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Though the relation of Fourier series with topological algebras is rather loose, I should like to contribute to this volume in honour of Wiesław Żelazko by reviewing a few facts on two questions:

1. how do the partial sums of Fourier-Lebesgue series behave?
2. when is a closed subalgebra of $L^1(\mathbb{T})$ (convolution algebra) generated by the idempotents which it contains?

In both cases the classical frame is $L^1(\mathbb{T})$, and I shall say a few words on the slightly simpler analogue, $L^1(\mathbb{D})$, where $\mathbb{D} = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$.

1. Partial sums. Convergence and divergence of Fourier series, and more generally the behaviour of their partial sums $S_n(t)$, can be considered in three ways: at a given point t, or almost everywhere, or everywhere. Already Lebesgue established that $S_n(t) = \mathrm{o}(\log n)$ $(n \to \infty)$ at each given point t and uniformly when the function is integrable and bounded $(f \in L^\infty(\mathbb{T}))$ and that it is a best possible result [14, 15]. Hardy proved that $S_n(t) = \mathrm{o}(\log n)$ $(n \to \infty)$ almost everywhere when $f \in L^1(\mathbb{T})$ and conjectured that it is also a best possible result ("I have not proved rigorously that it is so, but it seems to me very probable" ([4], § 5 and 6). In Hardy's Collected Papers the comment after Hardy's article (volume III, p. 125) says that Hardy's result had been proved earlier by Lebesgue, and that it has been shown that the result $S_n = \mathrm{o}(\log n)$ is best possible; both statements are wrong, and Hardy's conjecture is still open.

Convergence almost everywhere holds when $f \in L^2(\mathbb{T})$: this is Carleson's theorem (1966) [3]. It extends to $f \in L^p(\mathbb{T})$ when $p > 1$ (Hunt) and more generally when

$$\int_{\mathbb{T}} |f(t)| \log^+ |f(t)| \, \log^+ \log^+ \log^+ |f(t)| \, \mathrm{d}t < \infty$$

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(Antonov 1996 [1]). Sjölin and Soria (2003) extended this result to the case of $\mathbb{D}$ (series of Fourier-Walsh) and other orthogonal expansions [19].

On the other hand there exist integrable functions whose Fourier series diverge almost everywhere [7] and even everywhere [8]: these are Kolmogorov's examples of 1923 and 1926. About the rapidity of divergence the best result is due to Konyagin (1999 [9, 10]): as soon as

$$\lambda_n = \mathrm{o}(\sqrt{\log n}/\sqrt{\log\log n}) \quad (n \to \infty),$$

there exists $f \in L^1(\mathbb{T})$ such that

$$\overline{\lim_{n\to\infty}} \frac{S_n(t)}{\lambda_n} = \infty$$

everywhere.

Can Konyagin's result be improved in taking $\lambda_n = \mathrm{o}(\log n)$ (that would prove Hardy's conjecture) or Hardy's result be improved in proving that $S_n(t) = \mathrm{o}\big((\log n)^p\big)$ for some, or for all p such that $\frac{1}{2} < p < 1$? It seems worth trying to improve Hardy's result. Since Fourier-Walsh series may be more manageable than ordinary Fourier series, I made an attempt in that direction, that is, in considering $L^1(\mathbb{D})$ instead of $L^1(\mathbb{T})$. This attempt is unsuccessful but supports the idea that $\sqrt{\log n}$ may play a role in the estimate. Here is what I know.

Notations:

$$X = \mathbb{D} = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$$
$$x = (x_0, x_1, \ldots) \in X$$
$$\mathrm{d}x = \text{Haar measure, also } m(\mathrm{d}x)$$
$$r_j(x) = (-1)^{x_{j-1}}, \text{ Rademacher functions } \ (j = 1, 2, \ldots)$$
$$w_0 = 1,\ w_1 = r_1,\ w_2 = r_2,\ w_3 = r_1 r_2,\ w_4 = r_3,\ \ldots$$
$$w_n = r_{j_1} \ldots r_{j_n} \text{ when } \ n = 2^{j_1 - 1} + \ldots + 2^{j_\nu - 1},\ j_1 > j_2 > \ldots > j_\nu \geq 1,$$
$$\text{Walsh functions } \ (n = 0, 1, \ldots)$$
$$f \in L^1(X)$$
$$c_n = \int f w_n = \int_X f(x) w_n(x)\, \mathrm{d}x$$
$$S(f) = \sum_{n=0}^{\infty} c_n w_n$$
$$S_N(f, x) = \sum_{n=0}^{N-1} c_n w_n(x)$$

These are the partial sums under investigation. The partial sums of order 2^k constitute a dyadic martingale, f_k:

$$f_k(x) = S_{2^k}(f, x).$$

When $N = 2^k + l, 0 \leq l \leq 2^k$ the partial sum of order N satisfies

$$S_N(f, x) = f_k(x) + r_{k+1}(x) S_l(r_{k+1} f, x). \tag{1}$$

Assume first $f \geq 0$ and $\int f = 1$, let $\lambda > 1$ and

$$E(\lambda, f) = \{x : \sup_k f_k(x) \leq \lambda\}.$$

If we stop the martingale f_k as soon as $f_k > \lambda$ we obtain a bounded positive martingale $\widetilde{f}_k$ whose limit $\widetilde{f}$ satisfies $\int \widetilde{f} = 1$ and $\widetilde{f} > \lambda$ on $X \setminus E(\lambda, f)$, therefore $m(X \setminus E(\lambda, f)) < \frac{1}{\lambda}$ and

$$m\big(E(\lambda, f)\big) > 1 - \frac{1}{\lambda}. \tag{2}$$

Now let us assume only that f is real-valued, $f \in L^1(X)$ and $\int |f| = 1$. We point out that

$$-\lambda \leq f_k(x) \leq \lambda$$

for all k when $x \in E(\lambda, |f|)$, and the same holds when we replace f by g such that $|g| = |f|$. Let us write

$$E_k = E_k(\lambda, |f|),$$

and observe that E_k belongs to the σ-field generated by $r_1, r_2, \ldots r_k$, and $E_k \downarrow E$.

Given $N = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_\nu}, k_1 > k_2 > \cdots > k_\nu \geq 0$, we write

$$\nu = l(N) \quad (\leq \log_2 N).$$

Using (1) and (2) we have immediately the analogue of Hardy's result:

Proposition 1. *Given $\lambda > 1$, the inequality*

$$|S_N(f, x)| \leq \lambda l(N)$$

holds on the set $E = E(\lambda, |f|)$, whose Haar measure exceeds $1 - 1/\lambda$.

Corollary 1. $S_n(f, x) = \mathrm{o}(\log N)$ $(N \to \infty)$ *a.e. when $f \in L^1(X)$.*

(As usual, the estimate o( ) is derived from O( ) by subtracting from f a Walsh polynomial).

Here is the way in which $\sqrt{\log N}$ appears.

Proposition 2. *For every $\lambda > 0$, $\mu > 0$ and N integer,*

$$m\{x \in E(\lambda, |f|) : |S_N(f, x)| > \mu\lambda\sqrt{l(N)}\} \ < 2\exp\left(-\frac{\mu^2}{2}\right).$$

Corollary 2. *Given any increasing sequence of integers of N_j,*

$$S_{N_j}(f, x) = \mathrm{o}(\sqrt{l(N_j)\log j}) \quad (j \to \infty) \quad a.e.$$

when $f \in L^1(X)$.

Remark. Corollary 2 implies Corollary 1 but it gives nothing more when $N_j = j$, the important case. Proposition 2 expresses that $S_N(f, x)$ has a subgaussian distribution on E, with standard deviation $\lambda\sqrt{N}$; it is better than Proposition 1 when $l(N)$ is large, but no so good for small values of $l(N)$ (in particular when $N = 2^k$).

Proof of Proposition 2. (1) can be written as $S_N = A + r_{k+1}B$, where A and B are measurable with respect to the σ-field generated by $r_1, r_2, \ldots, r_k$. Given $u > 0$, let us write

$$\operatorname{Ch} uS_N = \operatorname{Ch} uA \, \operatorname{Ch} uB + r_{k+1} \operatorname{Sh} uA \, \operatorname{Sh} uB$$

and integrate on E_k. We obtain

$$\int_{E_k} \operatorname{Ch} uS_N = \int_{E_k} \operatorname{Ch} uA \, \operatorname{Ch} uB \leq \operatorname{Ch} u\lambda \int_{E_k} \operatorname{Ch} uB$$

and, returning to the original notations, this reads

$$\int_{E_{k_1}} \operatorname{Ch}\big(uS_N(f)\big) \leq \operatorname{Ch} u\lambda \int_{E_{k_1}} \operatorname{Ch}\big(uS_l(r_{k+1}f)\big) \leq \operatorname{Ch} u\lambda \int_{E_{k_2}} \operatorname{Ch}\big(uS_l(r_{k+1}f)\big),$$

hence, iterating and using $E \subset E_{k_1}$, and $\nu = \ell(N)$,

$$\int_E \operatorname{Ch} uS_N \leq (\operatorname{Ch} u\lambda)^\nu \leq \exp \frac{1}{2}\nu u^2 \lambda^2,$$

$$m(E \cap \{|S_N| > \mu\lambda\sqrt{\nu}\}) \operatorname{Ch} u\mu\lambda\sqrt{\nu} \leq \exp \frac{1}{2}\nu u^2 \lambda^2$$

and, choosing $u = \mu(\lambda\sqrt{\nu})^{-1}$, we obtain the inequality of Proposition 2.

2. Subalgebras and idempotents. Partial sums are only one way to deal with Fourier series. Summability processes of different kinds can be used in order to obtain convergence at a point, or almost everywhere, or in functional spaces. As an example, the method of the first arithmetical mean gives convergence almost everywhere and convergence in $L^1(\mathbb{T})$ for every $f \in L^1(\mathbb{T})$ (Fejér-Lebesgue; see f.e. [18] chapter III).

For $L^1(\mathbb{D})$ we simply consider the $f_k = S_{2^k}$.

It may be interesting, in particular from a computational point of view, to rearrange a series according to the size of the terms. Then a summability process will be defined by a family of functions $\Phi_k : \mathbb{R}^+ \to \mathbb{R}^+$ (usually $\Phi_k = 0$ in a neighbourhood of 0), tending to 1 as $k \to \infty$. Given a series of complex numbers $\sum u_n$, we are looking for

$$\lim_{k\to\infty} \sum \Phi_k(|u_n|)u_n$$

whenever it has a meaning. Let us denote it by $(\Phi_k)\sum u_n$.

When we deal with series of functions, $\sum a_n\varphi_n(x)$, the usual "decreasing rearrangement" consists in rearranging the series so that the magnitude of the coefficients decreases. The new partial sums are or the form $\sum \Phi_\varepsilon(|a_n|)a_n\varphi_n(x)$, where $\Phi_\varepsilon = \mathbf{1}_{[\varepsilon,\infty[}$. When $|\varphi_n(x)| = 1$ for all n and x, the limit is $(\Phi_\varepsilon)\sum a_n\varphi_n(x)$ when it exists.

Again, it is possible to look for such a limit at a given point, or almost everywhere, or in a functional space. For Fourier series we mainly have negative results:

1. There exists $f \in L^2(\mathbb{T})$ whose Fourier series, when rearranged decreasingly, diverges unboundedly almost everywhere (Körner 1996 [11]).

2. There exists $f \in C(\mathbb{T})$ with the same property, and actually this is generic in $C(\mathbb{T})$, space of continuous functions on $\mathbb{T}$ (it holds on a countable intersection of open dense subsets of $C(\mathbb{T})$) (Körner 1999 [12]).

3. For all spaces $L^p(\mathbb{T})$ except $L^2(\mathbb{T})$, and for $C(\mathbb{T})$, there exist functions f in the space, with Fourier series $S(f)$, such that, whatever the process (Φ_k), the limit $(\Phi_k)S(f)$ does not exist in the space.

The last result can be expressed in the following way:

4. In the convolution algebras $L^p(\mathbb{T})$ $(p \neq 2)$ and $C(\mathbb{T})$ there exist closed subalgebras that are not generated by the idempotents they contain (Kahane 1966 [5], Rider 1969 [17], Kahane and Katznelson 1978[6], Bachelis and Gilbert 1979 [2], Oberlin 1982 [16]).

For $p = 1$ this answers a question of W. Rudin in his book of 1962 [18], p.231. It is possible to say a little more for $L^1(\mathbb{T})$ and for $C(\mathbb{T})$. To every closed subalgebra A is associated an equivalence relation R on $(\mathbb{Z})$, such that $n \sim m$ (R) means $\widehat{f_n} = \widehat{f_m}$ for all $f \in A$. Conversely, given an equivalence relation R on $\mathbb{Z}$, does there exist a unique closed subalgebra A having R as its associated equivalence relation? In other words, is the maximal subalgebra A^R consisting of the f such that $\widehat{f_n} = \widehat{f_m}$ when $n \sim m$ (R) the same as the minimal subalgebra, generated by the idempotents $\sum e_n$ $(n \in$ class of $R)$? Of course, the answer is positive if each finite class of R consists of a single point. Here is what is proved in [6].

5. In the case of $L^1(\mathbb{T})$ there is an equivalence relation on $\mathbb{Z}$, R, whose finite classes consist of one or two points, such that $A^R \neq A_R$. In the case of $C(\mathbb{T})$ there is an equivalence relation R whose finite classes consist of two points exactly, such that $A^R \neq A_R$.

The proof relies heavily on real and imaginary Riesz products, and cannot be translated to other $L^p(\mathbb{T})$ $(1 < p < \infty, p \neq 2)$. It can be adapted to the case of any compact abelian group instead of $\mathbb{T}$. For example, the equivalence relation R can be defined in the following way in the case of $C(\mathbb{D})$. We start from $S_0 = \{2^1, 2^2, 2^3, \ldots, 2^{100}\}$ and define by induction blocks of integers $S_j (j \geq 0)$ and pairs $\{n, \lambda\ (n)\}$, $n \in S_j$ $(j \geq 0)$; the $\{n, \lambda\ (n)\}$ will be the classes of R.

It suffices that for each $j \geq 0$ the $\lambda(n)$ are distinct powers of 2 larger that $\sup S_j$, and S_{j+1} is defined as the set of all sums $\sum \lambda(n)$ $(n \in S_j)$ containing at least two terms. In the case of $L^1(\mathbb{D})$ one has to add as individual points all finite sums $\sum \lambda(n)$ $(n \in \bigcup_0^\infty S_j)$ not considered previously.

Let me conclude by a remark. The problem on subalgebras and idempotents was appealing in the sixties, as part of the saga of spectral synthesis (how to reconstruct a space from building blocks, here the idempotents). It may reappear in the future, but it stimulated no new work in the last twenty years. In the meantime wavelets appeared, and, roughly speaking, all good spaces of functions have good spaces of coefficients and approximation by partial sums of wavelet expansions work as well as in L^2.

On the other hand, I indicated a series of references on the behaviour of partial sums, or rearranged partial sums, almost everywhere. This appears as a living subject, and a useful additional reference is the expository article of Körner (2001) [13], including the works on rearranged Fourier series by Olevskiĭ, and on rearranged wavelet series by Tao. For ordinary partial sums, the remaining gap between Antonov and Konyagin is a real challenge, including for $L^1(\mathbb{D})$, where the analogue of Konyagin's result was established quite recently [2′, 2″].

Final remarks. I am grateful to the referee for a careful reading and for the references to Bochkarev's works [2′, 2″]. The second paper obtains the analogue of Konyagin's result with $\lambda_n = o(\sqrt{\log n})$ only, therefore in a stronger form. The first paper announced the result almost everywhere, but it is mainly concerned with almost everywhere divergence when smoothness conditions are required (in the English version, read "diverges" instead of "converges" in the main theorem).

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