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LOCALLY CONVEX DOMAINS OF INTEGRAL OPERATORS

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Abstract. We have shown in [1] that domains of integral operators are not in general locally convex. In the case when such a domain is locally convex we show that it is an inductive limit of L^1-spaces with weights.

We begin with some definitions and facts concerning integral operators and their domains. For more details and references on the subject we refer to [1]. In what follows we only consider real valued functions and do not address the obvious modifications needed in the complex case.

For finite measure spaces (T, dt), (S, ds) we denote by $L^0(T)$ and $L^0(S)$ the corresponding spaces of measurable, finite a.e. functions, equipped with complete metric vector topologies of convergence in measure. These topologies may be defined e.g., by means of pseudonorms of the form $u \to \int \frac{|u|}{1+|u|}$, with integration over S or over T, which we respectively denote by ρ_S and ρ_T. For a measurable function on $T \times S$, $k(t,s)$, we consider the integral operator K with kernel k, given by the formula $Ku(t) = \int k(t,s)u(s)ds$. We view this operator as an (unbounded) linear operator from $L^0(S)$ into $L^0(T)$. The operator is defined on the set of all functions $u \in L^0(S)$ for which $|K||u| \in L^0(T)$ (i.e., $\int_S |k(t,s)||u(s)|ds < \infty$ a.e.) This set is denoted D_K and referred to as the proper domain of K (as distinguished from the extended domain which is not considered here). D_K is a complete, solid metric vector space with the topology defined by the pseudonorm $\rho_K(u) = \rho_T(|K||u|) + \rho_S(u)$ and $K : D_K \to L^0(T)$ is continuous. We refer to this topology as *the metric topology* of D_K. A sequence $\{u_n\}$ converges to 0 in D_K if and only if both u_n and $|K||u_n|$ (and hence Ku_n) converge to 0 in measure. It is clear that for the purpose of study of D_K we may restrict our attention to nonnegative kernels, $k \geq 0$, the restriction made throughout the paper. Since (except for the purely atomic case) the topology of convergence in measure is not locally convex, there is no a priori reason to

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expect D_K to be in general locally convex. A trivial situation when this is not the case is when the kernel k vanishes on a set of the form $T \times S'$ where S' is a non-atomic subset of S of positive measure. In this case D_K contains as a direct summand the space $L^0(S')$ and cannot be locally convex. To avoid this trivial case we assume from now on that for a.e. $s \in S$, $k(t,s) > 0$ on a subset of T of positive measure. But even then, we constructed in [1] examples of kernels $k > 0$ a.e. on $T \times S$ with D_K not locally convex.

The simplest case of local convexity is that of a normed space. In this case we have the following theorem proved in [1].

THEOREM 1. *If D_K is a normed space, then it is an L^1-space with a weight, i.e., there is a $w \in L^0(S)$, $w > 0$ a.e. such that $D_K = \{u \in L^0(S); \int |u| w ds < \infty\} = L^1_w$.*

In the remainder of this note we shall use the terms *weight* and the corresponding notation L^1_w as they are introduced in Theorem 1. The set of all weights is partially ordered and directed by the relation $w_1 \preceq w$ if and only if $w_1 \geq cw$ a.e., where $c > 0$ is a constant. Note that this order reverses the pointwise relation $\leq$. Accordingly we have the equivalence relation $w \sim w_1$ if and only if $w \preceq w_1$ and $w_1 \preceq w$. We denote by $\mathcal{W}$ the family of all equivalence classes of weights. This set is again partially ordered and directed in an obvious way. To the partial order of w-s corresponds the partial order by continuous inclusion of L^1_w-s: $w \preceq w_1$ if and only if $L^1_w \subset_c L^1_{w_1}$. We find it convenient to think of L^1_w as a topological space associated with the equivalence class $[w]$, rather than a normed space associated with a specific weight w. To simplify, we do not include this convention in our notation, writing L^1_w instead of $L^1_{[w]}$. Observe that the space of all measurable functions L^0 can be written as the union of all spaces L^1_w, $[w] \in \mathcal{W}$. Hence we can think of L^0 as the direct (inductive) limit of the topological spaces L^1_w, i.e., the union of L^1_w-s endowed with the strongest (largest) topology making all the inclusions $L^1_w \subset \bigcup L^1_w$ continuous (see [2]). We refer to this topology as the *DL-topology*. Clearly the DL-topology is stronger (larger) than the original metric topology of convergence in measure. We could also introduce, using the above representation of L^0, the largest locally solid SDL, largest locally convex CDL and locally solid and locally convex $SCDL$ *topologies* on L^0. It is easily verified that the SDL-topology coincides with the metric topology of L^0.

As mentioned above, we are interested in describing the domain of K in the case when it is locally convex. It is perhaps useful to consider some simple examples of locally convex domains.

EXAMPLE 1. If T is purely atomic, $T = \{t_n\}$, $n = 1, 2, \ldots$ say, then D_K is the intersection of the sequence of spaces L^1 with weights $w_n(s) = k(t_n, s)$.

This example suggested a conjecture made in [1] that a locally convex D_K is the intersection of a sequence of spaces $L^1_{w_n}$ with appropriately chosen weights w_n.

EXAMPLE 2. Suppose that S is purely atomic, say $S = \{s_n\}$, $n = 1, 2, \ldots$ and that k has the property that for a.e. t the set $\{n; k(t, s_n) > 0\}$ is finite. Then D_K can be identified with the space of all sequences, l^0, with the product topology, i.e., the topology of coordinate-wise convergence.

EXAMPLE 3. Suppose that S and T are purely non-atomic and that $k(t,s) = \sum_{n=1}^{\infty} \psi_n(t)\varphi_n(s)$ where $\psi_n, \varphi_n > 0$ and that the sum is finite a.e. Then $D_K \subseteq \bigcap L^1_{\varphi_n}$ with the equality occurring if and only if $\limsup\{t; \psi_n(t) > 0\}$ is of measure 0.

The last two examples seem to put in doubt the conjecture formulated above.

Instead we are going to prove that a locally convex domain can be represented as a direct limit of spaces L^1_w with $[w]$ restricted to a directed set of equivalence classes determined by the operator K which will be defined presently.

We say that the operator K is non-singular if there is $u \in D_K$ such that $u > 0$ a.e. It is easily seen that if K is non-singular, then so is the transposed operator $K^\star : L^0(T) \to L^0(S)$ with the kernel $k^\star(s,t) = k(t,s)$. To avoid discussion of some trivial cases we assume that K is non-singular and that $K^\star v > 0$ a.e. whenever $v > 0$ a.e. This amounts to the assumption that for a.e. s the set $\{t; k(t,s) > 0\}$ is of positive measure.

We denote by $\mathcal{W}_K$ the set of all equivalence classes of functions of the form $w = K^\star v$, $v > 0$ a.e., with the equivalence relation $w \sim w_1$ defined above, after Theorem 1. We notice that $\mathcal{W}_K \subset \mathcal{W}$ and inherits from $\mathcal{W}$ its partial order. To see that $\mathcal{W}_K$ is directed it suffices to notice that for $w = K^\star v$ and $w_1 = K^\star v_1$ we have $[w] \preceq [\tilde{w}]$ and $[w_1] \preceq [\tilde{w}]$ where $\tilde{w} = K^\star \min\{v, v_1\}$, a definition independent of the choice of the representatives w and w_1. We further observe the following simple facts:

1) $D_K = \bigcup\{L^1_w; [w] \in \mathcal{W}_K\}$.
2) For every $w \in \mathcal{W}_K$ we have continuous inclusion $L^1_w \subset_c D_K$.

Both facts follow from the observation that $\int_T v(t)K|u|(t)dt = \int_S K^\star v(s)|u(s)|ds$. Every function in $L^0(T)$, in particular $K|u|$ for $u \in D_K$, belongs to L^1_v for some $v > 0$, $v \in D_{K^\star}$. Then $[K^\star v] = [w] \in \mathcal{W}_K$ and $u \in L^1_w$. If $u_n \to 0$ in L^1_w then $Ku_n \to 0$ in L^1_v where $K^\star v = w$ and both u_n and $K|u_n|$ converge to 0 in measure, hence $u_n \to 0$ in D_K.

The reason for considering the set of equivalence classes $\mathcal{W}_K$ rather than the set of all weights coming into competition is an attempt to keep an order among the corresponding L^1-spaces. Already in the argument in [1] leading to Theorem 1 above, there are infinitely many equivalent weights defining D_K which are now organized into $\mathcal{W}_K$ consisting of one element. Notice that conclusion 1) above remains valid with $\mathcal{W}_K$ replaced by a larger set of indices, $\mathcal{W}'_K$ consisting of all equivalence classes $[w] \in \mathcal{W}$ such that $[w_1] \preceq [w] \preceq [w_2]$ with $[w_1], [w_2] \in \mathcal{W}_K$ (see example 5 below). Note that $\mathcal{W}_K$ is cofinal in $\mathcal{W}'_K$.

On the union $D_K = \bigcup\{L^1_w; [w] \in \mathcal{W}_K\} = \bigcup\{L^1_w; [w] \in \mathcal{W}'_K\}$ one can introduce the direct limit topology, i.e., the strongest (largest) topology making all the inclusions $L^1_w \subset D_K$ continuous. We refer to this topology as *DL-topology* on D_K. Similarly as above, we can introduce on D_K three weaker (smaller) topologies SDL, CDL and $SCDL$. Similarly as in the case of L^0, SDL-topology on D_K coincides with its metric topology. We are now going to show (Theorem 4) that in the case when the metric topology on D_K is locally convex, then all five possible topologies on D_K coincide.

These considerations are based on a classical result of Maurey-Nikishin which we state in the form of the following lemma.

LEMMA. *Let C be be a convex set of nonnegative functions which is bounded in $L^0(T)$. Then there exists a function $v \in L^0(T)$, $v > 0$ a.e., such that $\int_T v(t)f(t)dt \le 1$ for all $f \in C$.*

As a corollary we get the following theorem.

THEOREM 2. *If the metric topology on D_K is locally convex and if $B \subset D_K$ is a bounded set in this topology, then B is contained and bounded in L^1_w for some $[w] \in \mathcal{W}_K$.*

Proof. Since the metric space D_K is solid and is assumed to be locally convex, the set $|B| = \{|u|; u \in D_K\}$ is bounded in D_K and so is its convex hulls $\mathrm{conv}|B|$. By continuity of K the convex set $C = K(|B|)$ is bounded in $L^0(T)$. Let v be as in the lemma. Then for every $u \in B_+$ we have $\int_T v(t) \int_S k(t,s)u(s)dsdt \le 1$ implying that $v \in D_{K^\star}$ and that with $w = K^\star v$, we have $u \in L^1_w$ and $\|u\|_w \le 1$ for all $u \in \mathrm{conv}|B|$ and hence for all $u \in B$. ∎

We next consider some situations where the conclusion of Theorem 2 is of interest.

Suppose that D_K with its metric topology is a normed (and therefore Banach) space, as in Theorem 1. Then the unit ball in the norm is bounded in D_K and, by Theorem 2, is contained in L^1_w. This together with the reverse inclusion, $L^1_w \subset D_K$ implies that $D_K = L^1_w$, the conclusion in Theorem 1.

We now consider convergent sequences in D_K with its metric topology assumed to be locally convex. If $u_n \to u$ in D_K, then the set $\{u, u_1, u_2, \dots\}$ is bounded in D_K and therefore, by Theorem 2, is contained in L^1_w for some $w \in \mathcal{W}_K$. We have the following theorem.

THEOREM 3. *If the metric topology on D_K is locally convex and if (u_n), $u_n \in D_K$ is a sequence convergent in this topology, $u_n \to u$, then $\{u, u_1, u_2, \dots\} \subset L^1_w$ for some $[w] \in \mathcal{W}_K$ and (u_n) contains a subsequence convergent to u in $L^1_{\tilde{w}}$, with some $[\tilde{w}] \in \mathcal{W}$, perhaps distinct from $[w]$.*

Proof. As already indicated, the first part of the statement follows readily from Theorem 2. To prove the second part, we may assume (using the first part) that $u = 0$. Then $\rho_K(u_n) \to 0$ and for some subsequence $(\tilde{u}_n)$ of (u_n) we have $\sum \rho_K(\tilde{u}_n) < \infty$. It follows that the set $\{\sum_1^\infty |\tilde{u}_n|, |\tilde{u}_1|, |\tilde{u}_1| + |\tilde{u}_2|, \dots\}$ is bounded in D_K and therefore contained in some $L^1_{\tilde{w}}$. Since $\tilde{u}_n \to 0$ in measure and $|\tilde{u}_n| \le \sum_1^\infty |\tilde{u}_n|$ for all n, the Lebesgue dominated convergence theorem implies that $\tilde{u}_n \to 0$ in $L^1_{\tilde{w}}$. ∎

The next result is a corollary to Theorem 3.

THEOREM 4. *If the metric topology on D_K is locally convex, then it coincides with DL-topology.*

Proof. Since, as already remarked, the metric topology on D_K is weaker than DL-topology, it suffices to show that every sequence convergent to 0 in the metric converges to 0 in DL-topology. If this was not so, we could find a sequence $u_n \to 0$ in the metric ρ_K and a neighborhood of 0 in the DL-topology, say U such that $u_n \notin U$ for all n. By Theorem 3 (u_n) contains a subsequence $\tilde{u}_n \to 0$ in some L^1_w, $w \in \mathcal{W}$ and $\tilde{u}_n \in U \cap L^1_w$ for large n, a contradiction. ∎

We conclude with two examples and a remark.

EXAMPLE 4. D_K is a Banach space if and only if $\mathcal{W}$ contains its upper bound in the sense of the $\preceq$-order.

Indeed, if D_K is a Banach space, then, by Theorem 1, $D_K = L^1_{\tilde{w}}$ for some $[\tilde{w}] \in \mathcal{W}$. Then for every $[w] \in \mathcal{W}$, $L^1_w \subset L^1_{\tilde{w}}$ and $[\tilde{w}]$ is the upper bound of $\mathcal{W}$. The reverse implication is immediate.

Example 5. One could expect the situation as in the previous example to occur in the case when $k(t,s) \geq \sum_{j=1}^{N} \psi_j(t)\varphi_j(s)$, $\psi_j, \varphi_j > 0$ a.e.

In the case of equality, i.e., of a finite-dimensional K, we have $\mathcal{W} = [\sum \varphi_j]$ consists of a single element. In general the set of equivalence classes $\mathcal{W}$ has as an upper bound $[\sum l_j]$ but it is not at all clear that it has the least upper bound, and even if it does, whether this bound belongs to $\mathcal{W}'_K$. For instance, if $N = 1$, we have the situation where $D_K = L^1_\varphi$ if $[\varphi] \in \mathcal{W}'_K$, otherwise there exists a cofinal sequence in $\mathcal{W}'_K$, i.e., D_K is a strong direct limit of L^1 spaces with weights (once it is known to be locally convex).

Remark. The conclusion of Theorem 4 with DL-topology replaced by SDL-topology or by CDL-topology could be obtained more directly without the use of the theorem of Murey-Nikishin. This, however, appears to be needed to obtain Theorem 2 and Theorem 3 which seem to be of independent interest.

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References

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