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UNITARY DILATION FOR POLAR DECOMPOSITIONS OF p-HYPONORMAL OPERATORS

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Abstract. In this paper, we introduce the angular cutting and the generalized polar symbols of a p-hyponormal operator T in the case where U of the polar decomposition $T = U|T|$ is not unitary and study spectral properties of it.

1. Introduction. As one of generalizations of normal operators, semi-hyponormal operators were introduced. In the study of semi-hyponormal operators, there exist many important techniques concerning polar decompositions $T = U|T|$ with unitary U (for example, symbols, angular cutting and spectral mapping theorems). Many useful results have been obtained under the assumption that U is unitary for the polar decomposition $T = U|T|$. Using a unitary dilation $\hat{U}$ of U of a semi-hyponormal operator $T = U|T|$, we study spectral properties of the operator $U\varphi(|T|)$ for a semi-hyponormal operator $T = U|T|$ and a strictly monotone increasing continuous function $\varphi(\cdot)$.

In [8, Chapter 6, Section 3.2] Xia gave spectral mapping theorems of semi-hyponormal operators $T = U|T|$ with unitary U for the following mapping: $T \to \xi(U)\varphi(|T|)$. In [7] Itoh extended these results to p-hyponormal operators. It is important for these results that

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U is unitary. There exist important examples without this form (for example, unilateral weighted shift). In this paper we study spectral mapping theorems for a semi-hyponormal operator T which does not have a polar decomposition $T = U|T|$ with unitary U.

Let T be an operator in a Hilbert space $\mathcal{H}$. T is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. If $p = 1$ and $p = \frac{1}{2}$, then T is said to be hyponormal and semi-hyponormal, respectively. By Löwner's Theorem, for $0 < p < q \leq 1$, if T is q-hyponormal, then T is p-hyponormal. If T is p-hyponormal, then $\ker(T) \subset \ker(T^*)$ (see [4]). Hence we may assume that U is isometry if $T = U|T|$ is p-hyponormal. Let $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote the spectrum, the point spectrum and the approximate point spectrum of T, respectively. Put $\sigma_r(T) = \sigma(T) \setminus \sigma_a(T)$. It is known from [8, Chapter 1, Section 3.1] that $\lambda \in \sigma_r(T)$ if and only if there exists a positive number c such that $||(T - \lambda I)x|| \geq c||x||$ $(x \in \mathcal{H})$ and $(T - \lambda I)(\mathcal{H}) \neq \mathcal{H}$.

2. Results. Let $T = U|T|$ be an operator in a Hilbert space $\mathcal{H}$. If U is isometry, let $\hat{U} = \begin{pmatrix} U & I - UU^* \\ 0 & U^* \end{pmatrix}$ and $\hat{T} = \hat{U} \begin{pmatrix} |T| & 0 \\ 0 & 0 \end{pmatrix}$. Then we have $|\hat{T}| = |T| \oplus 0$. If $T = U|T|$ is p-hyponormal, then we assume that U is isometry and hence we can define $\hat{U}$ and $\hat{T}$.

Proposition 1. *Let $T = U|T|$ be an operator in a Hilbert space $\mathcal{H}$. If there exists a unitary operator $\hat{U}$ on $\mathcal{H} \oplus \mathcal{H}$ such that $\hat{U}(x \oplus 0) = Ux \oplus 0$, then the following equalities hold for T and $\hat{T} = \hat{U}(|T| \oplus 0)$:*

(1) $\sigma(T) \setminus \{0\} = \sigma(\hat{T}) \setminus \{0\}$,
(2) $\sigma_a(T) \setminus \{0\} = \sigma_a(\hat{T}) \setminus \{0\}$,
(3) $\sigma_r(T) \setminus \{0\} = \sigma_r(\hat{T}) \setminus \{0\}$.

Proof. The proof of (1) is very similar to that of (3). Hence there is no need to cite [8, Chapter 2, Lemma 3.5]. If (3) holds, then so does (2). It is sufficient to show that (3) holds. If $\lambda \in \sigma_r(T) \setminus \{0\}$, there exists $c > 0$ such that $||(T - \lambda I)x|| \geq c||x||$ $(x \in \mathcal{H})$ and $(T - \lambda I)(\mathcal{H}) \neq \mathcal{H}$. Then

$$||(\hat{T} - \lambda I)(x \oplus y)||^2 = ||(T - \lambda I)x||^2 + || - \lambda y||^2 \geq c_1^2 ||x \oplus y||^2$$

where $c_1 = \min\{c, |\lambda|\}$. Since $(T - \lambda I)(\mathcal{H}) \neq \mathcal{H}$, it is easy to check $(\hat{T} - \lambda I)(\mathcal{H} \oplus \mathcal{H}) \neq \mathcal{H} \oplus \mathcal{H}$, so that $\lambda \in \sigma_r(\hat{T})$.

Conversely if $\lambda \in \sigma_r(\hat{T}) \setminus \{0\}$, then there exists $c > 0$ such that $||(\hat{T} - \lambda I)(x \oplus y)|| \geq c||x \oplus y||$ $(x \oplus y \in \mathcal{H} \oplus \mathcal{H})$ and $(\hat{T} - \lambda I)(\mathcal{H} \oplus \mathcal{H}) \neq \mathcal{H} \oplus \mathcal{H}$. Then

$$||(T - \lambda I)x|| = ||(\hat{T} - \lambda I)(x \oplus 0)|| \geq c||x \oplus 0|| = c||x||.$$

Suppose that $(T - \lambda I)(\mathcal{H}) = \mathcal{H}$. For $x, y \in \mathcal{H}$, there exists $x_1 \in \mathcal{H}$ such that $(T - \lambda I)x_1 = x$, so that

$$x \oplus y = (\hat{T} - \lambda I)(x_1 \oplus -(1/\lambda)y).$$

This is a contradiction. Hence $\lambda \in \sigma_r(T)$.

Theorem 2. *Let $T = U|T|$ be a semi-hyponormal operator and φ be a strictly monotone increasing continuous function on $[0, ||T||]$. If φ is operator monotone on $[0, ||T||]$ and*

$\varphi(0) = 0$, *then*

$$\sigma(U\varphi(|T|)) = \{e^{i\theta}\varphi(r) : e^{i\theta}r \in \sigma(U|T|)\},$$
$$\sigma_a(U\varphi(|T|)) = \{e^{i\theta}\varphi(r) : e^{i\theta}r \in \sigma_a(U|T|)\},$$
$$\sigma_r(U\varphi(|T|)) = \{e^{i\theta}\varphi(r) : e^{i\theta}r \in \sigma_r(U|T|)\}.$$

Proof. If $0 \notin \sigma(T)$, then $T = U|T|$ with unitary U. In this case, the result follows from [8, Chapter 6, Theorem 3.2]. We assume that $0 \in \sigma(T)$. Since φ is operator monotone and $\hat{U}|\hat{T}|\hat{U}^* = |\hat{T}^*| \leq |\hat{T}|$, we get $\hat{U}^*\varphi(|\hat{T}|)\hat{U} = \varphi(\hat{U}^*|\hat{T}|\hat{U}) \leq \varphi(|\hat{T}|)$ which implies that $\hat{U}\varphi(|\hat{T}|)$ is semi-hyponormal. By Xia's theorem [8, Chapter 6, Theorem 3.2], we again have

$$\sigma_*(\hat{U}\varphi(|\hat{T}|)) = \{e^{i\theta}\varphi(r) : e^{i\theta}r \in \sigma_*(\hat{U}|\hat{T}|)\},$$

where σ_* denotes σ, σ_a or σ_r. By Proposition 1,

$$\sigma_*(\hat{U}\varphi(|\hat{T}|)) \setminus \{0\} = \sigma_*(U\varphi(|T|)) \setminus \{0\}$$

and

$$\sigma_*(\hat{U}|\hat{T}|) \setminus \{0\} = \sigma_*(U|T|) \setminus \{0\}.$$

Therefore, we have

$$\sigma_*(U\varphi(|T|)) \setminus \{0\} = \{e^{i\theta}\varphi(r) : e^{i\theta}r \in \sigma_*(U|T|)\} \setminus \{0\}.$$

Next since U satisfies $\|U|T|x\| = \||T|x\|$ for any vector x, we have

$$0 \in \sigma_a(U|T|) \text{ if and only if } 0 \in \sigma_a(U\varphi(|T|)).$$

Since $\sigma_r(T) = \sigma(T) \setminus \sigma_a(T)$, we have

$$0 \in \sigma_r(U|T|) \text{ if and only if } 0 \in \sigma_r(U\varphi(|T|)).$$

This completes the proof.

By comparison in case of hyponormal operators [2], the following example implies that in the semi-hyponormal case, spectral mapping theorems are complicated.

EXAMPLE. Let $\{e_k\}_{k=1}^{\infty}$ be the canonical orthnormal basis for the Hilbert space l^2. Let U be the unilateral shift.

Let $T = U$. Then T is hyponormal and $|T| = I$, so that T is semi-hyponormal. Let φ be a function on $\mathbf{R}$ such that $\varphi(x) = x + 1$. We remark $\varphi(0) = 1$. Then $U\varphi(I) = 2U$, so that $U\varphi(I)$ is semi-hyponormal. Therefore, $\sigma(U\varphi(I)) = \{z : |z| \leq 2\}$ and $\{e^{i\theta}\varphi(r) : r \cdot e^{i\theta} \in \sigma(T)\} = \{z : 1 \leq |z| \leq 2\}$, so that $\sigma(U\varphi(|T|)) \neq \{e^{i\theta}\varphi(r) : re^{i\theta} \in \sigma(U|T|)\}$.

DEFINITION 1. Let $T = U|T|$ be a semi-hyponormal operator such that U is not unitary. Let $\mathbf{T} = \{z : |z| = 1\}$ be the unit circle. For a unitary operator $\hat{U}$ on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, let $\hat{U} = \int_{\mathbf{T}} \lambda dE(\lambda)$ be the spectral decomposition $\hat{U}$. Let γ be an arc in $\mathbf{T}$ and $E(\gamma) \neq 0$. For $\hat{T} = \hat{U}|\hat{T}|$, denote

$$\mathcal{K}_\gamma = E(\gamma)\mathcal{K}, \quad \hat{U}_\gamma = \hat{U}_{|\mathcal{K}_\gamma} \text{ and } \hat{T}_\gamma = E(\gamma)\hat{T}_{|\mathcal{K}_\gamma}.$$

$\hat{T}_\gamma$ is called the *section of* $\hat{T}$ *cut by the arc* γ. We denote $\hat{T}_\gamma$ simply by T_γ.

THEOREM 3. *Let $T = U|T|$ be a semi-hyponormal operator such that U is not unitary. Let γ be an arc in the unit circle $\mathbf{T}$ with $E(\gamma) \neq 0$. Put $\mathcal{D}_\gamma = \{\lambda : \lambda \neq 0,\ \lambda/|\lambda| \in \gamma\}$. Then*

$$\sigma_p(T_\gamma) \setminus \{0\} = \sigma_p(T) \cap \mathcal{D}_\gamma, \tag{1}$$

in addition, x is an eigenvector of T_γ corresponding to the eigenvalue λ if and only if x is an eigenvector of T and $\lambda \in \mathcal{D}_\gamma$,

$$\sigma(T_\gamma) \subset \overline{\mathcal{D}}_\gamma \cap \sigma(T), \tag{2}$$

moreover, if γ is an open arc, then

$$\sigma_a(T_\gamma) \cap \mathcal{D}_\gamma = \sigma_a(T) \cap \mathcal{D}_\gamma, \tag{3}$$

$$\sigma_r(T_\gamma) \cap \mathcal{D}_\gamma = \sigma_r(T) \cap \mathcal{D}_\gamma, \tag{4}$$

thus

$$\sigma(T_\gamma) \cap \mathcal{D}_\gamma = \sigma(T) \cap \mathcal{D}_\gamma. \tag{5}$$

Proof. By [8, Chapter 6, Theorem 3.9], $\sigma(T_\gamma) \subset \sigma(\hat{T}) = \sigma(T)$. An application of [8, Chapter 1, Theorem 3.2] and Proposition 1 complete the proof.

REMARK. For an open arc γ, $\sigma_*(T_\gamma) \cap \mathcal{D}_\gamma$ is independent of the choice of a unitary operator $\hat{U}$, where σ_* is σ, σ_a or σ_r.

For a unitary operator U and an operator A, if

$$\mathcal{S}_U^{\pm}(A) = \text{s-}\lim_{n \to \pm\infty} U^{-n} A U^n$$

exist, then $\mathcal{S}_U^{\pm}(A)$ are called the *general polar symbols* of A related to U. If $T = U|T|$ is semi-hyponormal with unitary U, then $\mathcal{S}_U^{\pm}(|T|)$ exist (cf. [8]). For $0 \leq k \leq 1$, we define

$$T_k = U(k\mathcal{S}_U^{+}(|T|) + (1-k)\mathcal{S}_U^{-}(|T|)).$$

The operator T_k is called the *generalized polar symbol* of T with respect to k. In case U is not unitary, let $\hat{U}$ be the unitary dilation of U. For operators $\hat{U}$ and $\hat{T}$, since $\hat{T} = \hat{U}|\hat{T}|$ is semi-hyponormal with unitary $\hat{U}$, there exists $\mathcal{S}_{\hat{U}}^{\pm}(|\hat{T}|)$. For $0 \leq k \leq 1$, we also define

$$\hat{T}_k = \hat{U}(k\mathcal{S}_{\hat{U}}^{+}(|\hat{T}|) + (1-k)\mathcal{S}_{\hat{U}}^{-}(|\hat{T}|)).$$

Then Xia proved the following.

THEOREM A ([8, Chapter 4, Theorem 4.1]). *Let $T = U|T|$ be semi-hyponormal with unitary U. Then*

$$\sigma(T) = \bigcup_{0 \leq k \leq 1} \sigma(T_k).$$

We have the following result.

THEOREM 4. *Let $T = U|T|$ be semi-hyponormal such that U is not unitary. Then*

$$\sigma(T) = \bigcup_{0 \leq k \leq 1} \sigma(\hat{T}_k).$$

Proof. Since $\hat{T} = \hat{U}\begin{pmatrix} |T| & 0 \\ 0 & 0 \end{pmatrix}$ is semi-hyponormal with unitary $\hat{U}$, by Theorem A we have

$$\sigma(\hat{T}) = \bigcup_{0\le k\le 1} \sigma(\hat{T}_k).$$

Since U is not unitary, $\sigma(T) = \sigma(\hat{T})$. Hence the proof is complete.

Let $T = U|T|$ is p-hyponormal with U which is not unitary. If $S = U|T|^{2p}$, then S is semi-hyponormal. Let $|\hat{S}| = \begin{pmatrix} |T|^{2p} & 0 \\ 0 & 0 \end{pmatrix}$. Then we can define the semi-hyponormal operator $\hat{S} = \hat{U}|\hat{S}|$ with unitary $\hat{U}$.

THEOREM 5. *Let $T = U|T|$ be a p-hyponormal operator such that U is not unitary. Let $S = U|T|^{2p}$ and $\hat{S} = \hat{U}\begin{pmatrix} |T|^{2p} & 0 \\ 0 & 0 \end{pmatrix}$. Then $\mathcal{S}^{\pm}_{\hat{U}}(|\hat{S}|)$ exist and*

$$\sigma(T) = \bigcup_{0\le k\le 1} \sigma(T_{[k]}),$$

where

$$T_{[k]} = \hat{U}(k\mathcal{S}^{+}_{\hat{U}}(|\hat{S}|) + (1-k)\mathcal{S}^{-}_{\hat{U}}(|\hat{S}|))^{\frac{1}{2p}}.$$

Proof. Since $\hat{S} = \hat{U}\begin{pmatrix} |T|^{2p} & 0 \\ 0 & 0 \end{pmatrix}$ is semi-hyponormal with unitary $\hat{U}$, from Theorem A it follows that

$$\sigma(\hat{S}) = \bigcup_{0\le k\le 1} \sigma(\hat{S}_k).$$

Proposition 1 implies $\sigma(S) = \sigma(\hat{S})$. Since by Theorem 3 of [5]

$$\sigma(S) = \{r^{2p}e^{i\theta} : re^{i\theta} \in \sigma(T)\},$$

it follows that $re^{i\theta} \in \sigma(T)$ if and only if $r^{2p}e^{i\theta} \in \bigcup_{0\le k\le 1} \sigma(\hat{S}_k)$. Since

$$\hat{S}_k = \hat{U}(k\mathcal{S}^{+}_{\hat{U}}(|\hat{S}|) + (1-k)\mathcal{S}^{-}_{\hat{U}}(|\hat{S}|))$$

is normal, it follows that $r^{2p}e^{i\theta} \in \sigma(\hat{S}_k)$ if and only if $re^{i\theta} \in \sigma(T_{[k]})$. So the proof is complete.

Finally, we introduce Xia spectrum for a p-hyponormal operator $T = U|T|$ such that U is not unitary.

For a pair (A, B) of operators, $(z, w) \in \mathbf{C}^2$ belongs to the joint approximate point spectrum $\sigma_{ja}(A, B)$ if there exists a sequence $\{x_n\}$ of unit vectors such that $(A-z)x_n \to 0$ and $(B-w)x_n \to 0$ as $n \to \infty$. It is well known that if (A, B) is a commuting pair, then $\sigma_{ja}(A, B)$ is non-empty (cf. [1, Proposition 2]).

DEFINITION 2. Let $T = U|T|$ be a p-hyponormal operator such that U is not unitary. Let $S = U|T|^{2p}$, $\hat{S} = \hat{U}\begin{pmatrix} |T|^{2p} & 0 \\ 0 & 0 \end{pmatrix}$ and

$$|T_{[k]}| = (k\mathcal{S}^{+}_{\hat{U}}(|\hat{S}|) + (1-k)\mathcal{S}^{-}_{\hat{U}}(|\hat{S}|))^{\frac{1}{2p}}.$$

Since $\hat{U}$ commutes with every $|T_{[k]}|$ $(0 \leq k \leq 1)$, we see that $\sigma_{ja}(\hat{U}, |T_{[k]}|)$ is non-empty. We define the *Xia spectrum* $\sigma_X(T)$ of T by

$$\sigma_X(T) = \bigcup_{0 \leq k \leq 1} \sigma_{ja}(\hat{U}, |T_{[k]}|).$$

The following corollary is a direct consequence of Theorem 5 and the fact that $re^{i\theta} \in \sigma(T_{[k]})$ iff $(e^{i\theta}, r) \in \sigma_{ja}(\hat{U}, |T_{[k]}|)$. So we omit the proof.

Corollary 6. *Let $T = U|T|$ be a p-hyponormal operator such that U is not unitary. Let $S = U|T|^{2p}$, $\hat{S} = \hat{U}\begin{pmatrix} |T|^{2p} & 0 \\ 0 & 0 \end{pmatrix}$ and*

$$|T_{[k]}| = (k\mathcal{S}^{+}_{\hat{U}}(|\hat{S}|) + (1-k)\mathcal{S}^{-}_{\hat{U}}(|\hat{S}|))^{\frac{1}{2p}}.$$

Then $re^{i\theta} \in \sigma(T)$ if and only if $(e^{i\theta}, r) \in \sigma_X(T)$.

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