TOPOLOGICAL ALGEBRAS, THEIR APPLICATIONS,
AND RELATED TOPICS
BANACH CENTER PUBLICATIONS, VOLUME 67
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2005

DUALITY, UNIQUENESS OF TOPOLOGY AND AUTOMATIC CONTINUITY OF *-HOMOMORPHISMS IN BORNOLOGICAL LOCALLY C*-ALGEBRAS

GEORGE F. NASSOPOULOS

Department of Mathematics, University of Athens
Panepistemiopolis, Athens 157 84, Greece
E-mail: gnassop@math.uoa.gr

1. Introduction. This paper is a sequel to [8]. In that paper, the compound structure of the spectrum of a (commutative, unital) locally C*-algebra—which has no non-trivial counterpart in the usual norm case—is investigated, leading to the notion of a filtered space. To deepen our understanding, there may exist more than one compatible topology in a given involutive algebra which convert it into a locally C*-algebra [1]. Thus, the natural spectrum topology alone no longer determines the whole structure of the algebra in question but only at a purely algebraic level. For each of the various topologies, it is a *filtration* of selected compact subsets of the spectrum which is needed, too. In fact, the spectrum normally resides in the category *Tych* of Tychonoff (or completely regular) spaces and within admits a dual decomposition of the Arens-Michael one for the algebras considered, so producing the filtration we are looking for. The resulting pair of the spectrum-space and the specific filtration provides the associated filtered space. On the other hand, the completeness (and hence the locally C*-algebra structure) of the function algebras involved is characterized by means of filtered spaces. These facts enable us to extend the classical Gel'fand duality between locally C*-algebras and filtered spaces, in the form of a dual equivalence between convenient categories, as it often occurs for other duality theorems in Functional Analysis (see section 2).

The purpose of this paper is to utilize one of the basic properties of metrizable algebras, that of being bornological, in order to extend to them nice structural properties of C*-algebras. So the bornological locally C*-algebras seem to constitute the generalization proper of the classical C*-algebras. Specifically, the duality established in [8] is now placed in a more familiar setting and formulated between the spectrally barrelled

2000 *Mathematics Subject Classification*: Primary 46M15; Secondary 46K10, 18B30, 54B30.
Research of the author supported in part by University of Athens under grant 70/3/5634.
The paper is in final form and no version of it will be published elsewhere.

locally C*-algebras and their spectra, the compactly generated Nachbin-Shirota (Tychonoff) spaces; the bornological locally C*-algebras and their spectra, the compactly generated realcompact spaces; and by further specialization between the Fréchet locally C*-algebras and compactly generated hemicompact spaces, respectively. Informally, the filtration may be here cancelled, as being the greatest one consisting of all the compact subsets, and so no confusion is possible about the induced compact-open topology. In full agreement to the classical case it is then deduced that (i) there exists at most one topology in a given involutive algebra which converts it into a bornological locally C*-algebra, namely the compact-open topology in the functional representation of it; and (ii) every *-homomorphism $h : A \to B$ of a bornological algebra A to any locally C*-algebra B is necessarily continuous, as all the characters of A are. Another application also of interest is a generalization of the classical Banach-Stone Theorem to compactly generated Tychonoff spaces, in particular to locally compact ones (Corollary 5).

Traditionally compactly generated spaces are defined within the category *Haus* of Hausdorff spaces and continuous maps. They form a full coreflective subcategory *CGHaus* of it, complete, cocomplete and cartesian closed viz., a convenient category [10, 4]. From the categorical point of view, however, this machinery can be equally well applied to certain full epireflective subcategories of *Haus* containing the class of all compact spaces, as for instance to *Tych* of Tychonoff spaces, to $RCom_2$ of realcompact spaces and so on. The point is here that a space is compactly generated *in the reflective subcategory* if it is of course homeomorphic to the inductive limit of the system of all the compact subsets of it ordered by inclusion, but now the inductive limit exists and is taken to be in the subcategory in question. This means that it is constructed as the image by the reflector (the left adjoint to the embedding functor) of the ordinary inductive limit in *Haus*. A description of the reflection in *Tych* of any Hausdorff space X goes indeed back to the Stone-Čech Reduction Theorem, the resulting space being the "largest Tychonoff quotient" of X. It follows that a space is a *compactly generated Tychonoff space* if it possesses the *finest Tychonoff topology* for which all the inclusion maps of its compact subsets remain continuous. In its turn, the reflection in $RCom_2$ of a Tychonoff space T compactly generated or not is provided by the Hewitt realcompactification of T [2]. Note also that whenever the inductive limit in *Haus* already lies in the subcategory considered, it is the convenient one (but not conversely). Thus the spaces that belong to both categories *CGHaus* and *Tych* (resp. $RCom_2$) with illustrative examples locally compact, or metrizable spaces (resp. separable metric spaces, especially $\mathbb{R}^n$) are compactly generated Tychonoff (resp. realcompact) spaces.

I am indebted to the referee for useful suggestions on a previous draft of this paper.

2. Preliminaries. To fix the notation, by a *locally C*-algebra* (or pro-C*-algebra) we mean a complete Hausdorff topological *-algebra A over $\mathbb{C}$, the topology of which is determined by the set $P(A)$ of continuous C*-(algebra-) seminorms p on A with $p(1) = 1$ in case A is unital. A morphism $h : A \to B$ of locally C*-algebras is any (uniformly) continuous multiplicative linear map, involution and unit-preserving. The commutative, unital, locally C*-algebras form a category $\mathcal{A}$, $\mathcal{A}_1$ being the full subcategory of C*-algebras.

Given a locally C*-algebra A, it admits the so called Arens-Michael decomposition [5] as the projective limit of its ingredient C*-algebras A_p, so that $A = \varprojlim_p A_p$ in $\mathcal{A}$. Here the set $P(A)$ is properly directed in the pointwise ordering of the real-valued functions, while the algebras A_p, $p \in P(A)$ are the quotients of A (as well as of A_b, the C*-subalgebra of bounded elements of A, i.e., those satisfying $sup\{p(\alpha) \; | p \in P(A)\} < +\infty$) by the nullideals (or kernels) of the seminorms p, respectively. They are indeed complete in the induced norms and hence C*-algebras ([9], Satz 3.6, Folgerung 5.4 and following Remark). Formally a locally C*-algebra is denoted by $(A; A_p, p \in P(A))$, the projections can be chosen to be the quotient epimorphisms $\pi_p : A \to A_p$ for all $p \in P(A)$.

Now the spectrum $\Omega(A)$ is realized by all the continuous and therefore, hermitian characters (as A_b is dense in A) to be compatible with the individual structure of the commutative algebra A. Equipped with the topology of simple convergence it is a Tychonoff space. As it is shown in ([8], Spectral Decomposition Theorem), the spectrum is in fact a compactly generated Tychonoff space being the inductive limit of the spectral system formed by the spectra $\Omega(A_p)$, $p \in P(A)$ of the ingredient C*-algebras, that is, it admits a dual decomposition $\Omega(A) = \varinjlim_p \Omega(A_p)$ in *Tych*. Obviously, this decomposition is necessary for a duality, but in general fails in the category *Haus* (see [3], p. 58, Proposition 1.2 and Rem. 1.4 for a counterexample). The idea of the given proof is based on the fact that for every continuous function $f : \varinjlim_p \mathit{\Omega}(A_p) \to \mathbb{C}$ there exists a unique element $\alpha = (\alpha_p, p \in P(A))$ in A such that $f = \hat{\alpha} \; o \; \theta$, where $\hat{\alpha}$ stands for the Gel'fand transform of α and θ for the map provided by the universal property of the inductive limits (a continuous bijection in *Haus*).

By a filtration of a space X we mean a directed system $X_s, s \in S$ of compact subsets of it such that $X = \varinjlim_s X_s$ in *Tych*. A pair $(X; X_s, s \in S)$ where X is a Tychonoff space and $X_s, s \in S$ a specific filtration of it is said to be a *filtered space* (see also [10], section 10). The spectrum $(\Omega(A); \Omega(A_p), p \in P(A))$ of a locally C*-algebra $(A; A_p, p \in P(A))$ of course provides the motivating example. Here the spectra $\Omega(A_p), p \in P(A)$ are identified with their homeomorphic images in $\Omega(A)$, the compact, equicontinuous subsets of it ([6], p. 32). On the other hand, every compactly generated Tychonoff space is viewed as a filtered space (at least) with respect to the greatest filtration consisting of the totality of its compact subsets, whereas any compact space is also so considered with the trivial filtration of the space alone, which is cofinal in the greatest one. Furthermore, the function algebra $C(X)$ endowed with the topology of uniform convergence on a directed system $X_s, s \in S$ of compact subsets covering X, i.e., the induced one by the supremum seminorms corresponding to the X_s's, is plainly complete and hence a locally C*-algebra if (and only if) $X_s, s \in S$ is a filtration of the Tychonoff space X. By a morphism $f : (X; X_s, s \in S) \to (Y; Y_r, r \in R)$ of filtered spaces we mean a continuous map f of X to Y subject to the condition: "For every $s \in S$ there exists some $r \in R$ such that $f(X_s) \subseteq Y_r$". Denote by $\mathcal{X}$ the resulting category of filtered spaces. Then the above described assignments between locally C*-algebras and filtered spaces are clearly functorial and a duality is established, extending the classical Gel'fand duality, by passing to the respective limits.

THEOREM ([8]). *The category $\mathcal{A}$ of commutative, unital, locally C*-algebras is dually equivalent to the category $\mathcal{X}$ of filtered spaces, the accomplished functors being the spectral functor $\Omega : \mathcal{A} \to \mathcal{X}^{op}$ and its equivalence-inverse, the function algebra functor $\mathcal{C} : \mathcal{X}^{op} \to \mathcal{A}$.*

Needless to say, both the accomplished Gel'fand and Dirac natural transformations define isomorphisms between locally C*-algebras the former, filtered spaces the latter.

3. The main results. Our interest is now concentrated on special locally C*-algebras, the spectrally barrelled, bornological and Fréchet ones, and the interactions upon their spectra. The above stated duality is restricted to them and is placed in a more familiar setting . In fact, the category *CGTych* of compactly generated Tychonoff spaces is fully embedded into the category $\mathcal{X}$ of filtered spaces, each space of the former exclusively considered with the greatest filtration. This being the case, the greatest filtration may be informally omitted, keeping in mind that the topology induced by it in the representative function algebras is the compact-open.

For the sake of comprehension we recall from ([5], Chap. VIII, sections 1 and 2) the following notion and some of the basic properties of it.

Definition 1. A commutative, unital, locally m-convex algebra A is called *spectrally barrelled* if the properties of being equicontinuous, relatively weakly compact and weakly bounded are equivalent for any subset of its spectrum space $\Omega(A)$.

In general, a spectrally barrelled locally m-convex algebra need not be a barrelled locally convex space.

Proposition 2. *The inductive limit of a system of spectrally barrelled locally m-convex algebras is spectrally barrelled. In particular, every i-bornological locally C*-algebra is spectrally barrelled.*

Proof. Let A be the inductive limit of a system $\{A_\gamma, \gamma \in \Gamma\}$ of such algebras with universal morphisms $g_\gamma : A_\gamma \to A$. Then one has for the spectra $\mathit{\Omega}(A) = \varprojlim_\gamma \mathit{\Omega}(A_\gamma)$ up to a homeomorphism (e.g. [3], p. 59, Proposition 1.3). If $B \subset \mathit{\Omega}(A)$ is weakly bounded in the dual space A', then its projection $B_\gamma = \{\chi_\gamma = \chi \; o \; g_\gamma \mid \chi \in B\}$ is a weakly bounded subset of $\mathit{\Omega}(A_\gamma)$ for each $\gamma \in \Gamma$ and $B \subset \Pi_\gamma B_\gamma$. By the Tychonoff Theorem B is relatively weakly compact, as this holds for all B_γ. Furthermore, for each 0-neighbourhood V in $\mathbb{C}$, the inverse image $g_\gamma^{-1}(\bigcap_{\chi \in B} \chi^{-1}(V)) = \bigcap_{\chi_\gamma \in B_\gamma} \chi_\gamma^{-1}(V)$ is a 0-neighbourhood in A_γ for all $\gamma \in \Gamma$, because the B_γ's are also equicontinuous. It follows that $\bigcap_{\chi \in B} \chi^{-1}(V)$ is a 0-neighbourhood in A, so that B is equicontinuous, too. Hence A is spectrally barrelled.

The final assertion follows at once from the fact that an i-bornological locally C*-algebra A is expressed as the inductive limit of a suitable system of Banach subalgebras of A ([12], p. 201, Corollary to Theorem 1).

Theorem 3. *The full subcategory $\mathcal{SB}$ of spectrally barrelled algebras in $\mathcal{A}$ is dually equivalent to the full subcategory of CGTych, consisting of Nachbin-Shirota spaces.*

Proof. By the definition of a spectrally barrelled algebra, the weakly compact subsets and the weakly closed equicontinuous subsets coincide in the spectrum space. But this holds if and only if the filtration of the spectrum of a locally C*-algebra A consisting of the compact, equicontinuous subsets of $\Omega(A)$ is the greatest one. By the above made convention the spectrum of a spectrally barrelled locally C*-algebra is merely a compactly generated Tychonoff space. On the other hand, the weakly bounded subsets and the relatively weakly compact subsets being identical, $\Omega(A)$ is a Nachbin-Shirota space, as well, in the terminology of [5]. (Conversely, the function algebra associated with such a space is the desired one).

The next duality improves and strengthens well-known results by putting them in the right setting.

THEOREM 4. *The full subcategory $\mathcal{B}$ of (i-)bornological algebras in $\mathcal{A}$ is dually equivalent to the full subcategory of CGTych consisting of realcompact spaces.*

Proof. By Proposition 2 and Theorem 3 the spectrum of an i-bornological locally C*-algebra A is a compactly generated Tychonoff space. But now the Gel'fand transformation of the i-bornological A onto the function algebra $\mathcal{C}(\Omega(A))$ is a homeomorphism for the compact-open topology. An appeal to the algebra version ([12], p. 207, Theorem 5) of the Nachbin-Shirota Theorem therefore shows that $\Omega(A)$ is furthermore a (compactly generated) realcompact space. Conversely, the function algebra for the compact-open topology, associated with a compactly generated realcompact space is an i-bornological locally C*-algebra.

REMARK. The classical Nachbin-Shirota Theorem of course implies that an i-bornological locally C*-algebra is *a fortiori* a bornological (and in presence of completeness, a barrelled) locally convex space, too. Consequently, there is no reason hereafter to distinguish between the two notions for locally C*-algebras.

As an application of the dualities stated, one gets:

COROLLARY 5. *The function algebra $\mathcal{C}(\mathbb{R}^n)$ for the compact-open topology is a bornological (and so a spectrally barrelled) locally C*-algebra and a Fréchet one for $n = 1$. Moreover, two compactly generated Tychonoff spaces, in particular two locally compact or metrizable spaces T and T' are homeomorphic if (and only if) their function algebras $\mathcal{C}(T)$ and $\mathcal{C}(T')$ are isomorphic.*

Proof. The first argument in the proof of Theorem 3 also shows that one more duality is in order between the full subcategory of $\mathcal{A}$ consisting of algebras having a continuous Gel'fand transformation for the compact-open topology and the category *CGTych* itself. This makes the second assertion clear.

COROLLARY 6 ([7], Corollary 5.6). *Each character on a bornological locally C*-algebra A is always continuous.*

Proof. Since the spectrum is here a realcompact space, the characters of the function algebra $\mathcal{C}(\Omega(A))$ (and hence of A) are exhausted to the evaluations at points of $\Omega(A)$ and of course, they are all continuous.

PROPOSITION 7. *On an involutive algebra A there exists at most one topology converting it into a bornological locally C*-algebra.*

Proof. Suppose that there are two such topologies in A. The respective spectra of A to them both consist of the totality of its characters, have the same weak topology and the common greatest filtration. They are therefore, identical. An application of the

Duality Theorem 4 so entails the coincidence of the two topologies and its determination, whenever this exists, as the compact-open topology in its functional representation.

Proposition 8. *Let A be a bornological and B any locally C*-algebra. Then every *-homomorphism $h : A \to B$ from A to B is necessarily continuous.*

Proof. By the projective topology of B, the continuity of h is equivalent to that of the composites $\omega \circ h$ for all characters ω in $\Omega(B)$. But this is immediate from Corollary 6.

Theorem 9 (compare [11], Theorem 4). *The full subcategory $\mathcal{M}$ of Fréchet algebras in $\mathcal{A}$ is dually equivalent to the full subcategory of CGTych consisting of hemicompact spaces.*

Proof. Let A be a Fréchet locally C*-algebra. Then there exists a countable cofinal subset in $P(A)$ and hence an increasing sequence of its seminorms, which determines the topology of A. Dually, the ensuing sequence of compact subsets of the realcompact spectrum $\Omega(A)$ of A from it is then cofinal in the greatest filtration of $\Omega(A)$. Consequently, $\Omega(A)$ is in particular, a hemicompact space. Of course, the compact-open topology in the function algebra $\mathcal{C}(\Omega(A))$ is induced by the sequence quoted, as well. To this end, one may ignore any filtration, as no confusion is possible. It is also clear that the function algebra corresponding to a hemicompact space in *CGTych* being metrizable is a Fréchet locally C*-algebra.

References

[1] G. R. Allan, *On a class of locally convex algebras,* Proc. London Math. Soc. 17 (1967), 91–114.

[2] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

[3] A. Guichardet, *Leçons sur certaines algèbres topologiques*, Gordon and Breach, Paris, 1967.

[4] S. MacLane, *Categories for the Working Mathematician,* second edition, GTM 5, Springer, Berlin, 1998.

[5] A. Mallios, *Topological Algebras, Selected Topics,* NHMS 124, North-Holland, Amsterdam, 1986.

[6] E. A. Michael, *Locally multiplicative-convex topological algebras*, Mem. Amer. Math. Soc. 11, Providence, 1952.

[7] P. D. Morris and D. E. Wulbert, *Functional representations of topological algebras*, Pacific J. Math. 22 (1967), 323–337.

[8] G. F. Nassopoulos, *Spectral decomposition and duality in locally C*-algebras*, preprint, University of Athens, 2002.

[9] K. Schmüdgen, *Über LMC*-Algebren*, Math. Nachr. 68 (1975), 167–182.

[10] N. E. Steenrod, *A convenient category of topological spaces,* Michigan Math. J. 14 (1967), 133–152.

[11] S. Warner, *The topology of compact convergence on continuous function spaces*, Duke Math. J. 25 (1958), 265–282.

[12] S. Warner, *Inductive limits of normed algebras*, Trans. Amer. Math. Soc. 82 (1956), 190–216.