mathcal{A}_\alpha)$ be a directed family of unital Banach $*$-algebras with norms $\|\cdot\|_\alpha$. Let $\mathcal{A} := \prod_\alpha \mathcal{A}_\alpha$ be the product of $\mathcal{A}_\alpha$'s with algebraic operations and involution defined coordinatewise. Then $\mathcal{A}$ endowed with the m^*-seminorms

$$p_\alpha(x) := \|x_\alpha\|_\alpha, \ \forall\ x = (x_\alpha) \in \mathcal{A},$$

is a complete locally m-convex $*$-algebra. In particular, each $\mathcal{A}_\alpha$ is bicontinuously imbedded into $\mathcal{A}$ and $\mathcal{I}_{\|\cdot\|_\alpha} = \mathcal{A}_\alpha = \mathfrak{D}(\|\cdot\|_\alpha)$, so that each $\|\cdot\|_\alpha$ becomes a spectral, (semi)finite unbounded m^*-seminorm of $\mathcal{A}$. Hence Corollary 5.3 implies the existence of a $\|\cdot\|_\alpha$-spectral well-behaved $*$-representation of $\mathcal{A}$ whenever the Banach $*$-algebra $\mathcal{A}_\alpha$ is hermitian.

Furthermore, let $\mathfrak{D}(p_\infty) = \{x \in \mathcal{A} : \sup_\alpha p_\alpha(x) < \infty\}$, with $p_\infty(x) := \sup_\alpha p_\alpha(x)$, $x \in \mathfrak{D}(p_\infty)$. Consider the Banach $*$-subalgebra $\mathcal{A}_0 = \bigoplus_\alpha \mathcal{A}_\alpha$ (algebraic direct sum) of $\mathcal{A}$, under the norm $\|\cdot\|_0 = p_\infty\lceil_{\mathcal{A}_0}$. Then, $\mathcal{I}_{\|\cdot\|_0} = \mathcal{A}_0$, therefore $\|\cdot\|_0$ is a spectral (semi)finite unbounded m^*-seminorm of $\mathcal{A}$. Moreover, $\mathcal{A}$ is hermitian iff each $\mathcal{A}_\alpha$ is hermitian (this is not true, in general, for a complete locally m-convex $*$-algebra, but here we have that $\mathcal{A}/\ker p_\alpha = \mathcal{A}_\alpha$, up to an isometric $*$-isomorphism; i.e., each normed $*$-algebra $A/\ker p_\alpha$ is (automatically) complete). Thus assuming hermiticity for $\mathcal{A}$, each $\mathcal{A}_\alpha$ is hermitian and this implies hermiticity for $\mathcal{A}_0$ too, since $sp_{\mathcal{A}_0}((x_\alpha)) = \bigcup_\alpha sp_{\mathcal{A}_\alpha}(x_\alpha)$, for all $(x_\alpha) \in \mathcal{A}_0$. So again from Corollary 5.3, $\mathcal{A}$ admits a $\|\cdot\|_0$-spectral well-behaved $*$-representation.

Summing up, when a directed family $(\mathcal{A}_\alpha)$ of unital C^*-algebras is given, then since every C^*-algebra is hermitian, one concludes from the above that the pro-C^*-algebra ($\Leftrightarrow$locally C^*-algebra) $\mathcal{A} := \prod_\alpha \mathcal{A}_\alpha = \varprojlim_\alpha \mathcal{A}_\alpha$ has a plethora of q-spectral well-behaved $*$-representations, with respect to unbounded semifinite spectral C^*-seminorms q of $\mathcal{A}$, where q varies in the set consisting of $\|\cdot\|_0$ and all $\|\cdot\|_\alpha$'s rising as before.

Full proofs of the preceding results, as well as further new results and examples on spectral well-behaved $*$-representations will be presented elsewhere.

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