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QUASINORMAL OPERATORS ARE HYPERREFLEXIVE

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Dedicated to Professor Wiesław Żelazko on his 70th birthday

Abstract. We will prove the statement in the title. We also give a better estimate for the hyperreflexivity constant for an analytic Toeplitz operator.

Throughout the paper $\mathcal{H}$ will denote a complex separable Hilbert space. Let $B(\mathcal{H})$ be the algebra of all linear and bounded operators acting on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is called *quasinormal* if T commutes with T^*T. For any operator T we denote by $\mathcal{W}(T)$ the WOT closed algebra generated by T and the identity, by $\operatorname{Lat} T$ the lattice of all invariant subspaces for T, and by $\operatorname{AlgLat} T$ the algebra of all operators which leave elements of $\operatorname{Lat} T$ invariant. Denote by $d(A, \mathcal{W})$ the standard distance from an operator A to an algebra $\mathcal{W} \subset B(\mathcal{H})$ given by: $d(A, \mathcal{W}) = \inf\{\|A - T\| : T \in \mathcal{W}\}$.

An algebra $\mathcal{W} \subset B(\mathcal{H})$ is called *hyperreflexive* if there is a constant a such that

$$d(A, \mathcal{W}) \leq a \sup\{\|P^{\perp} A P\| : P \in \operatorname{Lat} \mathcal{W}\} \text{ for any } A \in B(\mathcal{H}). \tag{1}$$

The supremum on the right hand side of (1) we will denote by $\alpha(A, \mathcal{W})$. The smallest constant for which the inequality (1) holds is denoted by $\kappa(\mathcal{W})$ and called the *hyperreflexivity constant*. Recall that an algebra $\mathcal{W}$ is *reflexive* if $\mathcal{W} = \operatorname{AlgLat} \mathcal{W}$, which is a weaker property than hyperreflexivity. An operator T is called *hyperreflexive* or *reflexive* if the algebra $\mathcal{W}(T)$ is so.

There are only a few results concerning hyperreflexivity. Davidson [6] showed that the unilateral shift is hyperreflexive with constant at most 19. Rosenoer [10] proved hyperreflexity of commutative von Neumann algebras with constant not greater than 2. This gives the hyperreflexivity of normal operators with constant at most 3. In [12] Wogen proved that every quasinormal operator is reflexive. In this paper we show that

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quasinormal operators are hyperreflexive with constant smaller than 259. We also improve the estimation of the hyperreflexivity constant for the unilateral shift.

Let $\mathbb{T}$ be the unit circle on the complex plane $\mathbb{C}$. Denote $L^2 = L^2(\mathbb{T}, m)$ and $L^\infty = L^\infty(\mathbb{T}, m)$, where m is the normalized Lebesgue measure on $\mathbb{T}$. Let H^2, H^∞ be the Hardy spaces corresponding to L^2, L^∞, respectively, and P_{H^2} a projection from L^2 onto H^2. For each $\varphi \in L^\infty$ we define $T_\varphi : H^2 \to H^2$ by $T_\varphi f = P_{H^2}(\varphi f)$ for $f \in H^2$. The operator T_φ is called a *Toeplitz operator*. Let $\mathcal{T}$ denote the space of all Toeplitz operators and $\mathcal{A} = \{T_\varphi : \varphi \in H^\infty\}$ the algebra of all analytic Toeplitz operators. The unilateral shift S can be realized as the multiplication operator by independent variable T_z. Recall that $\mathcal{A} = \mathcal{W}(S)$.

Let $\pi\colon B(H^2) \to \mathcal{T}$ be the projection defined by Arveson in [1]. Recall that for any $A \in \mathcal{T}$ the operator $\pi(A)$ belongs to the weak* closed convex hull of $\{S^{*n}AS^n : n > N\}$ for any N.

To get a better estimate of the hyperreflexivity constant for the unilateral shift we need the following.

LEMMA 2. *Let* $A \in B(H^2)$. *If* $\pi(A) = 0$, *then* $d(A, \mathcal{A}) \leq 5.2\, \alpha(A, \mathcal{A})$.

Proof. In [6] Davidson showed the above inequality with constant 9. We get a better estimate by calculating more precisely the right hand side of (1). Assume that $\|A\| = 1$. Take an integer N and denote $P_N = S^N S^{*N}$. For any $\varepsilon > 0$ there is a unit vector x such that $\|Ax\| > 1 - \varepsilon$ and $x = P_N^\perp x$. Replacing N by a larger integer we can obtain $\|P_N^\perp Ax\| \geq 1 - \varepsilon$, $\|P_N Ax\| < \varepsilon$, $\|P_N A^* Ax\| \leq \varepsilon$. Denote $y = \|P_N^\perp Ax\|^{-1} P_N^\perp Ax$. Since $\pi(A) = 0$ belongs to the weak* closed convex hull of $S^{*n}AS^n$, thus 0 belongs to the convex hull of $\{\langle S^{*n}AS^n x, y\rangle : n > N\}$. Hence we can take $n > N$ such that $\mathrm{Re}\langle Az^n x, z^n y\rangle < \varepsilon$.

For any $a \in (0,1)$ define $\omega = a - z^n$ and $h = \frac{b}{1 - az^n} = b\sum_{k=0}^\infty a^n z^{nk}$, where $b = \sqrt{1-a^2}$. The vector hy belongs to span of $\{hz^l : 0 \leq l \leq N\}$. Hence hy is orthogonal to ωH^2. Thus for any $T \in \mathcal{W}(S)$ we have $\langle T\omega x, hy\rangle = 0$. Note that $\|\omega x\| = \sqrt{1+a^2}$, $\|hy\| = 1$. Since $h = b + az^n h$, thus

$$\begin{aligned}|\langle A\omega x, hy\rangle| &= |ab\langle Ax, y\rangle + a^2\langle Ax, z^n hy\rangle - b\langle Az^n x, y\rangle - a\langle Az^n x, z^n hy\rangle| \\ &\geq |ab\langle Ax, y\rangle - ab\langle Az^n x, z^n y\rangle| - a^2|\langle Ax, z^n hy\rangle| \\ &\qquad - b|\langle Az^n x, y\rangle| - a^2|\langle Az^n x, z^{2n} hy\rangle|.\end{aligned}$$

We have the following estimates:

$$\begin{aligned}|\langle Ax, y\rangle - \langle Az^n x, z^n y\rangle| &\geq \mathrm{Re}(\langle Ax, y\rangle - \langle Az^n x, z^n y\rangle) \\ &\geq \langle Ax, \|P_N^\perp Ax\|^{-1} P_N^\perp Ax\rangle - \varepsilon = 1 - 2\varepsilon, \\ |\langle Ax, z^n hy\rangle| &\leq |\langle P_N Ax, z^n hy\rangle| \leq \|P_N Ax\| \leq \varepsilon, \\ |\langle Az^n x, y\rangle| &\leq |\langle z^n x, P_N A^* y\rangle| \\ &\leq \|P_N^\perp Ax\|^{-1}(\|P_N A^* Ax\| + \|P_N A^* P_N Ax\|) \\ &\leq 2\varepsilon(1-\varepsilon)^{-1} = O(\varepsilon).\end{aligned}$$

Thus

$$|\langle A\omega x, hy\rangle| \geq ab(1 - 2\varepsilon) - a^2\varepsilon - b\, O(\varepsilon) - a^2.$$

Hence

$$|\langle A\|\omega x\|^{-1}\omega x, hy\rangle| \geq a(\sqrt{1-a^2}-a)(1+a^2)^{-\frac{1}{2}}\|A\|.$$

Taking $a = 0.39$ we get

$$d(A,\mathcal{A}) \leq \|A\| \leq 5.2\,|\langle A\|\omega x\|^{-1}\omega x, hy\rangle| \leq 5.2\,\alpha(A,\mathcal{A}).$$

The last inequality holds since $\alpha(A,\mathcal{A}) = \sup\{|\langle Ax,y\rangle| : \langle Sx,y\rangle = 0 \text{ for all } S \in \mathcal{W}, \|x\| = \|y\| = 1\}$, see [2]. ∎

Considering an arbitrary operator $A \in B(H^2)$, as in [6] we get the same inequality as in Lemma 2 but we have to double the constant and add 1. Hence

$$d(A,\mathcal{A}) < 12\,\alpha(A,\mathcal{A}).$$

Thus

PROPOSITION 3. *The algebra of all analytic Toeplitz operators is hyperreflexive with constant smaller than* 12.

REMARK 4. In [11] Rosenoer proved that the tensor product of a hyperreflexive von Neumann algebra $\mathcal{M}$ and the algebra of analytic Toeplitz operators $\mathcal{A}$ is hyperreflexive with constant at most $40c+19$, where c is the hyperreflexivity constant for $\mathcal{M}$. He used 19 as an estimate for $\mathcal{A}$. According to Proposition 3 we can get the constant $26c+12$.

Recall after [3, 7] that an algebra $\mathcal{W} \subset B(\mathcal{H})$ has *property* $\mathbb{A}_1(1)$ if $\mathcal{W}$ is weak* closed and for any weak* continuous functional φ on $\mathcal{W}$ and for a given $\varepsilon > 0$ there are $g, h \in \mathcal{H}$ such that $\varphi(T) = \langle Tg, h\rangle$ for every $T \in \mathcal{W}$ and they satisfy $\|g\| \cdot \|h\| \leq (1+\varepsilon)\|\varphi\|$.

In [9] Kraus and Larson proved that if $\mathcal{W}$ is hyperreflexive and has property $\mathbb{A}_1(1)$, then any weak* closed algebra $\mathcal{L} \subset \mathcal{W}$ is hyperreflexive and $\kappa(\mathcal{L}) \leq 1 + 2\kappa(\mathcal{W})$.

THEOREM 5. *Quasinormal operators are hyperreflexive with constant not greater than* 259.

Proof. By Brown's result [4] every quasinormal operator is unitarily equivalent to $(A \otimes S) \oplus N$, where A is positive with $\mathit{ker}\,A = \{0\}$, N is normal and S is the unilateral shift. Since hyperreflexivity is kept with the same constant by unitary equivalence it is enough to consider the above model.

Since $\mathcal{W}(A)$ is a commuting von Neumann algebra, it is hyperreflexive and $\kappa(A) \leq 2$ (see [10, Theorem 3.5]). Moreover, $\mathcal{W}(A)$ has property $\mathbb{A}_1(1)$ ([5, Proposition 60.1]). Now by Remark 4 the tensor product $\mathcal{W}(A) \otimes \mathcal{W}(S)$ is hyperreflexive with constant $\kappa(\mathcal{W}(A) \otimes \mathcal{W}(S)) \leq 64$. Since every normal operator is hyperreflexive with constant less than or equal to 3 (see [10, Theorem 3.6]), then [8, Corollary 6.4] implies that

$$\kappa((\mathcal{W}(A) \otimes \mathcal{W}(S)) \oplus \mathcal{W}(N)) \leq 129.$$

Note that

$$\mathcal{W}((A \otimes S) \oplus N) \subset (\mathcal{W}(A) \otimes \mathcal{W}(S)) \oplus \mathcal{W}(N).$$

Since $(\mathcal{W}(A) \otimes \mathcal{W}(S)) \oplus \mathcal{W}(N)$ has property $\mathbb{A}_1(1)$, applying [9, Theorem 3.3] we get hyperreflexivity of $(A \otimes S) \oplus N$ and

$$\kappa((A \otimes S) \oplus N) \leq 259. \ ∎$$

COROLLARY 6. *Every isometry is hyperreflexive.*

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