A CLASSIFICATION OF PROJECTORS

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Abstract. A positive operator \( A \) and a closed subspace \( S \) of a Hilbert space \( \mathcal{H} \) are called compatible if there exists a projector \( Q \) onto \( S \) such that \( AQ = Q^*A \). Compatibility is shown to depend on the existence of certain decompositions of \( \mathcal{H} \) and the ranges of \( A \) and \( A^{1/2} \). It also depends on a certain angle between \( A(S) \) and the orthogonal of \( S \).

1. Introduction. Consider the set \( \mathcal{Q} \) of all (bounded linear) projectors on a Hilbert space \( \mathcal{H} \). Sometimes the elements of \( \mathcal{Q} \) are named oblique projectors in order to emphasize that they are not necessarily orthogonal. Since the early years of matrix and operator theories, projectors have played a relevant role in many studies on spectral theory, approximation, optimization, orthogonal decompositions, least square methods, and so on. Very recently, several applications of oblique projectors to signal processing [10], [13], [36]; sampling [11], [57]; wavelets [3], [56]; information theory [57]; integral equations [51], [52]; statistics [54]; least square approximation [28], [29], [60] and parallel computing [17] have been found. For these multiple manifestations, many results on projectors are rediscovered once and again by different specialists. It seems that a short survey on several old and new results on oblique projectors may be helpful for the interested reader.

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For each closed subspace $S$ of $H$ let $Q^S$ denote the set of all projectors with range $S$. For each (bounded linear semidefinite) positive operator $A$ on $H$ consider the set $\mathcal{P}(A, S) = \{ Q \in Q^S : AQ = Q^* A \}$, i.e., all $Q$ with range $S$ which are Hermitian with respect to the sesquilinear form $\langle \xi, \eta \rangle_A = \langle A \xi, \eta \rangle$. Of course, $\mathcal{P}(A, S)$ can be empty (see examples below); we say that $A, S$ are compatible if $\mathcal{P}(A, S)$ is not empty. This condition can be read in terms of different space decompositions, range inclusions and angles between certain closed subspaces of $H$. It is known [19] that, if $A$ and $S$ are compatible then a distinguished element $P_{A, S}$ of $\mathcal{P}(A, S)$ exists which has optimal properties. We show explicit formulas for $P_{A, S}$ which are computationally useful.

Many results on oblique projectors can be found in the papers by Afriat [1], Davis [22], Ljance [43], Mizel and Rao [45], Halmos [33], Greville [32], Gerisch [30], Pták [49]. Projectors which are Hermitian with respect to a positive matrix have been studied by Mitra and Rao [44] and Baksalary and Kala [9]. More recently, Hassi and Nordstrom [35] studied projectors which are Hermitian with respect to a self-adjoint operator but with emphasis on the case in which $\mathcal{P}(A, S)$ is a singleton. In [47], Pasternak-Winiarski studied the analyticity of the map $A \rightarrow P_{A, S}$, where $A$ runs over the set of positive invertible operators. The map $(A, S) \rightarrow P_{A, S}$ is studied by Andruchow, Corach and Stojanoff [6], for positive invertible $A$. For general selfadjoint $A$, several results on $\mathcal{P}(A, S)$ can be found in [19] and the present paper can be seen as its continuation. Additional results by the authors are contained in [20] and [21]. The latter makes a link between oblique projectors and abstract splines in the sense of Atteia [8]. It is natural that this type of least square approximation results appears in this context, because $P_{A, S}$ is a kind of orthogonal projector for an appropriate inner product. In particular, oblique projectors, mainly in the finite-dimensional setting, appear frequently under the form of ”scaled projectors”, i.e., projectors which are Hermitian with respect to a positive diagonal matrix. The reader is referred to the papers by Stewart [53], O’Leary [46], Hanke and Neumann [34], Gonzaga and Lara [31], Wei [60], Forsgren [28], Vavasis [14], among many others, for results on and applications of scaled projectors. A relationship between scaled and $A$-Hermitian projectors, also in the infinite-dimensional setting, can be found in [7].

The contents of the paper are the following. Section 2 begins with some preliminaries and a short survey of known results on $\mathcal{P}(A, S)$ and $P_{A, S}$, taken from [19], [20] and [21]. Then, we prove several characterizations of compatibility in terms of decompositions of $H$ and of the ranges of $A$ and $A^{1/2}$, of certain range inclusions and also of the angle between the closure of $A(S)$ with the orthogonal complement of $S$. Most of these results are new and the proof of the remainder has been greatly simplified. We collect in Section 3 several formulas for $P_{A, S}$ using results from Greville [32], Kerzman and Stein [38], [39], Ljance [43], Pták [49] and Buckholtz [16].

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2. Oblique projectors. In what follows $\mathcal{H}$ denotes a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, $L(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$, $GL(\mathcal{H})$ denotes the group of invertible operators on $\mathcal{H}$, $L(\mathcal{H})^+$ the cone of positive operators, $GL(\mathcal{H})^+ = GL(\mathcal{H}) \cap$
$L(\mathcal{H})^+$ and $Q = \{Q \in L(\mathcal{H}) : Q^2 = Q\}$ the set of oblique projectors. For an operator $W$ its image is denoted by $R(W)$ and its nullspace by $N(W)$. Recall that if $\mathcal{H}, \mathcal{K}$ are two Hilbert spaces and $C \in L(\mathcal{H}, \mathcal{K})$ has closed range, then there exists a unique $C^\dagger \in L(\mathcal{K}, \mathcal{H})$ such that $CC^\dagger C = C$, $C^\dagger CC^\dagger = C^\dagger$ and $CC^\dagger$, $C^\dagger C$ are Hermitian; $C^\dagger$ is called the Moore-Penrose inverse of $C$ (see [23] and [12] for details).

The following result by R. G. Douglas will be frequently used in this paper. Given Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{G}$ and operators $A \in L(\mathcal{H}, \mathcal{G})$, $B \in L(\mathcal{K}, \mathcal{G})$ then the following conditions are equivalent:

i) the equation $AX = B$ has a solution in $L(\mathcal{K}, \mathcal{H})$;

ii) $R(B) \subseteq R(A)$;

iii) there exists $\lambda > 0$ such that $BB^* \leq \lambda AA^*$.

In this case, there exists a unique $D \in L(\mathcal{K}, \mathcal{H})$ such that $AD = B$ and $R(D) \subseteq R(A^*)$; moreover, $\|D\|^2 = \inf\{\lambda > 0 : BB^* \leq \lambda AA^*\}$. We shall call $D$ the reduced solution of $AX = B$. The reader is referred to [26] and [27] for the proof of the Douglas theorem and related results. Let us remark that if $R(A)$ is closed then the reduced solution of $AX = B$ is $A^\dagger B$: this follows quite easily from the properties of the Moore-Penrose pseudoinverse.

For a fixed closed subspace $\mathcal{S}$ of $\mathcal{H}$, operators in $\mathcal{H}$ are represented as $2 \times 2$ matrices according to the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$; more precisely, for each $B \in L(\mathcal{H})$ the identity

$$B = PBP + PB(I - P) + (I - P)BP + (I - P)B(I - P)$$

where $P$ is the orthogonal projector onto $\mathcal{S}$, can be matricially rephrased as $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, where $b_{11} = PBP|_\mathcal{S} \in L(\mathcal{S})$, $b_{12} = PB(I - P)|_\mathcal{S} \subseteq L(\mathcal{S}^\perp, \mathcal{S})$, $b_{21} = (I - P)BP|_\mathcal{S} \in L(\mathcal{S}, \mathcal{S}^\perp)$ and $b_{22} = (I - P)B(I - P)|_\mathcal{S} \subseteq L(\mathcal{S}^\perp)$. In particular, $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, any projector $Q$ onto $\mathcal{S}$ has the form $Q = \begin{pmatrix} 1 & e \\ 0 & 0 \end{pmatrix}$ for some $e \in L(\mathcal{S}^\perp, \mathcal{S})$ and any $A \in L(\mathcal{H})^+$ can be expressed as $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, where $a \in L(\mathcal{S})^+$, $b \in L(\mathcal{S}^\perp, \mathcal{S})$, $c \in L(\mathcal{S}^\perp)^+$ and $|\langle b\xi, \xi \rangle|^2 \leq |\langle a\xi, \xi \rangle |^2 |\langle c\eta, \eta \rangle |^2$ for every $\xi \in \mathcal{S}$, $\eta \in \mathcal{S}^\perp$ [50]. As a consequence (see [4]) it follows that the image of the positive square root of $a$ contains the image of $b : R(a^{1/2}) \supseteq R(b)$.

Given a closed subspace $\mathcal{S}$ let $Q^\mathcal{S}$ be the subset of $Q$ of all projectors with range (i.e. image) $\mathcal{S}$. Of course $Q$ is the disjoint union of all $Q^\mathcal{S}$. On the other hand, any positive (bounded linear) operator $A$ on $\mathcal{H}$ defines a (Hermitian semidefinite) positive sesquilinear form

$$\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad \langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle, \quad \xi, \eta \in \mathcal{H}.\$$

A (bounded linear) operator $T$ on $\mathcal{H}$ is called $A$-Hermitian if $\langle T\xi, \eta \rangle_A = \langle \xi, T\eta \rangle_A$ for all $\xi, \eta \in \mathcal{H}$, i.e. if $AT = T^*A$. We shall not study the existence of an $A$-adjoint of an operator (see [41] and [25] for this type of problems). However, the following result shows that this existence is not irrelevant, even in the case of projectors.
Lemma 2.1. Given $Q \in Q$ and $A \in L(H)^+$, there exists $W \in L(H)$ such that $AQ = W^*A$ (i.e., $Q$ admits an $A$-adjoint) if and only if

$$R(A) = R(A) \cap N(Q^*) \oplus R(A) \cap R(Q^*).$$

Proof. If $\xi \in R(A)$ then $\xi = A\eta$, for some $\eta \in H$. Since $Q^*$ decomposes $H$ as the direct sum $R(Q^*) \oplus N(Q^*)$ there exists $w \in H$ such that $\xi = A\eta = Q^*w + z$, where $z \in N(Q^*)$. But $Q^*\xi = Q^*A\eta = Q^*w \in R(A)$, because $R(Q^*A) = R(AW) \subseteq R(A)$. Then $Q^*\xi = Q^*w \in R(A) \cap R(Q^*)$. Also $z = A\eta - Q^*w \in R(A) \cap N(Q^*)$, because $Q^*w \in R(A)$. This proves decomposition (1).

If formula (1) holds, then $R(Q^*A) = Q^*(R(A) \cap R(Q^*)) = R(A) \cap R(Q^*)$, so that $R(Q^*A) \subseteq R(A)$. By the Douglas theorem there exists a solution $W$ of the equation $AX = Q^*A$. ■

Denote by $Q_A$ the set of all $A$-Hermitian projectors on $H$ and $P(A,S) = Q^S \cap Q_A$. In [19] it is remarked that every $Q \in Q$ belongs to some $P(A,S)$. Thus, $Q = \cup P(A,S)$ where $S$ runs over the class of all closed subspaces of $H$ and $A$ over a class of positive operators $A$. The sets $P(A,S)$ are the object of our study.

We follow the terminology proposed by Ben-Israel and Greville [12]: the operator $Q : \xi \mapsto Q\xi$ which performs the projection is named projector, while $Q\xi$ is the projection of $\xi$ (under $Q$).

In what follows $S$ denotes a closed subspace of $H$ and $A$ denotes a positive operator on $H$. Define

$$S^{\perp_A} := \{\xi \in H : \langle \xi, \eta \rangle_A = 0 \quad \forall \eta \in S\}.$$

The identities $S^{\perp_A} = A^{-1}(S^{\perp}) = (AS)^{\perp}$ will be used without further mention. Observe that, if $A$ is invertible, then $\langle \cdot, \cdot \rangle_A$ is an inner product which is equivalent to $\langle \cdot, \cdot \rangle_A$; so that the subspace $S$ admits a closed $A$-complement in $(H, \langle \cdot, \cdot \rangle_A)$, namely $S^{\perp_A}$; thus, $H = S \oplus S^{\perp_A}$. However, if $A$ is not invertible, such a complement may not exist. In fact, $S \cap S^{\perp_A}$ may be non-trivial and $S + S^{\perp_A}$ may be a proper non-closed subspace of $H$ (see below).

The next theorem collects several well-known facts on projectors which are due to many mathematicians: Afriat [1], Greville [32], Pták [49], Chung [18], Buckholtz [16]. Indeed, the use of projectors is so extended that many results appear once again in papers in functional analysis, statistics, matrix analysis, signal processing, and so on.

Theorem 2.2. If $S$ and $N$ are closed subspaces of a Hilbert space $H$ then the following properties are equivalent:

1. $H = S \oplus N$,
2. there exists $Q \in Q$ such that $R(Q) = S$, and $N(Q) = N$,
3. $P_S - P_N \in GL(H)$,
4. $\|P_S + P_N - I\| < 1$,
5. $P_{S^{\perp}}|_N$ is injective and $P_{S^{\perp}}(N) = S^{\perp}$.

In that case $P_SP_{N^{\perp}}$ has a closed range,

$$\|P_SP_N\| = \|P_NP_S\| < 1, \quad P_S + P_N - P_NP_S \in GL(H), \quad P_{N^{\perp}}P_S - I \in GL(H).$$
and the projector onto $S$ with nullspace $N$ is
\[
P_{S//N} = (P_SP_{N\perp})^\dagger = (I - P_{N\perp}P_S)^{-1}P_N
\]
\[
= (I - P_SP_{N\perp})^{-1}P_S(I - P_SP_{N\perp})
\]
\[
= (I - P_SP_S)^{-1}(I - P_N)
\]
\[
= P_S(P_S + P_{N - P_N}P_S)^{-1}.
\]

In particular, $\|P_{S//N}\| = (I - \|P_NP_S\|^2)^{-1/2}$.

**Remark 2.3.** There is a formula, due to Kerzman and Stein [38], [39], which expresses, given a projector $Q$, the unique orthogonal projector $P$ such that $R(P) = R(Q)$. Some of the expressions of $P_{S//N}$ given above follow from Kerzman-Stein’s formula.

**Definition 2.4.** Let $S$ be a closed subspace of $\mathcal{H}$ and let $A \in L(\mathcal{H})^+$. We say that the pair $(A, S)$ is compatible if the set $\mathcal{P}(A, S)$ is not empty.

The following result, due to M. G. Krein [40], will be used, implicitly or explicitly, several times.

**Lemma 2.5** (Krein). Let $Q$ be a projector with $R(Q) = S$. Then $Q$ is $A$-Hermitian if and only if $N(Q) \subseteq A^{-1}(S^\perp)$. In particular, $Q \in \mathcal{P}(A, S)$ if and only if $N(Q) \subseteq A^{-1}(S^\perp)$, so that $(A, S)$ is compatible if and only if $\mathcal{H} = S + A^{-1}(S^\perp)$.

**Proof.** Suppose that $AQ = Q^*A$ and consider $\xi$ such that $\xi \in N(Q)$, then $\langle A\xi, Q\theta \rangle = \langle Q^*A\xi, \theta \rangle = \langle AQ\xi, \theta \rangle = 0$, for all $\theta \in \mathcal{H}$. Therefore $A\xi \in R(Q)^\perp$, or, equivalently, $\xi \in A^{-1}(R(Q)^\perp)$. Conversely, suppose that $N(Q) \subseteq A^{-1}(R(Q)^\perp)$ and consider $\xi, \eta \in \mathcal{H}$. Decompose $\xi = \nu + \rho$ and $\eta = \nu' + \rho'$, where $Q\rho = \rho, Q\rho' = \rho'$ and $Q\nu = Q\nu'$ = 0. Then $\langle AQ\xi, \eta \rangle = \langle AQ\rho, \nu' + \rho' \rangle = \langle A\rho, \rho' \rangle$ and $\langle Q^*A\xi, \eta \rangle = \langle A\rho, Q(\nu' + \rho') \rangle = \langle A\rho, \rho' \rangle$. Thus $AQ = Q^*A$. □

Observe that two projectors $Q_1, Q_2$ on $\mathcal{H}$ such that $R(Q_1) = R(Q_2)$ and $N(Q_1) \subseteq N(Q_2)$ are equal: every $\xi \in \mathcal{H}$ can be written as $\xi = \rho + \nu$ with $\rho \in R(Q_1), \nu \in N(Q_1)$; then $Q_1\xi = \rho$ and $Q_2\xi = \rho + Q_2\nu = \rho$ because $\nu \in N(Q_1) \subseteq N(Q_2)$. Using this remark, we prove the next result.

**Corollary 2.6.** The set $\mathcal{P}(A, S)$ is parametrized by the set of all direct complements of $S$ contained in $A^{-1}(S^\perp)$.

**Remark 2.7.** If $S \cap N(A) = \{0\}$ the pair $(A, S)$ is compatible if and only if $\overline{A(S)} \oplus S^\perp$ is closed. Indeed if $\mathcal{M}, \mathcal{N}$ are closed subspaces, then $\mathcal{M} + \mathcal{N}$ is closed if and only if $\mathcal{M}^\perp + \mathcal{N}^\perp$ is closed (see theorem 4.8 of [37]); if $(A, S)$ is compatible then $S \oplus A(S)^\perp = \mathcal{H}$, a fortiori $S + A(S)^\perp$ is closed. Then $S^\perp + \overline{A(S)}$ is closed. Moreover $S^\perp \cap \overline{A(S)} = (S + A(S)^\perp)^\perp = \{0\}$. Conversely, if $S^\perp \oplus \overline{A(S)}$ is closed, then $S^\perp + \overline{A(S)} = S^\perp + \overline{A(S)} = (S \cap A(S)^\perp)^\perp = (S \cap N(A))^\perp = \mathcal{H}$. Again, if $\mathcal{H} = S^\perp + \overline{A(S)}$ then $S + A(S)^\perp$ is closed and $(S + A(S)^\perp)^\perp = S^\perp \cap \overline{A(S)} = \{0\}$.

The following remarks may be helpful to understand the meaning of compatibility. With the $2 \times 2$ matrix representation mentioned above, if $Q$ is a projector onto $S$ then
$Q \in \mathcal{P}(A, S)$ if and only if
\[
\begin{pmatrix}
a & b \\
b^* & c
\end{pmatrix}
\begin{pmatrix}
1 & e \\
x^* & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
x & 0
\end{pmatrix}
\begin{pmatrix}
a & b \\
b^* & c
\end{pmatrix}.
\]

It is easy to see that the four equations reduce to a single one, namely, $ax = b$. By Douglas theorem, $ax = b$ has a solution if and only if $R(b) \subseteq R(a)$ and, in this case, there is a unique $d \in L(S^\perp, S)$ such that $ad = b$ and $R(d) \subseteq R(a)$.

As we saw, if $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in L(\mathcal{H}^+) \text{ then } R(a^{1/2}) \supseteq R(b)$. In general, $R(a) \subseteq R(a^{1/2}) \subseteq \overline{R(a)}$. Then, there is no much place for $a, b$ to satisfy $R(b) \subseteq R(a^{1/2})$ and not satisfy $R(b) \subseteq R(a)$. In fact, given $S$, the set $\mathcal{Y}_S = \{ B \in L(\mathcal{H}^+) : (B, S) \text{ is compatible} \}$ is everywhere dense in $L(\mathcal{H})^+$. Moreover, $GL(\mathcal{H})^+$ is dense in $L(\mathcal{H})^+$ and $GL(\mathcal{H})^+ \subseteq \mathcal{Y}_S$.

Indeed, from the comments above, if $A \in GL(\mathcal{H})^+$, then $a \in GL(S)^+$, so that the equation $ax = b$ has the unique solution $x = a^{-1}b$. Then $\mathcal{P}(A, S) = \{ P_{A, S} \}$, where
\[
P_{A, S} = \begin{pmatrix} 1 & a^{-1}b \\ 0 & 0 \end{pmatrix}.
\]

The following result, which contains another parametrization of $\mathcal{P}(A, S)$, in terms of the set of solutions in $L(S^\perp, S)$ of the equation $ax = b$, follows from the remarks above.

**Theorem 2.8.** The pair $(A, S)$ is compatible if and only if $R(b) \subseteq R(a)$. In this case
\[
\mathcal{P}(A, S) = \{ P + PV(I - P) : V \in L(S^\perp, S), PAPV = PA|_{S^\perp} \} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} : ax = b \right\}.
\]

We summarize the conditions which are equivalent to compatibility in the next statement:

**Theorem 2.9.** Given a closed subspace $S$ of $\mathcal{H}$ and a positive operator $A$ on $\mathcal{H}$, the following conditions are equivalent:

1. $\mathcal{P}(A, S)$ is non-empty;
2. $S + S^\perp_A = \mathcal{H}$;
3. there exists a closed subspace $W \subseteq S^\perp_A$ such that $S \oplus W = \mathcal{H}$;
4. for the representation $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ of $A$ under the decomposition $\mathcal{H} = S \oplus S^\perp$, we have $R(b) \subseteq R(a)$.

**Example 2.10.** If $A \in L(\mathcal{H})^+$ has a dense non-closed image in $\mathcal{H}$, then
\[
B = \begin{pmatrix} A & A^{1/2} \\ A^{1/2} & I \end{pmatrix}
\]
belongs to $L(\mathcal{H} \oplus \mathcal{H})^+$ because $B = TT^*$ for $T : H \rightarrow \mathcal{H} \oplus \mathcal{H}$ defined by $T \xi = (A^{1/2} \xi, \xi)$. On the other hand, $R(A^{1/2})$ properly contains $R(A)$, so that $B$ and $\mathcal{H} \oplus \{0\}$ are not compatible. In the same order of ideas, let $C = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in L(\mathcal{H} \oplus \mathcal{H})^+$. Then $(C, \mathcal{H} \oplus \{0\})$ is a compatible pair and $R(C) = R(A)$ is non-closed.
Suppose that \((A, S)\) is compatible. Define \(P_{A, S}\) the unique member of \(\mathcal{P}(A, S)\) determined by the reduced solution \(d\) of \(ac = b\): \(P_{A, S} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}\). Then \(\mathcal{P}(A, S)\) is an affine manifold identified with \(\{T \in L(\mathcal{H}) : T|_S = 0, T(S^\perp) \subseteq N\}\). In particular, \(\mathcal{P}(A, S)\) has a unique element if and only if \(N = \{0\}\). If \(N \neq \{0\}\), then \(\|P_{A, S}\| \leq \|Q\|\) for all \(Q \in \mathcal{P}(A, S)\). For a proof of these facts, see \([19]\).

**Theorem 2.11.** Let \(A\) and \(S\) be compatible. Denote by \(N = (AS)^\perp \cap S = N(A) \cap S\). Then \(N(P_{A, S}) = (AS)^\perp \cap N\).

**Proof.** Both projectors have the same image, namely \(S\). It suffices to show that \(N(P_{A, S}) \subseteq (AS)^\perp \cap N\). Recall that \(P_{A, S} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}\) where \(A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}\) and \(d\) is the reduced solution of \(ax = b\), i.e., \(ad = b\) and \(R(d) \subseteq \overline{R(a)}\). If \(\xi = \sigma + \sigma^\perp \in N(P_{A, S})\) then \(\sigma + d\sigma^\perp = 0\) and \(\xi = -d\sigma^\perp + \sigma^\perp\). We must prove \(-d\sigma^\perp + s^\perp \in W = (AS)^\perp \cap N\).

First, let us show \(-d\sigma^\perp + \sigma^\perp \in (AS)^\perp\); or, equivalently, that \(A(-d\sigma^\perp + \sigma^\perp) \in S^\perp\); but \((-d\sigma^\perp + \sigma^\perp) = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \begin{pmatrix} -d\sigma^\perp \\ \sigma^\perp \end{pmatrix} = \begin{pmatrix} 0 \\ -b^*d\sigma^\perp + c\sigma^\perp \end{pmatrix} \in S^\perp\). Next, we must show that \(-d\sigma^\perp + \sigma^\perp \in (S \cap N(A))^\perp\). By the definition of \(d\), \(-d\sigma^\perp = \lim a\sigma_n\) for some sequence \(\{\sigma_n\}\) in \(S\). Given \(\sigma \in S \cap N(A), a\sigma = A\sigma = 0\), so that \((-d\sigma^\perp + \sigma^\perp, \sigma) = \langle -d\sigma^\perp, \sigma \rangle = \lim \langle a\sigma_n, \sigma \rangle = \lim \langle a\sigma_n, a\sigma \rangle = 0\). This finishes the proof.

**Remark 2.12.** Under additional hypotheses on \(A\), other characterizations of compatibility and formulas for \(P_{A, S}\) can be used. We mention a sample of these, taken from \([19]\) and \([20]\):

1. If \(R(PAP)\) is closed (or, equivalently, if \(R(PA^{1/2})\) or \(A^{1/2}(S)\) are closed), then \((A, S)\) is compatible. Indeed, if \(A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}\), the positivity of \(A\) implies that \(R(b) \subseteq R(a^{1/2})\) (see, e.g., \([4]\)). If \(R(PAP) = R(a)\) is closed, then \(R(b) \subseteq R(a^{1/2}) = R(a)\) so that \((A, S)\) is compatible, by Theorem 2.8. In this case,

   \[
   P_{A, S} = \begin{pmatrix} 1 & a^\dagger b \\ 0 & 0 \end{pmatrix},
   \]

since \(a = PAP\) has closed range, and \(a^\dagger b\) is the reduced solution of \(ax = b\). In particular, if \(N = N(a) = N(A) \cap S = \{0\}\) (i.e. \(R(a) = S\)), one gets

   \[
   P_{A, S} = (PAP)^\dagger PA.
   \]

Otherwise, \(P_{A, S} = P_N + (PAP)^\dagger PA\).

2. If \(A\) has closed range then the following conditions are equivalent:
   (a) The pair \((A, S)\) is compatible.
   (b) \(R(PAP)\) is closed.
   (c) \(R(AP)\) is closed.
   (d) \(S^\perp + R(A)\) is closed.

3. If \(P, Q\) are orthogonal projectors with \(R(P) = S\), then \((Q, S)\) is compatible if and only if \(R(QP)\) is closed. Moreover, if \((Q, S)\) is compatible, then \(\mathcal{H} = S + Q^{-1}(S^\perp) = \)
Remark 2.13. Consider the following conditions:

1. The pair \((A, S)\) is compatible;
2. \(A(S)\) is closed in \(R(A)\);
3. \(A^{-1}(\overline{A(S)}) = S + N(A)\);
4. \(A^{1/2}(S)\) is closed in \(R(A^{1/2})\);
5. \(S + N(A)\) is closed;
6. \(P_{\overline{R(A)}}(S)\) is closed, where \(P_{\overline{R(A)}}\) is the orthogonal projector onto \(\overline{R(A)}\).

A precise description of the relationships among them is provided by the implications: 

1 \(\leftrightarrow\) 2 \(\iff\) 3 \(\rightarrow\) 4 \(\rightarrow\) 5 \(\leftrightarrow\) 6. Moreover, \((A, S)\) is compatible if and only if \(P_{\overline{R(A)}}(S)\) is closed and \((A, P_{\overline{R(A)}}(S))\) is compatible.

The next result is a characterization of compatibility in terms of orthogonal decompositions of \(R(A)\) and \(R(A^{1/2})\).

Proposition 2.14. Given \(A \in L(H)^+\), the following conditions are equivalent:

1. The pair \((A, S)\) is compatible.
2. \(R(A) = A(S) \oplus S^\perp \cap R(A)\).
3. \(R(A^{1/2}) = A^{1/2}(S) \oplus A^{1/2}(S)^\perp \cap R(A^{1/2})\).
4. If \(M = A^{1/2}(S)\), then \(R(P_M A^{1/2}) \subseteq R(A^{1/2} P)\).

Proof. 1 \(\leftrightarrow\) 2: If \((A, S)\) is compatible then \(H = S + A^{-1}(S^\perp)\) so that 

\[
R(A) = A(S) + A(A^{-1}(S^\perp)) = A(S) + S^\perp \cap R(A);
\]

conversely, if \(R(A) = A(S) \oplus S^\perp \cap R(A)\), then \(H = A^{-1}(R(A)) = A^{-1}(A(S)) + A^{-1}(S^\perp \cap R(A))\). But \(A^{-1}(S^\perp \cap R(A)) = A^{-1}(S^\perp)\) and \(A^{-1}(A(S)) = S + N(A)\), so that \(H = S + N(A) + A^{-1}(S^\perp) = S + A^{-1}(S^\perp)\), because \(N(A) \subseteq A^{-1}(S^\perp)\).

1 \(\leftrightarrow\) 3: similar to (1) \(\leftrightarrow\) (2).

3 \(\leftrightarrow\) 4: If \(y \in R(A^{1/2})\) then \(y = y_1 + y_2\) for unique \(y_1 \in A^{1/2}(S)\) and \(y_2 \in A^{1/2}(S)^\perp\); but, then, \(P_M(y) = y_1 \in A^{1/2}(S) = R(A^{1/2} P)\). The converse is similar. \(\blacksquare\)

As a consequence of Proposition 2.14, it is easy to see that \((A, S)\) is compatible if and only if \(A^{1/2}(S)\) is closed in \(R(A^{1/2})\) and

\[
R(A^{1/2}) = A^{1/2}(S) \cap R(A^{1/2}) \oplus A^{1/2}(S)^\perp \cap R(A^{1/2}).
\]

More generally, given a closed subspace \(S\) of \(H\) and \(W = A^{-1/2}(A^{1/2}(S))\), then \((A, W)\) is compatible if and only if \(R(A^{1/2}) = A^{1/2}(S) \cap R(A^{1/2}) \oplus A^{1/2}(S)^\perp \cap R(A^{1/2})\); in fact, if \((A, W)\) is compatible then, by Proposition 2.14, \(R(A^{1/2}) = A^{1/2}(W) + A^{1/2}(W)^\perp \cap
R(A^{1/2}). On one hand, \(A^{1/2}(W) = \overline{A^{1/2}(S)} \cap R(A^{1/2})\); on the other hand, since \(A^{1/2}(S) \subseteq A^{1/2}(W) \subseteq \overline{A^{1/2}(S)}\), we get \(A^{1/2}(S)^\perp = A^{1/2}(W)^\perp\). Thus,

\[
R(A^{1/2}) = \overline{A^{1/2}(S)} \cap R(A^{1/2}) + A^{1/2}(S)^\perp \cap R(A^{1/2}),
\]

and, of course, the sum is direct. The converse is similar.

A notion which is naturally related to oblique projectors is that of angle between subspaces. We consider here two non-equivalent definitions of angles and we show a characterization of the compatibility of \((A, S)\) in terms of these angles. For excellent treatments on angles in Hilbert spaces the reader is referred to the survey by Deutsch [24] or the book by A. Ben-Israel and T. N. E. Greville [12].

Given two subspaces \(S, T\), the cosine of the Friedrichs angle between them is defined by

\[
c(S, T) = \sup\{\langle \xi, \eta \rangle : \xi \in S \cap (S \cap T)^\perp, ~ \|\xi\| < 1, ~ \eta \in T \cap (S \cap T)^\perp, ~ \|\eta\| < 1\}.
\]

It is well known (see Theorem 13 of [24]) that the following conditions are equivalent:

1. \(c(S, T) < 1\);
2. \(S + T\) is closed;
3. \(S^\perp + T^\perp\) is closed;
4. \(c(S^\perp, T^\perp) < 1\).

The formulas \(\|P_S P_T\| = c(S, T)\) [24] and \(\|P_{S^\perp T^\perp}\| = (1 - c(T, S)^2)^{-1/2}\) [49] relate this notion with oblique projectors.

The minimal angle between \(S\) and \(T\) is the angle whose cosine is defined by

\[
c_o(S, T) = \sup\{\langle \xi, \eta \rangle : \xi \in S, ~ \|\xi\| < 1, ~ \eta \in T, ~ \|\eta\| < 1\}.
\]

Observe that \(c(S, T) \leq c_o(S, T)\) and \(c(S, T) = c_o(S, T)\) when \(S \cap T = \{0\}\).

**Theorem 2.15.** Consider \(A \in L(H)^+\). Then \((A, S)\) is compatible if and only if \(c_o(S^\perp, A(S)^\perp) < 1\).

**Proof.** If \((A, S)\) is compatible then \(H = S + A^{-1}(S^\perp)\), so that \(S + A^{-1}(S^\perp)\) is closed. By the remarks above and the identity \(A^{-1}(S^\perp) = (AS)^\perp\), we get \(c(S, A^{-1}(S^\perp)) < 1\) or equivalently, \(c(S^\perp, A(S)^\perp) < 1\). But \(S^\perp \cap A(S)^\perp = (S + A^{-1}(S^\perp))^\perp = H^\perp = \{0\}\). Therefore, \(c_o(S^\perp, A(S)^\perp) = c(S^\perp, A(S)^\perp) < 1\).

Conversely, if \(c_o(S^\perp, A(S)^\perp) < 1\) then \(S^\perp \cap A(S)^\perp = \{0\}\) and \(S^\perp + A(S)^\perp\) is closed; therefore, \(S + A(S)^\perp\) is closed; also \((S + A(S)^\perp)^\perp = S^\perp \cap A(S)^\perp = \{0\}\). Then \(S + A(S)^\perp = H\) and \((A, S)\) is compatible. \(\square\)

**Remark 2.16.**

1. If \(A\) has closed range then, by Remark 2.12, the pair \((A, S)\) is compatible if and only if \(R(AP)\) is closed. Note that this is equivalent to the angle condition \(c(N(A), S) < 1\).
2. If \(P, Q\) are orthogonal projectors with \(R(P) = S\), define \(N = N(Q) \cap S\) and \(M = S \ominus N\). Then, again by Remark 2.12,

\[
\|P_{Q,M}\| = \|P_{Q,M}\| = (1 - \|(1 - Q)P_M\|^2)^{-1/2} = (1 - c(N(Q), S)^2)^{-1/2}.
\]
3. Formulas for $P_{A,S}$. This section is devoted to presenting several explicit formulas for $P_{A,S}$ in terms of the orthogonal projectors onto $S$, $W = A(S)\perp \ominus (S \cap N(A))$ and $W\perp$. Afriat [1], Greville [32] and Pták [49] have proven this type of formulas, the first two in finite dimensional settings. Some of these formulas seem to have been known by V. E. Ljance [43]. Consider $A \in L(\mathcal{H})^+$ and $S$ a closed subspace of $\mathcal{H}$ such that $(A,S)$ is compatible. Denote $\mathcal{N} = S \cap A(S)\perp = S \cap N(A)$ and $W = A(S)\perp \ominus \mathcal{N}$. Then, as shown in Theorem 3.5 of [22], $W$ is the kernel of $P_{A,S}$ so that $P_{A,S} = P_S/\perp W$, the oblique projector onto $S$, along $W$. Afriat [1] and Greville [32] exhibited formulas for an oblique projector $Q$ in terms of the orthogonal projectors onto $R(Q)$ and $N(Q)$, by using the Moore-Penrose pseudoinverse. However, in order to use the same method in our infinite dimensional setting we need to know that the operator whose Moore-Penrose pseudoinverse is considered has closed range [23]. This justifies the need of a proof for the first part of the next result. The rest of the proof follows without change Greville’s arguments.

Lemma 3.1.

1. $(A,S)$ is compatible if and only if $P_{W\perp}P_S$ has closed range.
2. If the pair $(A,S)$ is compatible then
   \begin{enumerate}
   \item[(a)] $P_{A,S} = (P_{W\perp}P_S)^\dagger$.
   \item[(b)] $P_{A,S} = (I - P_SP_W)^{-1}P_S(I - P_WP_W)$.
   \item[(c)] $P_{A,S} = (I - P_WP_S)^{-1}(I - P_W) = P_S(P_S + P_W - P_WP_S)^{-1}$.
   \end{enumerate}

Proof. If $(A,S)$ is compatible then $\mathcal{H} = S \oplus W$ by the remarks above. Observe first that $R(P_{W\perp}P_S) = W\perp$: for this, it suffices to show the inclusion $W\perp \subseteq R(P_{W\perp}P_S)$, because the converse is evident. If $\xi \in W\perp$, then $\xi$ decomposes as $\xi = \sigma + \omega$, $\sigma \in S$ and $\omega \in W$, so that $\xi = P_{W\perp}x = P_{W\perp}\sigma \in P_{W\perp}S = R(P_{W\perp}P_S)$.

Conversely, if $P_{W\perp}P_S$ has closed range then $(P_{W\perp}P_S)^\dagger$ is a bounded linear operator. Greville’s arguments for matrices [32] can be used almost without changes to prove that $(P_{W\perp}P_S)^\dagger$ is an idempotent with range $S$ and kernel $W$. Then $\mathcal{H} = S \oplus W = S + A(S)\perp$ and $(A,S)$ is compatible. The formulas of part 2 follow from the fact that $P_{A,S} = P_S/\perp W$, using Theorem 2.2.

Corollary 3.2. If the pair $(A,S)$ is compatible and $\mathcal{N} = \{0\}$ then $P_{A,S} = (P_{A(S)}^\perp P_S)^\dagger = (I - P_SP_{A(S)\perp})^{-1}P_S(I - P_SP_{A(S)\perp}) = (I - P_{A(S)\perp}P_S)^{-1}(I - P_{A(S)\perp})$.

The shortened operator of $A$ to $S$ is $A/\perp S = \sup\{X \in L(\mathcal{H})^\dagger : X \leq A$ and $R(X) \subseteq S\}$. In [48], Pekarev proved $A/\perp S = A^{1/2}P_M A^{1/2}$, where $M = A^{1/2}(S)$. Let us show a formula for $P_{A,S}$ in the spirit of Pekarev’s. The relationship between the projectors in $P(A,S)$ and $A/\perp S$ is given by the formula $A/\perp S = AE$, which holds for every projector $E$ such that $I - E \in P(A,S)$ (see [19]). In particular $A/\perp S = A(I - P_{A,S})$ and, if $A$ were invertible, we can compute $P_{A,S} = A^{-1}(A - A/\perp S) = A^{-1/2}P_MA^{1/2}$.

In order to get a generalization of this formula, we consider firstly the injective case:
Proposition 3.3. Let $A \in L(\mathcal{H})^+$ injective such that $(A, S)$ is compatible. Then

$$P_{A, S} = A^{-1/2}P_M A^{1/2}$$

where $M = \overline{A^{1/2}(S)}$.

Proof. Observe that in this case $P(A, S) = \{P_{A, S}\}$ because $S \cap N(A) = \{0\}$. Define $Q = A^{-1/2}P_M A^{1/2}$. Then $Q$ is well defined because $A^{-1/2} : R(A^{1/2}) \to \mathcal{H}$ and $R(P_M A^{1/2}) \subseteq R(A^{1/2})$, by Theorem 2.14. It is easy to see that $Q^2 = Q$ and that $N(Q) = A(S)^\perp$: in fact, $\xi \in N(Q)$ if and only if $P_M A^{1/2} \xi = 0$, i.e., $A^{1/2} \xi \in A^{1/2}(S)^\perp$, or, what is the same, $\xi \in A^{-1/2}(A^{-1/2}(S)^\perp) = A^{-1}(S)^\perp$. On the other hand, by the definition of $Q$, $R(Q) \subseteq A^{-1/2}(M) = A^{-1/2}(A^{1/2}(S) \cap R(A^{1/2})) = A^{-1/2}(A^{1/2}(S)) = S$ because, by Theorem 2.13, $A^{1/2}(S)$ is closed in $R(A^{1/2})$: this proves that $R(Q) \subseteq S$. Conversely, if $\sigma \in S$, then $Q \sigma = A^{-1/2}P_M A^{1/2} \sigma = \sigma$. Then $R(Q) = S$ and $Q = P_{A, S}$. ■

We generalize this formula to any (not necessarily injective) $A \in L(\mathcal{H})^+$. For $B \in L(\mathcal{H})^+$ denote

$$B^\sharp = (B|_{R(B)})^{-1} : R(B) \to \overline{R(B)} \subseteq \mathcal{H}.$$ 

Observe that $B^\sharp$ is a linear, not necessarily bounded operator. If $R(B)$ is closed, then $B^\sharp P_{R(B)} = B^1$.

Proposition 3.4. Consider $A \in L(\mathcal{H})^+$ such that $(A, S)$ is compatible. Set $M = \overline{A^{1/2}(S)}$.

1. If $S \subseteq \overline{R(A)}$ then $P_{A, S} = (A^{1/2})^\sharp P_M A^{1/2}$.
2. If $S \cap N(A) = \{0\}$ then $P_{A, S} = (P_{R(A)}P_S) \dagger P_A P_{R(M)}(S) = (P_{R(A)}P_S) \dagger (A^{1/2})^\sharp P_M A^{1/2}$.

Proof. Observe that $P(A, S) = \{P_{A, S}\}$ because $S \cap N(A) = \{0\}$ in both cases.

1. If $S \subseteq \overline{R(A)}$ and $Q = (A^{1/2})^\sharp P_M A^{1/2}$ then $Q$ is well defined because $P_M(R(A^{1/2})) \subseteq R(A^{1/2})$, by Proposition 2.14. On one hand $P_M(R(A^{1/2})) \subseteq M \cap R(A^{1/2}) = A^{1/2}(S)$, because, by Remark 2.13, $A^{1/2}(S)$ is closed in $R(A^{1/2})$ thus, $R(Q) \subseteq (A^{1/2})^\sharp A^{1/2}(S) = S$. On the other hand, $Q \sigma = \sigma$, for all $\sigma \in S$, because $S \subseteq \overline{R(A)}$. Then $R(Q) = S$. It is easy to see that $N(Q) = A^{-1}(S)^\perp$; thus, $Q = P_{A, S}$.

2. If $S \cap N(A) = \{0\}$ then the subspace $S' = P_{R(A)}(S)$ is closed because $(A, S)$ is compatible, $S' \subseteq \overline{R(A)}$ and $(A, S')$ is compatible (see Proposition 2.13). Also $A^{1/2}(S') = \overline{A^{1/2}(S)} = M$, so that $P_{A, S'} = (A^{1/2})^\sharp P_M A^{1/2}$. Now, $R(P_{R(A)}P_S) = S'$ is closed and $N(P_{R(A)}P_S) = S^\perp$: the proof is straightforward. ■

In the general case the set $P(A, S)$ can be parametrized by means of the set of complements $\mathcal{L}$ of $N = S \cap N(A)$ in $S$. More precisely:

Proposition 3.5. Let $Q \in Q$ and consider $A \in L(\mathcal{H})^+$ such that $(A, S)$ is compatible. Let $N = S \cap N(A)$. Then $Q \in P(A, S)$ if and only if there exists a (unique) closed subspace $\mathcal{L} \subseteq S$ such that $S = N \oplus \mathcal{L}$, $\mathcal{L} + N(Q)$ is closed, $S + N(Q) = \mathcal{H}$ and

$$Q = P_{A, \mathcal{L}} + P_{N/((\mathcal{L} + N(Q)))}.$$

Proof. $\Leftarrow$ Observe that $N + \mathcal{L} + N(Q) = S + N(Q) = \mathcal{H}$ and $\mathcal{L} + N(Q)$ is closed so that $Q' = P_{N/((\mathcal{L} + N(Q)))}$ is a well defined (oblique) projector. If $Q = P_{A, \mathcal{L}} + Q'$ then it is easy to see that $Q \in P(A, S)$.  


⇒) Consider $Q \in \mathcal{P}(A, \mathcal{S})$ and let $W = P_N Q$; then $R(W) = \mathcal{N}$. From $Q P_N = P_N$ we get that $W^2 = W$. Let $T = Q - W$; $T$ is $A$-selfadjoint because $Q$ and $W$ are both $A$-selfadjoint; equality $T^2 = T$ follows from $Q W = W = W Q$. Therefore $Q = T + W$, with $T^2 = T$ and $W^2 = W$. Let $\mathcal{L} = \mathcal{S} \cap N(W)$. It follows easily that $T = P_{A, \mathcal{L}}$, $\mathcal{S} = \mathcal{L} + \mathcal{N}$ and $N(W) = \mathcal{L} + N(Q)$.

Let $C \in L(\mathcal{H})$ such that $R(C) = \mathcal{S}$ is closed, and $A \in L(\mathcal{H})^+$ with closed range. Formula (3) suggests the natural generalization, which is widely used in the finite dimensional case:

$$\tag{4} P_{A,S} = C(C^* A C)^+ C^* A.$$  

In general, the formula is false for many reasons. For instance, $(C^* A C)^+$ is unbounded if $R(C^* A C)$ is not closed; or $C(C^* A C)^+ C^* A$ may have range strictly contained in $\mathcal{S}$. However, the wide range of applications of the right side of formula (4) makes it desirable to establish its exact relationship with $P_{A,S}$. In fact, projectors like $C(C^* A C)^+ C^* A$ appear explicitly in papers on scaled projections [53], [46], [34], [31], [60], [14], linear least squares problems [28], [29], linear feasibility [28], [29], [17], signal processing [36], [10], [58] and so on.

A first observation is that one needs to verify if $R(C^* A C)$ is closed. An interesting fact, which generalizes item 2 of Remark 2.12, is that $R(C^* A C)$ is closed if and only if $(A, \mathcal{S})$ is compatible. Indeed, note that $R(C^* A C)$ is closed if and only if $R(A^{1/2} C C^* A^{1/2})$ is closed. Since $R(C) = \mathcal{S}$ is closed, there exist $a, b > 0$ such that $a P \leq C C^* \leq b P$, so that

$$a A^{1/2} P A^{1/2} \leq A^{1/2} C C^* A^{1/2} \leq b A^{1/2} P A^{1/2}.$$  

This implies, by the Douglas theorem, the identity

$$R((A^{1/2} C C^* A^{1/2})^{1/2}) = R(A^{1/2} P) = A^{1/2}(S),$$

which is closed if and only if $(A, \mathcal{S})$ is compatible, by Remark 2.12.

Suppose now that $(A, \mathcal{S})$ is compatible. If $\mathcal{N} = N(A) \cap \mathcal{S}$, we shall see that

$$\tag{5} P_{A,S} = P_N + C(C^* A C)^+ C^* A,$$

showing that formula (4) holds if and only if $N(A) \cap \mathcal{S} = \{0\}$.

Define $Q = C(C^* A C)^+ C^* A$. It is clear that $Q^2 = Q$, $R(Q) \subseteq R(C) = \mathcal{S}$ and $A Q = Q^* A$. Therefore, $Q$ is an $A$-selfadjoint projector onto a subspace of $\mathcal{S}$. Also, since $C$ and $(C^* A C)^+$ are injective on $R(C^*)$,

$$N(Q) = N(C^* A) = A^{-1}(N(C^*)) = A^{-1}(S^\perp).$$

The next step is to show that $R(Q) = \mathcal{S} \ominus \mathcal{N}$. Note that

$$R(C^* A) = C^*(R(A)) = C^*(R(A^{1/2})) = R(C^* A^{1/2}).$$

Hence $R(C^* A) = R((C^* A C)^+ C^* A^{1/2})$ and $R(Q) = R(C C^* A^{1/2}) = R(C C^* A)$. But

$$N(ACC^*) = N(C C^*) \perp (R(C C^*) \cap N(A)) = S^\perp \perp \mathcal{N},$$

so that $R(Q) = N(ACC^*) = \mathcal{S} \ominus \mathcal{N}$, as claimed. This fact clearly shows that $Q \in \mathcal{P}(A, \mathcal{S} \ominus \mathcal{N}) = \{P_{A,S \ominus \mathcal{N}}\}$ (since $(\mathcal{S} \ominus \mathcal{N})^+ = S^\perp$) and also proves formula (5).
It is shown in [6] that for every projector $Q$ onto a closed subspace $S$, there exists an invertible positive $A \in L(H)$ such that $Q = PA, S$. This can be rewritten as follows:

**Proposition 3.6.** Let $S \in H$ be a closed subspace and $C \in L(H)$ with $R(C) = S$. Let $A \in L(H)^+$ with closed range. Then

1. $(A, S)$ is compatible if and only if $R(C^*AC)$ is closed.
2. If $N(A) \cap S = \mathcal{N}$, then
   \[ PA, S = P_N + C(C^*AC)^\dagger C^*A. \]
3. For every $Q \in L(H)$ such that $Q^2 = Q$ and $R(Q) = S$, there exists an invertible positive $A \in L(H)$ such that
   \[ Q = C(C^*AC)^{-1}C^*A. \]

**Final comments and open problems.** The structure of the set $\mathbb{T}_S = \{ A \in L(H)^+ : (A, S) \text{ is compatible} \}$ is not completely known. We have observed that $GL(H)^+$ is contained in $\mathbb{T}_S$. Of course, if $S$ is finite-dimensional, then $\mathbb{T}_S = L(H)^+$.

The extension of compatibility questions to Hermitian operators instead of positive operators is a much more difficult problem. The reader can find in [35], [19] and [44] some results in this direction.

Compatibility is related to some problems arising from wavelet and frame theory. The paper [7] deals with some problems in this area.

A difficult and very useful problem consists in determining conditions which ensure the convergence of sequences like \{$PA_n, S$\} and \{$P_{A_n, S}$\}. A sample of this type of results can be found in [21].

Given $Q \in Q^S$, it is known that $\chi_Q = \{ A \in L(H)^+ : Q \in \mathcal{P}(A, S) \}$ is not empty and the set $\chi_Q \cap GL(H)^+$ is characterized [6]. However, in general, the structure of $\chi_Q$ is unknown and it would be interesting to have optimality criteria for choosing $A \in \chi_Q$.

**References**


