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CLOSED IDEALS IN THE BANACH ALGEBRA OF OPERATORS ON A BANACH SPACE

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Abstract. In general, little is known about the lattice of closed ideals in the Banach algebra $\mathscr{B}(E)$ of all bounded, linear operators on a Banach space E. We list the (few) Banach spaces for which this lattice is completely understood, and we give a survey of partial results for a number of other Banach spaces. We then investigate the lattice of closed ideals in $\mathscr{B}(F)$, where F is one of Figiel's reflexive Banach spaces not isomorphic to their Cartesian squares. Our main result is that this lattice is uncountable.

Introduction. It is a basic fact that, for each non-zero Banach space E, the set $\mathscr{F}(E)$ of finite-rank operators on E is the smallest non-zero ideal in $\mathscr{B}(E)$. In particular, its closure $\overline{\mathscr{F}}(E)$ is the smallest non-zero *closed* ideal in $\mathscr{B}(E)$. Many other closed ideals in $\mathscr{B}(E)$ have been identified and studied, for instance the ideals of compact, strictly singular, inessential, completely continuous, and weakly compact operators (*e.g.*, see [35] for details and numerous other examples).

The general structure of the lattice of closed ideals in $\mathscr{B}(E)$ is, nevertheless, far from being well understood. Investigating this question is, in our opinion, a key step towards a deeper understanding of Banach algebras of the form $\mathscr{B}(E)$ for a Banach space E.

The present paper should be seen as part of this programme. It provides a survey of the major known results about closed ideals in $\mathscr{B}(E)$ for various Banach spaces E,

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and it contains new information about the closed ideals in $\mathscr{B}(E)$ for one particular class of Banach spaces E, namely Figiel's reflexive Banach spaces not isomorphic to their Cartesian squares.

More precisely, the paper is organized as follows. After setting up notation, conventions, and some standard definitions in Section 1, we describe in Section 2 the (surprisingly few) Banach spaces E for which all the closed ideals in $\mathscr{B}(E)$ are known. Then, in Section 3, we survey some important partial results about closed ideals in $\mathscr{B}(E)$ for various particular Banach spaces E. Finally, in Section 4, we investigate the lattice of closed ideals in $\mathscr{B}(F)$, where F is one of Figiel's reflexive Banach spaces not isomorphic to their Cartesian squares. Our main result is that this lattice is uncountable, adding yet another surprising property to the already long list of pathologies possessed by these apparently 'nice' Banach spaces.

1. Notation. In this section we describe the notation, conventions, and some standard definitions that we rely on throughout the paper.

For a set X, we write $\mathbb{P}(X)$ for the power set of X, and we denote by $\mathbb{P}_{\mathrm{fin}}(X)$ the subset of $\mathbb{P}(X)$ consisting of the finite subsets of X.

All Banach spaces are supposed to be over the same scalar field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let E be a Banach space. We write E' for the dual Banach space of E. For $m \in \mathbb{N}$, we denote by $E^{\oplus m}$ the direct sum of m copies of E.

For each real number $p \geqslant 1$, the *ℓ_p-direct sum* of a sequence $(E_k)_{k=1}^{\infty}$ of Banach spaces is defined by

$$\Big(\bigoplus_{k=1}^{\infty} E_k\Big)_{\ell_p} := \Big\{(x_k) \ \Big|\ x_k \in E_k \ (k \in \mathbb{N}) \text{ and } \sum_{k=1}^{\infty} \|x_k\|^p < \infty\Big\}.$$

This is a Banach space for coordinatewise defined vector space operations and the norm

$$\|(x_k)\| = \Big(\sum_{k=1}^{\infty} \|x_k\|^p\Big)^{1/p} \qquad \Big((x_k) \in \Big(\bigoplus_{k=1}^{\infty} E_k\Big)_{\ell_p}\Big).$$

Similarly, the *c_0-direct sum* of a sequence $(E_k)_{k=1}^{\infty}$ of Banach spaces is given by

$$\Big(\bigoplus_{k=1}^{\infty} E_k\Big)_{c_0} := \{(x_k) \mid x_k \in E_k \ (k \in \mathbb{N}) \text{ and } \|x_k\| \to 0 \text{ as } k \to \infty\};$$

it is a Banach space for coordinatewise defined vector space operations and the norm

$$\|(x_k)\| = \sup\{\|x_k\| \mid k \in \mathbb{N}\} \qquad \Big((x_k) \in \Big(\bigoplus_{k=1}^{\infty} E_k\Big)_{c_0}\Big).$$

We use the term *operator* for a bounded, linear map between Banach spaces. The collection of all operators from a Banach space E to a Banach space F is denoted by $\mathscr{B}(E, F)$, or just $\mathscr{B}(E)$ in the case where $E = F$. We write I_E for the identity operator on E.

DEFINITION 1.1 (Pietsch [35]). An *operator ideal* is an assignment $\mathscr{I}$ which associates to each pair of Banach spaces E and F a linear subspace $\mathscr{I}(E, F)$ of $\mathscr{B}(E, F)$ satisfying:

(i) $\mathscr{I}(E, F)$ is non-zero for some Banach spaces E and F;

(ii) for any Banach spaces D, E, F, and G, the composite operator TSR belongs to $\mathscr{I}(D,G)$ whenever R belongs to $\mathscr{B}(D,E)$, S to $\mathscr{I}(E,F)$, and T to $\mathscr{B}(F,G)$.

We usually write $\mathscr{I}(E)$ instead of $\mathscr{I}(E,E)$.

For an operator ideal $\mathscr{I}$ and Banach spaces E and F, we write $\overline{\mathscr{I}}(E,F)$ for the closure (in the operator norm) of $\mathscr{I}(E,F)$ in $\mathscr{B}(E,F)$. The assignment $\overline{\mathscr{I}}$ thus defined is an operator ideal, called the *closure* of $\mathscr{I}$. We say that the operator ideal $\mathscr{I}$ is *closed* if $\mathscr{I} = \overline{\mathscr{I}}$.

We shall consider the following operator ideals (and their closures) in this note:

- $\mathscr{F}$, the finite-rank operators;
- $\mathscr{K}$, the compact operators;
- $\mathscr{W}$, the weakly compact operators;
- $\mathscr{X}$, the operators with separable images;
- $\mathscr{S}$, the strictly singular operators;
- $\mathscr{E}$, the inessential operators;
- $\mathscr{G}_{\mathscr{C}}$ (where $\mathscr{C}$ is a subset of $\mathscr{B}(E,F)$ for some Banach spaces E and F), the operator ideal generated by the set $\mathscr{C}$.

We regard the first four of these operator ideals as so well-known that no definitions are required. We proceed to define the final three.

Definition 1.2 (Kato [25]; Pietsch [35]). Let E and F be Banach spaces.

(i) An operator $T\colon E \to F$ is *strictly singular* if T is not bounded below on any infinite-dimensional subspace of E. In other words, for each $\varepsilon > 0$ and each infinite-dimensional subspace D of E, there is a unit vector $x \in D$ such that $\|Tx\| < \varepsilon$.

(ii) An operator $T\colon E \to F$ is *inessential* if, for each operator $S\colon F \to E$, $I_E - ST$ is a Fredholm operator, that is, the spaces $\ker(I_E - ST)$ and $E/\operatorname{im}(I_E - ST)$ are finite-dimensional.

We write $\mathscr{S}(E,F)$ and $\mathscr{E}(E,F)$ for the sets of strictly singular and inessential operators from E to F, respectively. The assignments $\mathscr{S}$ and $\mathscr{E}$ thus defined are closed operator ideals.

The following method of producing ideals in $\mathscr{B}(E)$ for a Banach space E plays a key rôle in many of the constructions that we shall present in this paper. Its roots go back to Porta (see [37] and [3]).

Definition 1.3. Let D, E, F, and G be Banach spaces. For each subset $\mathscr{C}$ of $\mathscr{B}(E,F)$, set

$$\mathscr{G}_{\mathscr{C}}(D,G) := \operatorname{span}\{STR \mid R \in \mathscr{B}(D,E),\ T \in \mathscr{C},\ S \in \mathscr{B}(F,G)\} \subseteq \mathscr{B}(D,G).$$

Suppose that $\mathscr{C}$ contains a non-zero operator. Then the assignment $\mathscr{G}_{\mathscr{C}}$ thus defined is an operator ideal, called the *operator ideal generated by* $\mathscr{C}$. It is clearly the smallest operator ideal such that $\mathscr{C} \subseteq \mathscr{G}_{\mathscr{C}}(E,F)$. In the case where $E = F$ and $\mathscr{C} = \{I_E\}$, we write $\mathscr{G}_E$ instead of $\mathscr{G}_{\mathscr{C}}$.

2. Classifications of the closed ideals in $\mathscr{B}(E)$.

In this section we shall give a survey of the Banach spaces E for which *all* closed ideals in $\mathscr{B}(E)$ are known. As it turns out, the list of such spaces is almost embarrassingly short. We shall also give remarks about some closely related spaces.

It is ancient folklore that the matrix algebras $M_n(\mathbb{K})$, $n \in \mathbb{N}$, are simple. In other words, we have the following result.

THEOREM 2.1. *For each finite-dimensional Banach space E, the two trivial ideals $\{0\}$ and $\mathscr{B}(E)$ are the only ideals in $\mathscr{B}(E)$.* ∎

The first result about closed ideals in $\mathscr{B}(E)$ for an infinite-dimensional Banach space E is due to Calkin who in [5] classified all the ideals in $\mathscr{B}(\ell_2)$. In particular he obtained the following conclusion.

THEOREM 2.2 (Calkin). *The ideal $\mathscr{K}(\ell_2)$ of compact operators is the only non-trivial closed ideal in $\mathscr{B}(\ell_2)$.* ∎

Calkin's theorem has subsequently been generalized in two different ways.

First, Gohberg, Markus, and Feldman showed in [18] that the same conclusion holds true for the other classical sequence spaces ℓ_p, where $1 \leqslant p < \infty$, and c_0. A simplified and unified proof of this result is given in [23].

THEOREM 2.3 (Gohberg, Markus, and Feldman). *For $E = \ell_p$, where $1 \leqslant p < \infty$, and $E = c_0$, the ideal $\mathscr{K}(E)$ of compact operators is the only non-trivial closed ideal in $\mathscr{B}(E)$.* ∎

Second, Gramsch [22] and Luft [31] have independently classified the closed ideals in $\mathscr{B}(H)$ for each infinite-dimensional Hilbert space H (not necessarily separable). To state their result explicitly, we require the following terminology.

DEFINITION 2.4. Let κ be an infinite cardinal.

(i) We denote by κ^+ the minimum cardinal strictly greater than κ.
(ii) We say that a Hilbert space H has *dimension* κ, written $\dim H = \kappa$, if some (and thus each) orthonormal basis of H has cardinality κ.
(iii) A subset Y of a metric space X is called *κ-bounded* if, for each $\varepsilon > 0$, there exists a subset Z of Y of cardinality strictly less than κ such that

$$Y \subseteq \bigcup_{z \in Z} \operatorname{ball}(z, \varepsilon),$$

where $\operatorname{ball}(z, \varepsilon)$ denotes the open ball in X with centre z and radius ε.
(iv) An operator $T\colon E \to F$, where E and F are Banach spaces, is called *κ-compact* if $T(B)$ is κ-bounded for each norm-bounded subset B of E. We write $\mathscr{K}_\kappa(E, F)$ for the set of κ-compact operators from E to F.

The assignment $\mathscr{K}_\kappa$ thus defined is a closed operator ideal. We note that the first two of these operator ideals are already well-known: $\mathscr{K}_{\aleph_0} = \mathscr{K}$ (the usual compact operators) and $\mathscr{K}_{\aleph_1} = \mathscr{X}$ (the operators with separable images). Moreover, the family of operator ideals $(\mathscr{K}_\kappa)$ forms an increasing chain in the sense that $\mathscr{K}_\kappa(E, F) \subseteq \mathscr{K}_\lambda(E, F)$ for each pair of Banach spaces E and F and each pair of cardinals κ and λ such that $\kappa \leqslant \lambda$ (see [22, Satz 2.1–2.2] or [31, Lemma 4.1–4.3]).

The important realisation of Gramsch [22, Theorem 3.3] and Luft [31, Corollary 6.2] is that, for each Hilbert space H, *every* proper closed ideal in $\mathscr{B}(H)$ has the form $\mathscr{K}_\kappa(H)$ for some cardinal $\kappa \leqslant \dim H$, and that these ideals are pairwise distinct. More precisely, they showed the following theorem.

THEOREM 2.5 (Gramsch and Luft). *Let H be an infinite-dimensional Hilbert space. The mapping $\kappa \mapsto \mathscr{K}_\kappa(H)$ is a lattice isomorphism from the set of cardinals κ such that $\aleph_0 \leqslant \kappa \leqslant \dim H$ onto the set of non-trivial closed ideals in $\mathscr{B}(H)$.*

In other words, the lattice of closed ideals in $\mathscr{B}(H)$ is a chain of the form

$$\{0\} \subsetneq \overline{\mathscr{F}}(H) = \mathscr{K}(H) = \mathscr{S}(H) = \mathscr{E}(H) = \mathscr{K}_{\aleph_0}(H) \subsetneq \mathscr{K}_{\aleph_1}(H) = \mathscr{X}(H) \subsetneq \cdots$$
$$\cdots \subsetneq \mathscr{K}_\kappa(H) \subsetneq \mathscr{K}_{\kappa^+}(H) \subsetneq \cdots \subsetneq \mathscr{K}_{\dim H}(H) \subsetneq \mathscr{K}_{(\dim H)^+}(H) = \mathscr{W}(H) = \mathscr{B}(H),$$

where κ is a cardinal satisfying $\aleph_1 < \kappa$ and $\kappa^+ < \dim H$. ∎

It seems to be unknown whether or not Gramsch and Luft's theorem can be generalized to non-separable ℓ_p- and c_0-spaces in the same way as Calkin's theorem was generalized by Gohberg, Markus, and Feldman.

QUESTION 2.6. Let $\mathbb{I}$ be an uncountable set of cardinality κ, say, and let E denote one of the following Banach spaces

$$\ell_p(\mathbb{I}) := \Big\{ f \colon \mathbb{I} \to \mathbb{K} \;\Big|\; \sum_{i \in \mathbb{I}} |f(i)|^p < \infty \Big\} \qquad (1 \leqslant p < \infty)$$

and

$$c_0(\mathbb{I}) := \{ f \colon \mathbb{I} \to \mathbb{K} \mid \text{the set } \{ i \in \mathbb{I} \mid |f(i)| \geqslant \varepsilon \} \text{ is finite for each } \varepsilon > 0 \}.$$

Is it true that each non-trivial closed ideal in $\mathscr{B}(E)$ has the form $\mathscr{K}_\lambda(E)$ for some cardinal λ with $\aleph_0 \leqslant \lambda \leqslant \kappa$?[1]

Until 2003, it appears that Theorems 2.1, 2.3, and 2.5 gave the full list of Banach spaces E for which all the closed ideals in $\mathscr{B}(E)$ are known. However, Charles J. Read and the present authors have recently added a new member to this family (see [29, Corollary 5.6]).

THEOREM 2.7 (Laustsen, Loy, and Read). *For $E := (\bigoplus_{k=1}^\infty \ell_2^k)_{c_0}$, there are exactly two non-trivial closed ideals in $\mathscr{B}(E)$. Specifically, the lattice of closed ideals in $\mathscr{B}(E)$ is given by*

$$\{0\} \subsetneq \overline{\mathscr{F}}(E) = \mathscr{K}(E) = \mathscr{S}(E) = \mathscr{E}(E) = \mathscr{W}(E) \subsetneq \overline{\mathscr{G}}_{c_0}(E) \subsetneq \mathscr{B}(E). \;\blacksquare$$

REMARK 2.8. Theorem 2.7 was inspired by the classification of the complemented subspaces in the Banach space $(\bigoplus_{k=1}^\infty \ell_2^k)_{c_0}$ given by Bourgain, Casazza, Lindenstrauss, and Tzafriri [4]. In fact, they prove analogous results for other Banach spaces than $(\bigoplus_{k=1}^\infty \ell_2^k)_{c_0}$. To state their results in a unified way, set $E := (\bigoplus_{k=1}^\infty E_k)_D$, where D and E_k are given in one of the following four ways:

[1] Added in proof: Daws has answered Question 2.6 in the positive (*Closed ideals in the Banach algebra of operators on classical nonseparable spaces*, Math. Proc. Cambridge Philos. Soc., to appear).

(i) $D = c_0$ and $E_k = \ell_2^k$ for each $k \in \mathbb{N}$;
(ii) $D = c_0$ and $E_k = \ell_1^k$ for each $k \in \mathbb{N}$;
(iii) $D = \ell_1$ and $E_k = \ell_2^k$ for each $k \in \mathbb{N}$;
(iv) $D = \ell_1$ and $E_k = \ell_\infty^k$ for each $k \in \mathbb{N}$.

Then it is shown in [4, §8] that, for each infinite-dimensional, complemented subspace F of E, either F is isomorphic to D or F is isomorphic to E.

In the light of these results and Theorem 2.7, it is natural to ask what the closed ideals in $\mathscr{B}(E)$ are in the cases (ii)–(iv).[2]

Another Banach space for which this question attracts attention is $E := (\bigoplus_{k=1}^\infty \ell_p^k)_{c_0}$ for a fixed $p > 1$. It follows from [30, p. 72f] that E contains a complemented subspace isomorphic to $(\bigoplus_{k=1}^\infty \ell_2^k)_{c_0}$, as well as the 'trivial' complemented subspaces isomorphic to c_0 and of finite dimension. Consequently, for $p \neq 2$, $\mathscr{B}(E)$ contains at least three distinct non-trivial closed ideals, but we do not know if there are any others.

These questions were first raised in [29]. We intend to address them in future work.

3. Partial results about closed ideals in $\mathscr{B}(E)$. Even though Banach spaces E for which the lattice of closed ideals in $\mathscr{B}(E)$ is completely understood are rare, quite a few interesting partial results are known. In this section we shall survey a number of such results. Note that the state of knowledge at 1974 and 1977, respectively, can be found in [6, Chapter 5] and [35, Chapter 5].

Porta [38] has shown that the lattice $\overline{\text{ideal}}(\mathscr{B}(E))$ of closed ideals in $\mathscr{B}(E)$ for a Banach space E can be very complicated indeed. Precisely, Porta's result asserts that it is possible for it to have a sublattice isomorphic to the lattice $\mathbb{P}_{\text{fin}}(\mathbb{N})$ of finite subsets of $\mathbb{N}$. The construction is as follows. Let $\mathbf{p} = \{p_1, p_2, \dots\}$ be a countably infinite set of real numbers greater than 1, set

$$E_{\mathbf{p}} := \Big(\bigoplus_{k=1}^{\infty} \ell_{p_k}\Big)_{\ell_2}, \tag{3.1}$$

and associate with each $\mathbf{s} \in \mathbb{P}_{\text{fin}}(\mathbb{N})$ the closed subspace

$$E_{\mathbf{p}}(\mathbf{s}) := \{(x_k) \in E_{\mathbf{p}} \mid x_k = 0 \ (k \notin \mathbf{s})\}$$

of $E_{\mathbf{p}}$. Then the mapping

$$\mathbf{s} \mapsto \overline{\mathscr{G}}_{E_{\mathbf{p}}(\mathbf{s})}(E_{\mathbf{p}}), \quad \mathbb{P}_{\text{fin}}(\mathbb{N}) \to \overline{\text{ideal}}(\mathscr{B}(E_{\mathbf{p}})),$$

is injective and inclusion-preserving in both directions, and Porta's claim follows. Note that the space $E_{\mathbf{p}}$ is nice in many ways; for instance, it is reflexive, and we can even arrange that it is isometrically isomorphic to its dual space by requiring that the set $\mathbf{p}$ is *self-conjugate* in the sense that, for each $p \in \mathbf{p}$, the conjugate index $p' := p/(p-1)$ is also in $\mathbf{p}$.

[2]Added in proof: Case (iii) has been resolved by Laustsen, Schlumprecht, and Zsák (*The lattice of closed ideals in the Banach algebra of operators on a certain dual Banach space*, preprint). They show that $\overline{\mathscr{F}}(E)$ and $\overline{\mathscr{G}}_{\ell_1}(E)$ are the only non-trivial closed ideals in $\mathscr{B}(E)$ for $E = (\bigoplus_{k=1}^\infty \ell_2^k)_{\ell_1}$.

Pietsch [36] has shown that it is possible to have uncountably many distinct closed ideals between the ideals of compact and strictly singular operators. Specifically, he has found a family $(\mathscr{X}_q)_{2\leqslant q<\infty}$ of closed operator ideals satisfying

$$\mathscr{K}(E,F) \subseteq \mathscr{X}_r(E,F) \subseteq \mathscr{X}_q(E,F) \subseteq \mathscr{S}(E,F) \qquad (2 \leqslant q < r < \infty)$$

for each pair of Banach spaces E and F, and has observed that if $\mathbf{p}$ is a countable, dense subset of the open interval $]1,\infty[$, then all the ideals $\mathscr{X}_q(E_{\mathbf{p}})$ $(2 \leqslant q < \infty)$ are mutually distinct, where $E_{\mathbf{p}}$ is the Banach space given by (3.1).

Porta [39] has studied the lattice of closed ideals in $\mathscr{B}(E)$ for $E = \ell_p \oplus \ell_q$, where $1 < p < q < \infty$. His main results are that

$$\{0\} \subsetneq \overline{\mathscr{F}}(E) = \mathscr{K}(E) \subsetneq \mathscr{S}(E) = \mathscr{E}(E) = \overline{\mathscr{G}}_{\ell_p}(E) \cap \overline{\mathscr{G}}_{\ell_q}(E) \begin{array}{c} \subsetneq \overline{\mathscr{G}}_{\ell_p}(E) \subsetneq \\ \\ \subsetneq \overline{\mathscr{G}}_{\ell_q}(E) \subsetneq \end{array} \mathscr{B}(E),$$

and that, in the cases where either $p = 2$ or $q = 2$, if $\mathscr{J}$ is any other closed ideal in $\mathscr{B}(E)$, then necessarily $\mathscr{K}(E) \subsetneq \mathscr{J} \subsetneq \mathscr{S}(E)$. Volkmann [47] has generalized Porta's results to arbitrary finite direct sums of ℓ_p-spaces. Indeed, set $E := \ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n}$, where $n \in \{2,3,\ldots\}$ and $1 \leqslant p_1 < p_2 < \cdots < p_n < \infty$. Then he has proved that the mapping

$$\mathbf{s} \mapsto \{T \in \mathscr{B}(E) \mid Q_k T J_k \in \mathscr{K}(\ell_{p_k}) \ (k \notin \mathbf{s})\}$$

is a lattice isomorphism from $\mathbb{P}(\{1,\ldots,n\})$ onto the set of closed ideals in $\mathscr{B}(E)$ containing the ideal $\mathscr{S}(E)$ of strictly singular operators, where $J_k \colon \ell_{p_k} \to E$ and $Q_k \colon E \to \ell_{p_k}$ denote the canonical embedding and projection, respectively. In particular, there are exactly n maximal ideals in $\mathscr{B}(E)$. Similar results hold if one of the spaces ℓ_{p_k} is replaced with c_0.

Building on work of Rosenthal [40] and Schechtman [41], Pietsch has demonstrated that there are infinitely many closed ideals in $\mathscr{B}(L_p[0,1])$ for each $p \in\,]1,\infty[\, \setminus \{2\}$. Moreover, he has shown that there are uncountably many closed ideals in $\mathscr{B}(C[0,1])$ (see [35, Theorems 5.3.9 and 5.3.11]). In both cases, the ideals have the form $\overline{\mathscr{G}}_F(E)$, where F ranges over a certain family of complemented subspaces of the underlying Banach space E.

Edelstein and Mityagin [15, p. 225] observed that the ideal $\mathscr{W}(J_p)$ of weakly compact operators is a maximal ideal of codimension one in $\mathscr{B}(J_p)$, where $1 < p < \infty$ and J_p is the p^{th} quasi-reflexive James space (*cf.* [24]). The first author [28, Theorem 4.16] has proved that $\mathscr{W}(J_p)$ is the only maximal ideal in $\mathscr{B}(J_p)$, and, using techniques similar to Volkmann's, he has applied this result to construct Banach spaces E such that $\mathscr{B}(E)$ has any specified finite number of maximal ideals of any specified codimensions. Further, the lattice of closed ideals in $\mathscr{B}(J_p)$ has the form

$$\{0\} \subsetneq \overline{\mathscr{F}}(J_p) = \mathscr{K}(J_p) = \mathscr{S}(J_p) = \mathscr{E}(J_p) \subsetneq \overline{\mathscr{G}}_{\ell_p}(J_p) \subsetneq \mathscr{W}(J_p) \subsetneq \mathscr{B}(J_p),$$

and if there are any other closed ideals $\mathscr{J}$ in $\mathscr{B}(J_p)$, then they have to satisfy

$$\overline{\mathscr{G}}_{\ell_p}(J_p) \subsetneq \mathscr{J} \subsetneq \mathscr{W}(J_p)$$

(see [28, p. 528]). For $p = 2$, there is at least one such closed ideal. Indeed, building on work of Giesy and James [17], Casazza, Lin, and Lohman [9, Theorem 13(i)] observed that J_2 contains a complemented subspace isomorphic to $E := (\bigoplus_{k=1}^{\infty} \ell_\infty^k)_{\ell_2}$, and the first author has shown that $\overline{\mathscr{G}}_E(J_2)$ is neither equal to $\overline{\mathscr{G}}_{\ell_2}(J_2)$ nor to $\mathscr{W}(J_2)$.

While solving the unconditional basic sequence problem, Gowers and Maurey [20] found the first example of a hereditarily indecomposable Banach space. They also showed that the ideal $\mathscr{S}(E)$ of strictly singular operators is a maximal ideal of codimension one in $\mathscr{B}(E)$ for each such space E; once again, this maximal ideal is unique (see [28, Proposition 7.2]).

A completely different class of Banach spaces E with the property that the ideal $\mathscr{S}(E)$ of strictly singular operators is the unique maximal ideal in $\mathscr{B}(E)$ has been described by Whitley [49, Theorem 6.2]. Namely, this is true for each Banach space E which is *complementably minimal* in the sense that each closed, infinite-dimensional subspace of E contains a subspace which is isomorphic to E and complemented in E. Schlumprecht's space S is an interesting example of a complementably minimal Banach space (see [42]).

Androulakis and Schlumprecht [1] have proved that non-compact, strictly singular operators exist both on Schlumprecht's space S and on the particular hereditarily indecomposable Banach space E that Gowers and Maurey constructed in [20], and so for $F = S$ and $F = E$, $\mathscr{S}(F)$ is not the only non-trivial closed ideal in $\mathscr{B}(F)$. In the second case Androulakis and Schlumprecht have even shown that the quotient $\mathscr{S}(E)/\mathscr{K}(E)$ is non-separable by embedding the Banach space ℓ_∞/c_0 into it (see [1, p. 670]). Nevertheless, it is unknown for both spaces how many ideals (if any) there are between $\mathscr{K}(F)$ and $\mathscr{S}(F)$.

It is a major open problem whether or not there exists a Banach space E such that the ideal of compact operators is a maximal ideal of codimension one in $\mathscr{B}(E)$. The reader is referred to Schlumprecht's paper [43] for the current state of this difficult problem together with an impressive new method of attack.

Shelah [44] gave the first example of a non-separable Banach space E for which the ideal $\mathscr{X}(E)$ of operators with separable images has codimension one in $\mathscr{B}(E)$. However, Shelah's construction relies on the assumption of an axiom outside ZFC, the so-called $\diamond_{\aleph_1}$ (which in turn is a consequence of a set-theoretic axiom known as $V = L$). Happily, this extra axiom is in fact not necessary: Shelah and Steprans [45] have shown that such Banach spaces exist within ZFC. Wark [48] has subsequently modified Shelah and Steprans's construction to obtain a reflexive example. Further, assuming either Martin's axiom or the continuum hypothesis, Koszmider [26] has recently found a compact separable scattered non-metrizable Hausdorff space K such that the Banach space $E = C(K)$ of continuous functions on K has this property. (Note that E is necessarily non-separable because K is non-metrizable.)

Argyros and Tolias [2] have constructed a non-separable, non-reflexive hereditarily indecomposable Banach space E for which $\mathscr{S}(E) = \mathscr{W}(E) = \mathscr{X}(E)$.

Mankiewicz [32] on the one hand, and Dales, Loy, and Willis [11] on the other, have found Banach spaces E such that ℓ_∞ is a quotient of $\mathscr{B}(E)$. It follows that, for each of these spaces E, $\mathscr{B}(E)$ has at least $2^{2^{\aleph_0}}$ maximal ideals of codimension one. When solving Banach's hyperplane problem, Gowers constructed a Banach space G such that ℓ_∞/c_0 is a quotient of $\mathscr{B}(G)$ (see [19] and [21]). The first author [28, Theorem 8.4] has classified the maximal ideals in $\mathscr{B}(G)$ by observing that each such ideal is the preimage of a maximal ideal in ℓ_∞/c_0.

We conclude this survey with some remarks about $\mathscr{B}(\ell_\infty)$. (Note that this is the same as $\mathscr{B}(L_\infty[0,1])$ because Pełczyński [34] has shown that the Banach spaces ℓ_∞ and $L_\infty[0,1]$ are isomorphic.) Gohberg, Markus, and Feldman [18, Theorem 5.3] observed that there are strictly singular, non-compact operators on ℓ_∞. It follows from [30, Proposition 2.f.4] that $\mathscr{W}(\ell_\infty)$ is the unique maximal ideal in $\mathscr{B}(\ell_\infty)$. Indeed, suppose that $T \in \mathscr{B}(\ell_\infty)$ is not weakly compact. Then, by the above-mentioned proposition, there is a subspace E of ℓ_∞ such that $E \cong \ell_\infty$ and T is bounded below on E. Let $J\colon E \to \ell_\infty$ be the inclusion mapping, and let $U\colon \ell_\infty \to E$ be an isomorphism. We see that $T(E)$ is a closed subspace of ℓ_∞ isomorphic to ℓ_∞. Since ℓ_∞ is injective, this implies that $T(E)$ is complemented in ℓ_∞. Take an idempotent operator Q on ℓ_∞ such that $\operatorname{im} Q = T(E)$. Then we have a commutative diagram

$$\begin{array}{ccc}
\ell_\infty & \xrightarrow{\quad I_{\ell_\infty} \quad} & \ell_\infty \\
\Big\downarrow U & & \Big\uparrow U^{-1} \\
 & & E \\
 & & \Big\uparrow (T|_E)^{-1} \\
E & \xrightarrow{\quad T|_E \quad} & T(E) \\
\Big\downarrow J & & \Big\uparrow Q \\
\ell_\infty & \xrightarrow{\quad T \quad} & \ell_\infty.
\end{array}$$

We conclude that the ideal generated by T is $\mathscr{B}(\ell_\infty)$, and so each proper ideal in $\mathscr{B}(\ell_\infty)$ is contained in $\mathscr{W}(\ell_\infty)$, as claimed.

The proof of [30, Theorem 2.f.12] shows that $\mathscr{W}(\ell_\infty) \subseteq \mathscr{S}(\ell_\infty)$. Further, each weakly compact set in ℓ_∞ is norm separable (see [33, Exercise 2.99]), and so $\mathscr{W}(\ell_\infty) \subseteq \mathscr{X}(\ell_\infty)$. The ideals $\mathscr{S}(\ell_\infty)$ and $\mathscr{X}(\ell_\infty)$ are both proper because ℓ_∞ is infinite-dimensional and non-separable, respectively. Hence we conclude that $\mathscr{S}(\ell_\infty) = \mathscr{W}(\ell_\infty) = \mathscr{X}(\ell_\infty)$ by maximality of $\mathscr{W}(\ell_\infty)$.

It follows from [14, p. 245] that ℓ_∞ has the (metric) approximation property, so that $\overline{\mathscr{F}}(\ell_\infty) = \mathscr{K}(\ell_\infty)$. Since non-compact operators from ℓ_∞ to c_0 exist (*e.g.*, see [12, Exercise 4(iii), p. 114]), $\mathscr{K}(\ell_\infty)$ is strictly contained in $\overline{\mathscr{G}}_{c_0}(\ell_\infty)$. The latter is a proper ideal and thus contained in $\mathscr{W}(\ell_\infty)$, but we do not know whether or not this inclusion is strict.

We summarize the above results as follows:

$$\{0\} \subsetneq \overline{\mathscr{F}}(\ell_\infty) = \mathscr{K}(\ell_\infty) \subsetneq \overline{\mathscr{G}}_{c_0}(\ell_\infty) \subseteq \mathscr{W}(\ell_\infty) = \mathscr{X}(\ell_\infty) = \mathscr{S}(\ell_\infty) = \mathscr{E}(\ell_\infty) \subsetneq \mathscr{B}(\ell_\infty),$$

and note that if there are any other closed ideals $\mathscr{J}$ in $\mathscr{B}(\ell_\infty)$, then they must satisfy

$$\mathscr{K}(\ell_\infty) \subsetneq \mathscr{J} \subsetneq \mathscr{W}(\ell_\infty).$$

We do not know whether such ideals exist. However, since ℓ_∞ contains subspaces isometric to E and E' for every separable Banach space E, it seems likely that further closed ideals can be found among $\overline{\mathscr{G}}_E(\ell_\infty)$ and $\overline{\mathscr{G}}_{E'}(\ell_\infty)$, where E is a separable Banach space.

4. Figiel spaces. In this section we shall study the lattice of closed ideals in $\mathscr{B}(F)$, where F is one of Figiel's reflexive Banach spaces not isomorphic to their Cartesian squares. These Figiel spaces are ℓ_q-direct sums of finite-dimensional ℓ_p-spaces for certain p's and q's (see Theorem 4.7, below, for details), and so they are nice in many ways. In particular, in the light of Theorem 2.7, one might expect that the lattice of closed ideals in $\mathscr{B}(F)$ for a Figiel space F would be 'well-behaved'. However, this is not the case: we shall show that this lattice is uncountable and has a highly complicated order structure.

For a real number $p \geqslant 1$ and a natural number n, we denote by ℓ_p^n the n-dimensional vector space over the scalar field $\mathbb{K}$ equipped with the norm

$$\|(\alpha_1, \dots, \alpha_n)\| = (|\alpha_1|^p + \cdots + |\alpha_n|^p)^{1/p} \qquad (\alpha_1, \dots, \alpha_n \in \mathbb{K}).$$

We write $\{e_1, \dots, e_n\}$ for the canonical basis of ℓ_p^n.

Throughout this section, we fix a sequence $(p_k)_{k=1}^\infty$ of real numbers greater than or equal to 1, a real number $q \geqslant 1$, and a sequence $(n_k)_{k=1}^\infty$ of natural numbers, and we set

$$F := \Big(\bigoplus_{k=1}^\infty \ell_{p_k}^{n_k}\Big)_{\ell_q}. \tag{4.1}$$

The following result is a consequence of [29, Example 3.9].

THEOREM 4.1 (Laustsen, Loy, and Read). *Let F be the Banach space defined in* (4.1), *and suppose that $\mathscr{J}$ is an ideal in $\mathscr{B}(F)$ not contained in $\overline{\mathscr{F}}(F)$. Then $\mathscr{J}$ contains the ideal $\mathscr{G}_{\ell_q}(F)$.*

It follows that

$$\overline{\mathscr{F}}(F) = \mathscr{K}(F) = \mathscr{S}(F) = \mathscr{E}(F) \subsetneq \overline{\mathscr{G}}_{\ell_q}(F),$$

and there are no closed ideals $\mathscr{J}$ in $\mathscr{B}(F)$ such that $\overline{\mathscr{F}}(F) \subsetneq \mathscr{J} \subsetneq \overline{\mathscr{G}}_{\ell_q}(F)$. ∎

DEFINITION 4.2. We associate with each subset s of $\mathbb{N}$ a Banach space $F(\mathrm{s})$ such that there is a canonical embedding $V_{\mathrm{s}}\colon F(\mathrm{s}) \to F$ and a canonical quotient map $W_{\mathrm{s}}\colon F \to F(\mathrm{s})$ satisfying

$$W_{\mathrm{s}} V_{\mathrm{s}} = I_{F(\mathrm{s})}. \tag{4.2}$$

More precisely, we split in three cases depending on the size of s.

First, suppose that s is empty. Then we set $F(\mathrm{s}) := \{0\}$, $V_{\mathrm{s}} := 0$, and $W_{\mathrm{s}} := 0$.

Second, suppose that s is non-empty and finite, say $\mathrm{s} = \{s(1), s(2), \dots, s(m)\}$, where $m \in \mathbb{N}$ and $1 \leqslant s(1) < s(2) < \cdots < s(m)$. Then we set

$$F(\mathrm{s}) := \ell_{p_{s(1)}}^{n_{s(1)}} \oplus \ell_{p_{s(2)}}^{n_{s(2)}} \oplus \cdots \oplus \ell_{p_{s(m)}}^{n_{s(m)}},$$

and we equip $F(\mathrm{s})$ with the ℓ_q^m-norm

$$\|(x_1, \dots, x_m)\| := (\|x_1\|^q + \cdots + \|x_m\|^q)^{1/q} \qquad (x_1 \in \ell_{p_{s(1)}}^{n_{s(1)}}, \dots, x_m \in \ell_{p_{s(m)}}^{n_{s(m)}}).$$

The embedding $V_{\mathbf{s}}\colon F(\mathbf{s}) \to F$ and the quotient map $W_{\mathbf{s}}\colon F \to F(\mathbf{s})$ are defined by

$$V_{\mathbf{s}}(x_k)_{k=1}^m := (y_j)_{j=1}^\infty \qquad \text{and} \qquad W_{\mathbf{s}}(z_k)_{k=1}^\infty := (z_{s(1)}, z_{s(2)}, \dots, z_{s(m)}),$$

where $(x_k)_{k=1}^m \in F(\mathbf{s})$ and $(z_k)_{k=1}^\infty \in F$, and where $(y_j)_{j=1}^\infty \in F$ is given by

$$y_j := \begin{cases} x_k & \text{if } j = s(k) \text{ for some } k \in \{1, \dots, m\} \\ 0 & \text{otherwise} \end{cases} \qquad (j \in \mathbb{N}).$$

Third, suppose that s is infinite, say $\mathbf{s} = \{s(k) \mid k \in \mathbb{N}\}$ with $1 \leqslant s(1) < s(2) < \cdots$. Then we set

$$F(\mathbf{s}) := \Big(\bigoplus_{k=1}^\infty \ell_{p_{s(k)}}^{n_{s(k)}}\Big)_{\ell_q},$$

and we define $V_{\mathbf{s}}\colon F(\mathbf{s}) \to F$ and $W_{\mathbf{s}}\colon F \to F(\mathbf{s})$ by

$$V_{\mathbf{s}}(x_k)_{k=1}^\infty := (y_j)_{j=1}^\infty \qquad \text{and} \qquad W_{\mathbf{s}}(z_k)_{k=1}^\infty := (z_{s(j)})_{j=1}^\infty,$$

where $(x_k)_{k=1}^\infty \in F(\mathbf{s})$ and $(z_k)_{k=1}^\infty \in F$, and where $(y_j)_{j=1}^\infty \in F$ is given by

$$y_j := \begin{cases} x_k & \text{if } j = s(k) \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \qquad (j \in \mathbb{N}).$$

It follows from (4.2) that in each case $P_{\mathbf{s}} := V_{\mathbf{s}} W_{\mathbf{s}}$ is an idempotent operator on F with image isomorphic to $F(\mathbf{s})$. We now give some basic rules of calculus for these operators.

LEMMA 4.3. *Let* s *and* t *be subsets of* $\mathbb{N}$. *Then:*

(i) $P_{\mathbf{s}\cap\mathbf{t}} = P_{\mathbf{s}} P_{\mathbf{t}}$;
(ii) $P_{\mathbf{s}\cup\mathbf{t}} = P_{\mathbf{s}} + P_{\mathbf{t}} - P_{\mathbf{s}\cap\mathbf{t}}$.

Proof. For each $x = (x_k)_{k=1}^\infty \in F$ and $j \in \mathbb{N}$, the j^{th} coordinate of $P_{\mathbf{s}} P_{\mathbf{t}} x$ is given by

$$\begin{aligned} (P_{\mathbf{s}} P_{\mathbf{t}} x)_j &= \left\{ \begin{array}{ll} (P_{\mathbf{t}} x)_j & \text{if } j \in \mathbf{s} \\ 0 & \text{otherwise} \end{array} \right\} \\ &= \left\{ \begin{array}{ll} \left\{ \begin{array}{ll} x_j & \text{if } j \in \mathbf{t} \\ 0 & \text{otherwise} \end{array} \right\} & \text{if } j \in \mathbf{s} \\ 0 & \text{otherwise} \end{array} \right\} = \left\{ \begin{array}{ll} x_j & \text{if } j \in \mathbf{s}\cap\mathbf{t} \\ 0 & \text{otherwise} \end{array} \right\} = (P_{\mathbf{s}\cap\mathbf{t}} x)_j. \end{aligned}$$

Similarly, the j^{th} coordinate of $(P_{\mathbf{s}} + P_{\mathbf{t}} - P_{\mathbf{s}\cap\mathbf{t}})x$ is given by

$$\begin{aligned} ((P_{\mathbf{s}} + P_{\mathbf{t}} - P_{\mathbf{s}\cap\mathbf{t}})x)_j &= \left\{ \begin{array}{ll} x_j & \text{if } j \in \mathbf{s} \\ 0 & \text{otherwise} \end{array} \right\} + \left\{ \begin{array}{ll} x_j & \text{if } j \in \mathbf{t} \\ 0 & \text{otherwise} \end{array} \right\} - \left\{ \begin{array}{ll} x_j & \text{if } j \in \mathbf{s}\cap\mathbf{t} \\ 0 & \text{otherwise} \end{array} \right\} \\ &= \left\{ \begin{array}{ll} x_j & \text{if } j \in \mathbf{s}\cup\mathbf{t} \\ 0 & \text{otherwise} \end{array} \right\} = (P_{\mathbf{s}\cup\mathbf{t}} x)_j. \end{aligned}$$

It follows that $P_{\mathbf{s}} P_{\mathbf{t}} x = P_{\mathbf{s}\cap\mathbf{t}} x$ and $(P_{\mathbf{s}} + P_{\mathbf{t}} - P_{\mathbf{s}\cap\mathbf{t}})x = P_{\mathbf{s}\cup\mathbf{t}} x$, as required. ∎

For each subset $\mathbb{S}$ of $\mathbb{P}(\mathbb{N})$, let $\mathscr{I}(\mathbb{S})$ be the ideal in $\mathscr{B}(F)$ generated by the set $\{P_{\mathbf{s}} \mid \mathbf{s} \in \mathbb{S}\}$, that is,

$$\mathscr{I}(\mathbb{S}) := \mathscr{G}_{\{P_{\mathbf{s}} \mid \mathbf{s}\in\mathbb{S}\}}(F);$$

we write $\overline{\mathscr{I}}(\mathbb{S})$ for the closure of $\mathscr{I}(\mathbb{S})$.

LEMMA 4.4. *Let $\mathbb{S}$ be a subset of $\mathbb{P}(\mathbb{N})$.*

(i) $\mathscr{I}(\mathbb{S}) = \{0\}$ *if and only if* $\mathbb{S} = \emptyset$ *or* $\mathbb{S} = \{\emptyset\}$.
(ii) $\mathscr{I}(\mathbb{S}) = \mathscr{F}(F)$ *if and only if* $\mathbb{S} \neq \emptyset$, $\mathbb{S} \neq \{\emptyset\}$, *and each* $\mathbf{s} \in \mathbb{S}$ *is finite.*
(iii) *Let* $\mathbf{s} \in \mathbb{S}$. *Then* $P_{\mathbf{t}} \in \mathscr{I}(\mathbb{S})$ *for each* $\mathbf{t} \subseteq \mathbf{s}$.
(iv) *Suppose that* $m \in \mathbb{N}$ *and* $\mathbf{s}_1, \ldots, \mathbf{s}_m \in \mathbb{S}$. *Then* $P_{\mathbf{s}_1 \cup \mathbf{s}_2 \cup \cdots \cup \mathbf{s}_m} \in \mathscr{I}(\mathbb{S})$.

Proof. Clauses (i) and (ii) are both clear, and clause (iii) follows immediately from Lemma 4.3(i).

Clause (iv) is obvious for $m = 1$. Now let $m \geqslant 2$, set $\mathbf{t} := \mathbf{s}_1 \cup \mathbf{s}_2 \cup \cdots \cup \mathbf{s}_{m-1}$, and assume inductively that $P_{\mathbf{t}} \in \mathscr{I}(\mathbb{S})$. Lemma 4.3(ii) implies that

$$P_{\mathbf{t} \cup \mathbf{s}_m} = P_{\mathbf{t}} + P_{\mathbf{s}_m} - P_{\mathbf{t} \cap \mathbf{s}_m},$$

and so $P_{\mathbf{t} \cup \mathbf{s}_m} \in \mathscr{I}(\mathbb{S})$ by the induction hypothesis and clause (iii). ∎

COROLLARY 4.5. *Let $\mathbb{S}$ be a subset of $\mathbb{P}(\mathbb{N})$ such that $\mathbb{S} \neq \emptyset$ and $\mathbb{S} \neq \{\emptyset\}$, and let $\mathbf{t} \in \mathbb{P}(\mathbb{N})$. Suppose that there are $m \in \mathbb{N}$ and $\mathbf{s}_1, \ldots, \mathbf{s}_m \in \mathbb{S}$ such that $\mathbf{t} \setminus (\mathbf{s}_1 \cup \mathbf{s}_2 \cup \cdots \cup \mathbf{s}_m)$ is finite. Then $P_{\mathbf{t}} \in \mathscr{I}(\mathbb{S})$.*

Proof. Set $\mathbf{t}_1 := \mathbf{t} \cap (\mathbf{s}_1 \cup \mathbf{s}_2 \cup \cdots \cup \mathbf{s}_m)$ and $\mathbf{t}_2 := \mathbf{t} \setminus (\mathbf{s}_1 \cup \mathbf{s}_2 \cup \cdots \cup \mathbf{s}_m)$. Then $\mathbf{t}_1$ and $\mathbf{t}_2$ are disjoint with $\mathbf{t}_1 \cup \mathbf{t}_2 = \mathbf{t}$, so that $P_{\mathbf{t}} = P_{\mathbf{t}_1} + P_{\mathbf{t}_2}$ by Lemma 4.3(ii). Lemma 4.4(iii) and (iv) imply that $P_{\mathbf{t}_1} \in \mathscr{I}(\mathbb{S})$. By assumption, $\mathbf{t}_2$ is finite, and so $P_{\mathbf{t}_2} \in \mathscr{F}(F) \subseteq \mathscr{I}(\mathbb{S})$ because $\mathscr{I}(\mathbb{S}) \neq \{0\}$. Now the result follows. ∎

LEMMA 4.6. *Let $m \in \mathbb{N}$, and let $\mathbb{S}_1, \ldots, \mathbb{S}_m$ be subsets of $\mathbb{P}(\mathbb{N})$. Then*

$$\mathscr{I}(\mathbb{S}_1 \cup \cdots \cup \mathbb{S}_m) = \mathscr{I}(\mathbb{S}_1) + \cdots + \mathscr{I}(\mathbb{S}_m).$$

Proof. Clearly, $P_{\mathbf{s}} \in \mathscr{I}(\mathbb{S}_1) + \cdots + \mathscr{I}(\mathbb{S}_m)$ for each $\mathbf{s} \in \mathbb{S}_1 \cup \cdots \cup \mathbb{S}_m$. This implies that $\mathscr{I}(\mathbb{S}_1 \cup \cdots \cup \mathbb{S}_m) \subseteq \mathscr{I}(\mathbb{S}_1) + \cdots + \mathscr{I}(\mathbb{S}_m)$ because $\mathscr{I}(\mathbb{S}_1) + \cdots + \mathscr{I}(\mathbb{S}_m)$ is an ideal.

For the reverse inclusion, it suffices to show that $\mathscr{I}(\mathbb{S}_j) \subseteq \mathscr{I}(\mathbb{S}_1 \cup \cdots \cup \mathbb{S}_m)$ for each $j \in \{1, \ldots, m\}$. However, this is immediate from the fact that $P_{\mathbf{s}} \in \mathscr{I}(\mathbb{S}_1 \cup \cdots \cup \mathbb{S}_m)$ whenever $\mathbf{s} \in \mathbb{S}_j$. ∎

To progress further, we shall specialize to the case where F is of the form considered by Figiel in [16]. He showed that, for each strictly decreasing sequence $(p_k)_{k=1}^{\infty}$ of real numbers greater than 2 and each real number q with $1 < q \leqslant \inf p_k$, there exists a sequence $(n_k)_{k=1}^{\infty}$ in $\mathbb{N}$ such that the Banach space F defined by (4.1) satisfies: for each $m \in \mathbb{N}$, $F^{\oplus(m+1)}$ is not isomorphic to any subspace of $F^{\oplus m}$. Casazza, Kottman, and Lin have subsequently observed that Figiel's proof in fact yields a stronger result (see [8]): one can use the argument given by Figiel in [16, p. 297f] word for word to prove the following theorem.

THEOREM 4.7. *Let $(p_k)_{k=1}^{\infty}$ be a strictly decreasing sequence of real numbers greater than 2, and let q be a real number with $1 < q \leqslant \inf p_k$. Then there exists a sequence $(n_k)_{k=1}^{\infty}$ in $\mathbb{N}$ such that, for each infinite subset $\mathbf{s}$ of $\mathbb{N}$ and each $m \in \mathbb{N}$, $F(\mathbf{s})^{\oplus(m+1)}$ is not isomorphic to any subspace of $F^{\oplus m}$, where F and $F(\mathbf{s})$ are the Banach spaces defined by (4.1) and Definition 4.2, respectively.* ∎

In the remainder of this section (except Lemma 4.14, which applies under more general conditions), we shall suppose that $(p_k)_{k=1}^\infty$, q, and $(n_k)_{k=1}^\infty$ are chosen in accordance with Theorem 4.7 and that the Banach spaces F and $F(\mathbf{s})$ are defined by (4.1) and Definition 4.2, respectively. Under these circumstances, the converse of Corollary 4.5 is true, as we shall now show. Our proof is inspired by that of [27, Lemma 3.2].

PROPOSITION 4.8. *Let $\mathbb{S}$ be a subset of $\mathbb{P}(\mathbb{N})$ such that $\mathbb{S} \neq \emptyset$ and $\mathbb{S} \neq \{\emptyset\}$, and let $\mathbf{t} \in \mathbb{P}(\mathbb{N})$. Then $P_{\mathbf{t}} \in \mathscr{I}(\mathbb{S})$ if and only if there are $m \in \mathbb{N}$ and $\mathbf{s}_1, \dots, \mathbf{s}_m \in \mathbb{S}$ such that $\mathbf{t} \setminus (\mathbf{s}_1 \cup \mathbf{s}_2 \cup \cdots \cup \mathbf{s}_m)$ is finite.*

Proof. The implication '$\Leftarrow$' was proved in Corollary 4.5.

Conversely, suppose that $P_{\mathbf{t}} \in \mathscr{I}(\mathbb{S})$, say

$$P_{\mathbf{t}} = \sum_{j=1}^m T_j P_{\mathbf{s}_j} R_j,$$

where $m \in \mathbb{N}$, $R_1, \dots, R_m, T_1, \dots, T_m \in \mathscr{B}(F)$, and $\mathbf{s}_1, \dots, \mathbf{s}_m \in \mathbb{S}$. Define operators

$$\Delta\colon x \mapsto (x, \dots, x),\ F \to F^{\oplus m}, \quad \text{and} \quad \Sigma\colon (x_1, \dots, x_m) \mapsto x_1 + \cdots + x_m,\ F^{\oplus m} \to F.$$

Then we have a commutative diagram:

$$\begin{array}{ccc} F & \xrightarrow{\quad P_{\mathbf{t}} \quad} & F \\ {\scriptstyle\Delta}\big\downarrow & & \big\uparrow{\scriptstyle\Sigma} \\ F^{\oplus m} & & F^{\oplus m} \\ {\scriptstyle R_1 \oplus \cdots \oplus R_m}\big\downarrow & & \big\uparrow{\scriptstyle T_1 \oplus \cdots \oplus T_m} \\ F^{\oplus m} & \xrightarrow{P_{\mathbf{s}_1} \oplus \cdots \oplus P_{\mathbf{s}_m}} & F^{\oplus m}. \end{array}$$

Set

$$R := (P_{\mathbf{s}_1} \oplus \cdots \oplus P_{\mathbf{s}_m})(R_1 \oplus \cdots \oplus R_m)\,\Delta\colon\ F \to F^{\oplus m}$$

and

$$T := \Sigma\,(T_1 \oplus \cdots \oplus T_m)(P_{\mathbf{s}_1} \oplus \cdots \oplus P_{\mathbf{s}_m})\colon\ F^{\oplus m} \to F,$$

so that $P_{\mathbf{t}} = TR$. It follows from [28, Lemma 3.9(ii)] that $Q := RTRT \in \mathscr{B}(F^{\oplus m})$ is idempotent with $\operatorname{im} Q \cong \operatorname{im} P_{\mathbf{t}}$. Clearly, $\operatorname{im} Q \subseteq \operatorname{im}(P_{\mathbf{s}_1} \oplus \cdots \oplus P_{\mathbf{s}_m})$.

For Banach spaces D and E, let us write $D \gtrsim E$ to indicate that D contains a subspace isomorphic to E. Then we have

$$\begin{aligned} F^{\oplus m} &\cong \operatorname{im}(P_{\mathbf{s}_1} \oplus \cdots \oplus P_{\mathbf{s}_m}) \oplus \ker(P_{\mathbf{s}_1} \oplus \cdots \oplus P_{\mathbf{s}_m}) \\ &\gtrsim \operatorname{im} Q \oplus \ker P_{\mathbf{s}_1} \oplus \cdots \oplus \ker P_{\mathbf{s}_m} \cong \operatorname{im} P_{\mathbf{t}} \oplus \operatorname{im} P_{\mathbb{N}\setminus\mathbf{s}_1} \oplus \cdots \oplus \operatorname{im} P_{\mathbb{N}\setminus\mathbf{s}_m} \\ &\gtrsim (\operatorname{im} P_{\mathbf{t}\setminus(\mathbf{s}_1\cup\mathbf{s}_2\cup\cdots\cup\mathbf{s}_m)})^{\oplus(m+1)} \cong F(\mathbf{t} \setminus (\mathbf{s}_1 \cup \mathbf{s}_2 \cup \cdots \cup \mathbf{s}_m))^{\oplus(m+1)}. \end{aligned}$$

By Theorem 4.7, this implies that the set $\mathbf{t} \setminus (\mathbf{s}_1 \cup \mathbf{s}_2 \cup \cdots \cup \mathbf{s}_m)$ is finite. ∎

The following lemma is a variation of the well-known Carl Neumann invertibility criterion (*e.g.*, see [10, Theorem 2.1.29(i)]).

LEMMA 4.9. *Let $\mathscr{J}$ be an ideal in a Banach algebra $\mathscr{A}$, and let $p \in \mathscr{A}$ be idempotent. Then $p \in \overline{\mathscr{J}}$ if and only if $p \in \mathscr{J}$.*

Proof. Only the implication '⇒' requires proof. Suppose that $p \in \overline{\mathscr{J}}$. If $p = 0$, then certainly $p \in \mathscr{J}$. Otherwise take $a \in \mathscr{J}$ such that $\|p - a\| < 1/\|p\|^2$. Then $\|p - pap\| < 1$, and so we may define

$$b := p + \sum_{n=1}^{\infty} (p - pap)^n \in \mathscr{A}.$$

We have

$$\begin{aligned} \mathscr{J} \ni bpap &= \Big(p + \sum_{n=1}^{\infty} (p - pap)^n\Big)(p - (p - pap)) \\ &= p + \sum_{n=1}^{\infty} (p - pap)^n - (p - pap) - \sum_{n=2}^{\infty} (p - pap)^n = p, \end{aligned}$$

as required. ∎

PROPOSITION 4.10. *Let $\mathbb{S}$ and $\mathbb{T}$ be subsets of $\mathbb{P}(\mathbb{N})$, and suppose that $\mathbb{S} \neq \emptyset$ and $\mathbb{S} \neq \{\emptyset\}$. Then the following three assertions are equivalent:*

(i) $\mathscr{I}(\mathbb{T}) \subseteq \mathscr{I}(\mathbb{S})$;
(ii) $\overline{\mathscr{I}}(\mathbb{T}) \subseteq \overline{\mathscr{I}}(\mathbb{S})$;
(iii) *for each $\mathbf{t} \in \mathbb{T}$, there are $m \in \mathbb{N}$ and $\mathbf{s}_1, \ldots, \mathbf{s}_m \in \mathbb{S}$ such that $\mathbf{t} \setminus (\mathbf{s}_1 \cup \mathbf{s}_2 \cup \cdots \cup \mathbf{s}_m)$ is finite.*

Proof. '(i)⇒(ii)' is clear, '(ii)⇒(i)' follows from Lemma 4.9, and '(i)⇔(iii)' is a consequence of Proposition 4.8. ∎

In [8, Corollary 5], Casazza, Kottman, and Lin proved that, for each $\mathbf{s}, \mathbf{t} \in \mathbb{P}(\mathbb{N})$, $F(\mathbf{s})$ is isomorphic to $F(\mathbf{t})$ if and only if the symmetric difference $\mathbf{s} \triangle \mathbf{t} := (\mathbf{s} \setminus \mathbf{t}) \cup (\mathbf{t} \setminus \mathbf{s})$ is finite. Taking $\mathbb{S} = \{\mathbf{s}\}$ and $\mathbb{T} = \{\mathbf{t}\}$ in Proposition 4.10 yields the following generalization of this result.

COROLLARY 4.11. *Let $\mathbf{s}$ and $\mathbf{t}$ be non-empty subsets of $\mathbb{N}$. Then $\mathscr{I}(\{\mathbf{s}\}) = \mathscr{I}(\{\mathbf{t}\})$ if and only if $\overline{\mathscr{I}}(\{\mathbf{s}\}) = \overline{\mathscr{I}}(\{\mathbf{t}\})$ if and only if $\mathbf{s} \triangle \mathbf{t}$ is finite.* ∎

The set $\mathbb{P}_{\text{fin}}(\mathbb{N})$ of finite subsets of $\mathbb{N}$ is an ideal in the Boolean algebra $\mathbb{P}(\mathbb{N})$, and so the quotient $\mathscr{Q} := \mathbb{P}(\mathbb{N})/\mathbb{P}_{\text{fin}}(\mathbb{N})$ is again a Boolean algebra. Let $\pi\colon \mathbb{P}(\mathbb{N}) \to \mathscr{Q}$ denote the quotient homomorphism. We note that, by definition,

$$\pi(\mathbf{s}) = \pi(\mathbf{t}) \iff \mathbf{s} \triangle \mathbf{t} \in \mathbb{P}_{\text{fin}}(\mathbb{N}) \qquad (\mathbf{s}, \mathbf{t} \in \mathbb{P}(\mathbb{N})). \tag{4.3}$$

Let $\text{ideal}(\mathscr{Q})$ denote the lattice of ideals in $\mathscr{Q}$, and let $\text{ideal}_0(\mathscr{B}(F))$ and $\overline{\text{ideal}}_0(\mathscr{B}(F))$ denote the lattices of non-zero ideals and closed non-zero ideals in $\mathscr{B}(F)$, respectively. Then we can define mappings

$$\Phi\colon\ \mathscr{Y} \mapsto \mathscr{I}(\pi^{-1}(\mathscr{Y})), \quad \text{ideal}(\mathscr{Q}) \to \text{ideal}_0(\mathscr{B}(F)),$$

and

$$\Psi\colon\ \mathscr{Y} \mapsto \overline{\mathscr{I}}(\pi^{-1}(\mathscr{Y})), \quad \text{ideal}(\mathscr{Q}) \to \overline{\text{ideal}}_0(\mathscr{B}(F)).$$

THEOREM 4.12. *The mappings Φ and Ψ are inclusion-preserving in both directions. In particular, Φ and Ψ are injective lattice homomorphisms.*

Proof. Let $\mathscr{Y}$ and $\mathscr{X}$ be ideals in $\mathscr{Q}$. We must show that

$$\mathscr{X} \subseteq \mathscr{Y} \iff \Phi(\mathscr{X}) \subseteq \Phi(\mathscr{Y}) \iff \Psi(\mathscr{X}) \subseteq \Psi(\mathscr{Y}).$$

Both implications '$\Rightarrow$' are easy, so it only remains to show that $\Psi(\mathscr{X}) \subseteq \Psi(\mathscr{Y})$ implies that $\mathscr{X} \subseteq \mathscr{Y}$.

To this end, set $\mathbb{S} := \pi^{-1}(\mathscr{Y})$ and $\mathbb{T} := \pi^{-1}(\mathscr{X})$. Then $\mathbb{S}$ and $\mathbb{T}$ are ideals in $\mathbb{P}(\mathbb{N})$ containing $\mathbb{P}_{\text{fin}}(\mathbb{N})$, and $\overline{\mathscr{I}}(\mathbb{T}) \subseteq \overline{\mathscr{I}}(\mathbb{S})$ by assumption. Given $\xi \in \mathscr{X}$, take $\mathbf{t} \in \mathbb{T}$ such that $\xi = \pi(\mathbf{t})$. Proposition 4.10 implies that there are $m \in \mathbb{N}$ and $\mathbf{s}_1, \dots, \mathbf{s}_m \in \mathbb{S}$ such that $\mathbf{r} := \mathbf{t} \setminus (\mathbf{s}_1 \cup \cdots \cup \mathbf{s}_m)$ is finite. Then $\mathbf{r} \in \mathbb{S}$, and since $\mathbb{S}$ is an ideal, we also have $\mathbf{s} := (\mathbf{s}_1 \cup \cdots \cup \mathbf{s}_m) \cap \mathbf{t} \in \mathbb{S}$. It follows that $\mathbf{t} = \mathbf{r} \cup \mathbf{s} \in \mathbb{S}$. We conclude that $\xi = \pi(\mathbf{t}) \in \pi(\mathbb{S}) = \mathscr{Y}$, as required. ∎

In particular, we see that there are uncountably many closed ideals in $\mathscr{B}(F)$. More precisely, we have the following result.

COROLLARY 4.13. *There is an uncountable subset $\mathbb{S}$ of $\mathbb{P}(\mathbb{N})$ such that $\overline{\mathscr{I}}(\{\mathbf{s}\}) \neq \overline{\mathscr{I}}(\{\mathbf{t}\})$ whenever $\mathbf{s}, \mathbf{t} \in \mathbb{S}$ are distinct.*

Proof. Take a maximal subset $\mathbb{S}$ of $\mathbb{P}(\mathbb{N})$ such that $\pi(\mathbf{s}) \neq \pi(\mathbf{t})$ whenever $\mathbf{s}, \mathbf{t} \in \mathbb{S}$ are distinct. It follows from (4.3) and Corollary 4.11 that the closed ideals $\overline{\mathscr{I}}(\{\mathbf{s}\})$, $\mathbf{s} \in \mathbb{S}$, are pairwise distinct. The set $\mathbb{S}$ is uncountable because $\mathbb{P}(\mathbb{N})$ is uncountable and $\mathbb{P}_{\text{fin}}(\mathbb{N})$ is countable. ∎

We shall next show that the mappings Φ and Ψ fail to be surjective. In fact, it is not hard to see that, for each non-zero ideal $\mathscr{Y}$ in $\mathscr{Q}$, we have strict inclusions $\mathscr{G}_{\ell_q}(F) \subsetneq \Phi(\mathscr{Y})$ and $\overline{\mathscr{G}}_{\ell_q}(F) \subsetneq \Psi(\mathscr{Y})$. However, it turns out that we can do a little better. This requires introduction of the Banach spaces

$$E_{r,s} := \Big(\bigoplus_{k=1}^{\infty} \ell_r^k\Big)_{\ell_s}, \tag{4.4}$$

where r and s are real numbers greater than or equal to 1.

LEMMA 4.14. *Let $(p_k)_{k=1}^{\infty}$ be a decreasing sequence of real numbers with $p := \inf p_k \geqslant 1$, let $(n_k)_{k=1}^{\infty}$ be an unbounded sequence of natural numbers, and let $q \geqslant 1$ be a real number. Then the Banach space F defined by* (4.1) *contains a complemented subspace isomorphic to $E_{p,q}$.*

Proof. By induction, we choose a strictly increasing sequence $(s(k))_{k=1}^{\infty}$ of natural numbers such that

$$n_{s(k)} \geqslant k \qquad \text{and} \qquad k^{1/p-1/p_{s(k)}} < 2 \qquad (k \in \mathbb{N}).$$

If $p < 2$, then we may also arrange that $p_{s(k)} \leqslant 2$ for each $k \in \mathbb{N}$. It follows from [46, Proposition 37.6(i)] that the Banach–Mazur distance between ℓ_p^k and $\ell_{p_{s(k)}}^k$ is less than 2 for each $k \in \mathbb{N}$. Since $\ell_{p_{s(k)}}^{n_{s(k)}}$ contains a 1-complemented copy of $\ell_{p_{s(k)}}^k$, we can take operators $R_k \colon \ell_p^k \to \ell_{p_{s(k)}}^{n_{s(k)}}$ and $T_k \colon \ell_{p_{s(k)}}^{n_{s(k)}} \to \ell_p^k$ such that $\|R_k\| = 1$, $\|T_k\| \leqslant 2$, and $T_k R_k = I_{\ell_p^k}$.

Set $\mathrm{s} := \{s(k) \mid k \in \mathbb{N}\}$. Then we can define operators

$$R\colon\ (x_k) \mapsto (R_k x_k), \quad E_{p,q} \to F(\mathrm{s}), \qquad \text{and} \qquad T\colon\ (x_k) \mapsto (T_k x_k), \quad F(\mathrm{s}) \to E_{p,q}.$$

Clearly, they satisfy $TR = I_{E_{p,q}}$, and so it follows from (4.2) and [28, Lemma 3.6] that $V_{\mathrm{s}}RTW_{\mathrm{s}}$ is an idempotent operator on F with image isomorphic to $E_{p,q}$. ∎

The following lemma is a simple modification of [29, Lemma 4.7].

Lemma 4.15. *Let D and E be Banach spaces, and let P be an idempotent operator on E. Then $P \in \mathscr{G}_D(E)$ if and only if $P \in \overline{\mathscr{G}}_D(E)$ if and only if, for some $n \in \mathbb{N}$, there is an idempotent operator Q on $D^{\oplus n}$ with $\operatorname{im} Q \cong \operatorname{im} P$.* ∎

Proposition 4.16. *Let $\mathscr{Y}$ be an ideal in $\mathscr{Q}$.*

(i) *If $\mathscr{Y} = \{\pi(\emptyset)\}$, then $\Phi(\mathscr{Y}) = \mathscr{F}(F)$ and $\Psi(\mathscr{Y}) = \overline{\mathscr{F}}(F)$.*
(ii) *If $\mathscr{Y} \neq \{\pi(\emptyset)\}$, then $\mathscr{G}_{E_{p,q}}(F) \subsetneq \Phi(\mathscr{Y})$ and $\overline{\mathscr{G}}_{E_{p,q}}(F) \subsetneq \Psi(\mathscr{Y})$, where $p := \inf p_k$ and the Banach space $E_{p,q}$ is defined by (4.4).*

In particular, the mappings Φ and Ψ are not surjective.

Proof. Clause (i) follows from Lemma 4.4(ii) and the fact that $\pi^{-1}(\{\pi(\emptyset)\}) = \mathbb{P}_{\mathrm{fin}}(\mathbb{N})$. To prove clause (ii), note that $\mathscr{Y} \neq \{\pi(\emptyset)\}$ means that we can take $\mathrm{s} \in \pi^{-1}(\mathscr{Y}) \setminus \mathbb{P}_{\mathrm{fin}}(\mathbb{N})$. Then we have

$$P_{\mathrm{s}} \in \Phi(\mathscr{Y}) \subseteq \Psi(\mathscr{Y}). \tag{4.5}$$

By Lemma 4.14, the Banach space $F(\mathrm{s})$ contains a complemented subspace isomorphic to $E_{p,q}$. Since $\operatorname{im} P_{\mathrm{s}} \cong F(\mathrm{s})$, there are operators $R\colon E_{p,q} \to F$ and $T\colon F \to E_{p,q}$ such that $I_{E_{r,s}} = TP_{\mathrm{s}}R$. Now the inclusions $\mathscr{G}_{E_{p,q}}(F) \subseteq \Phi(\mathscr{Y})$ and $\overline{\mathscr{G}}_{E_{p,q}}(F) \subseteq \Psi(\mathscr{Y})$ follow from (4.5) and the fact that $\Phi(\mathscr{Y})$ and $\Psi(\mathscr{Y})$ are operator ideals.

Assume towards a contradiction that $P_{\mathrm{s}} \in \overline{\mathscr{G}}_{E_{p,q}}(F)$. Then, by Lemma 4.15, we can take $n \in \mathbb{N}$ and an idempotent operator Q on $E_{p,q}^{\oplus n}$ with $\operatorname{im} Q \cong \operatorname{im} P_{\mathrm{s}}$. Casazza, Kottman, and Lin have shown that $E_{p,q} \cong E_{p,q}^{\oplus 2}$ (see [7, Corollary 7(i)]). Hence, using the notation $\gtrsim$ introduced in the proof of Proposition 4.8, we have

$$F \gtrsim E_{p,q} \cong E_{p,q}^{\oplus(2n)} \gtrsim (\operatorname{im} Q)^{\oplus 2} \cong F(\mathrm{s})^{\oplus 2},$$

contradicting Theorem 4.7. When combined with (4.5), this shows that the inclusions $\mathscr{G}_{E_{p,q}}(F) \subseteq \Phi(\mathscr{Y})$ and $\overline{\mathscr{G}}_{E_{p,q}}(F) \subseteq \Psi(\mathscr{Y})$ are strict. The final clause now follows from the facts that $\mathscr{F}(F) \subsetneq \mathscr{G}_{E_{p,q}}(F)$ and $\overline{\mathscr{F}}(F) \subsetneq \overline{\mathscr{G}}_{E_{p,q}}(F)$ because F contains a complemented subspace isomorphic to $E_{p,q}$ (*cf.* Lemma 4.14). ∎

Clearly, $E_{p,q}$ contains a complemented subspace isomorphic to ℓ_q, and so we have $\mathscr{G}_{\ell_q}(F) \subseteq \mathscr{G}_{E_{p,q}}(F)$. We shall next determine when this inclusion is strict. Certainly, a necessary condition is that $E_{p,q} \not\cong \ell_q$. It turns out that this condition is also sufficient (see Corollary 4.18, below).

We note that $E_{r,r} \cong \ell_r$ for each $r \geqslant 1$ and that $E_{2,s} \cong \ell_s$ whenever $s > 1$ (*e.g.*, see [30, p. 73]). Our next result establishes that $E_{r,s} \not\cong \ell_s$ in all other cases of interest to us (that is, whenever $r > \max\{2, s\}$). This is surely well-known to specialists in Banach space theory, but for the convenience of other readers, we include a proof. It relies on the notion of *cotype*. We refer to [13, Chapter 11] for an introduction to this subject.

LEMMA 4.17. *Let r and s be real numbers with $r > \max\{2, s\}$ and $s \geqslant 1$. Then the Banach space $E_{r,s}$ defined by* (4.4) *has cotype at least r. In particular, $E_{r,s}$ is not isomorphic to ℓ_s.*

Proof. Assume towards a contradiction that $E_{r,s}$ has cotype t, where $2 \leqslant t < r$. Then there is a constant $c > 0$ such that, for each $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in E_{r,s}$,

$$\Big(\sum_{j=1}^{k} \|x_j\|^t\Big)^{1/t} \leqslant c\left(\int_0^1 \Big\|\sum_{j=1}^{k} r_j(u)x_j\Big\|^2 \,\mathrm{d}u\right)^{1/2}, \tag{4.6}$$

where r_j denotes the j^{th} Rademacher function for each $j \in \mathbb{N}$. Take $k \in \mathbb{N}$, and let $J_k\colon \ell_r^k \to E_{r,s}$ denote the canonical k^{th} coordinate embedding. Then

$$\Big(\sum_{j=1}^{k} \|J_k(e_j)\|^t\Big)^{1/t} = k^{1/t} \qquad \text{and} \qquad \left(\int_0^1 \Big\|\sum_{j=1}^{k} r_j(u)J_k(e_j)\Big\|^2 \,\mathrm{d}u\right)^{1/2} = k^{1/r},$$

so that (4.6) implies that

$$c \geqslant k^{1/t-1/r} \to \infty \quad \text{as} \quad k \to \infty,$$

a contradiction.

The final clause is now immediate from the fact that ℓ_s has cotype $\max\{2, s\}$. ∎

COROLLARY 4.18. *Set $p := \inf p_k \geqslant \max\{2, q\}$. The inclusion $\mathscr{G}_{\ell_q}(F) \subseteq \mathscr{G}_{E_{p,q}}(F)$ is strict if and only if the inclusion $\overline{\mathscr{G}}_{\ell_q}(F) \subseteq \overline{\mathscr{G}}_{E_{p,q}}(F)$ is strict if and only if $p > 2$.*

Proof. By contraposition, we must prove that

$$\mathscr{G}_{\ell_q}(F) = \mathscr{G}_{E_{p,q}}(F) \iff \overline{\mathscr{G}}_{\ell_q}(F) = \overline{\mathscr{G}}_{E_{p,q}}(F) \iff p = 2.$$

The first implication '$\Rightarrow$' is clear.

For the second implication '$\Rightarrow$', suppose that $\overline{\mathscr{G}}_{\ell_q}(F) = \overline{\mathscr{G}}_{E_{p,q}}(F)$. Take an idempotent operator P on F with $\operatorname{im} P \cong E_{p,q}$ (*cf.* Lemma 4.14). Then $P \in \mathscr{G}_{E_{p,q}}(F) \subseteq \overline{\mathscr{G}}_{\ell_q}(F)$ by the assumption, and so Lemma 4.15 implies that, for some $n \in \mathbb{N}$, there is an idempotent operator Q on $\ell_q^{\oplus n}$ with $\operatorname{im} Q \cong \operatorname{im} P$. Since $\ell_q^{\oplus n} \cong \ell_q$, Pełczyński's theorem [30, Theorem 2.a.3] shows that either $\operatorname{im} Q$ is finite-dimensional or $\operatorname{im} Q \cong \ell_q$, and so the same is true for $\operatorname{im} P$. By Lemma 4.17, this implies that $p = 2$.

Finally, if $p = 2$, then $\ell_q \cong E_{p,q}$ (see above), and so obviously $\mathscr{G}_{\ell_q}(F) = \mathscr{G}_{E_{p,q}}(F)$ in this case. ∎

REMARK 4.19. All the closed ideals in $\mathscr{B}(F)$ that we have found are of the form $\overline{\mathscr{G}}_E(F)$ for some complemented subspace E of F. Further, all the complemented subspaces E of F that we have used are of the form

$$E = (E_1 \oplus E_2 \oplus \cdots \oplus E_k \oplus \cdots)_{\ell_q}, \tag{4.7}$$

where $E_1, E_2, \ldots, E_k, \ldots$ are uniformly complemented subspaces of $\ell_{p_1}^{n_1}, \ell_{p_2}^{n_2}, \ldots, \ell_{p_k}^{n_k}, \ldots$, respectively. We have no particular reason to believe that *all* closed ideals in $\mathscr{B}(F)$ have the special form mentioned above, nor that every complemented subspace of F is isomorphic to one of the form (4.7).

Finally, let us make some comments about maximal ideals in $\mathscr{B}(F)$.

Let X be a set. An ultrafilter $\mathscr{U}$ on X is *fixed* if $\mathscr{U} = \{\mathbf{u} \in \mathbb{P}(X) \mid x \in \mathbf{u}\}$ for some $x \in X$, and *free* otherwise. It is a standard fact that an ultrafilter is fixed if and only if it contains a finite set. There is a bijective correspondence between the set of ultrafilters on X and the set of maximal ideals in the Boolean algebra $\mathbb{P}(X)$. Specifically, an ultrafilter $\mathscr{U}$ on X corresponds to the maximal ideal $\{X \setminus \mathbf{u} \mid \mathbf{u} \in \mathscr{U}\}$ in $\mathbb{P}(X)$, and *vice versa*. It follows from these facts that each maximal ideal in the quotient $\mathbb{P}(X)/\mathbb{P}_{\mathrm{fin}}(X)$ has the form $\{\pi(X \setminus \mathbf{u}) \mid \mathbf{u} \in \mathscr{U}\}$ for some free ultrafilter $\mathscr{U}$ on X, where $\pi\colon \mathbb{P}(X) \to \mathbb{P}(X)/\mathbb{P}_{\mathrm{fin}}(X)$ is the quotient homomorphism.

In particular, this discussion applies to $X = \mathbb{N}$. When combined with Theorem 4.12, it leads to the following question.

QUESTION 4.20. Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$. Is the ideal $\overline{\mathscr{I}}(\{\mathbb{N} \setminus \mathbf{u} \mid \mathbf{u} \in \mathscr{U}\})$ a maximal ideal in $\mathscr{B}(F)$?

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