BOURGAIN ALGEBRAS OF G-DISC ALGEBRAS

T. TONEV and K. YALE

The University of Montana, Missoula, MT 59803, U.S.A.
E-mails: tonevtv@mso.umt.edu, ikyale@msn.com

1. Introduction. The norm topology of a commutative Banach algebra $A$ is too rough to reflect some of the delicate properties of $A$. Weaker topologies are consequently of importance and they can be used to construct algebras associated to $A$ and which contain important information about $A$. Bourgain algebras were introduced by J. Cima and R. Timoney [2] in their study of Dunford-Pettis property (DPP) for a certain class of function algebras. In effect, the algebra $A$ has the DPP whenever its Bourgain algebra is as large as possible. In this paper we determine the Bourgain algebras related with some $G$-disc algebras.

Given a commutative Banach algebra $A$, let $c_o^w(A)$ denote the family of all sequences of elements in $A$ which tend to 0 with respect to a given topology on $A$. For the weak topology $w$, $c_o^w(A)$ is the set of all weakly null sequences of elements $\varphi_n$ in $A$, i.e. sequences $\{\varphi_n\}$ such that $L(\varphi_n) \to 0$ as $n \to \infty$ for any bounded linear functional on $A$.

Let $A \subset B$ be two commutative Banach algebras and let the norm $\| \cdot \|_B$ to $A$. The Bourgain algebra $A_b^B$ of $A$ with respect to $B$ is the set of all $f \in B$ such that for every weakly null sequence $\{\varphi_n\}$ in $A$ there exist a sequence $\{g_n\}$ in $A$ such that $\|f \varphi_n - g_n\|_B \to 0$ as $n \to \infty$ [2].

Let $\pi_A : B \to B/A$ be the natural projection of $B$ onto $B/A$. For every fixed $f \in B$ let $P_f : A \to fA \subset B$ be the multiplication by $f \in B$ on $A$. Denote $S_f = \pi_A \circ P_f : A \to (fA + A)/A \subset B/A$ the Hankel type operator $S_f : g \mapsto \pi_A(fg)$. Note that $\pi_A$ and $S_f$ both are bounded linear maps onto $B/A$ and onto $(fA + A)/A \subset B/A$ correspondingly.

Observe that if $f \in A_b^B$ if $S_f$ maps every weakly null sequence of $A$ onto a null sequence with respect to the quotient norm topology of $\pi_A(fA) \subset B/A$. Thus $f \in A_b^B$ if the operator $S_f$ is completely continuous, i.e. $S_f(\varphi_n) = \pi_A(f \varphi_n) \to 0$, $n \to \infty$ for every
weakly null sequence \( \{ \varphi_n \}_n \) in \( A \). Equivalently, \( f \in A^B_b \) if and only if \( S_f(c^w_0(A)) \subset c^\| \| (B/A) \).

Let \( A^B_{wc} \) be the set of all \( f \) in \( B \) such that \( S_f \) is \textit{weakly compact} (rather than completely continuous). In a uniform algebra setting, B. Cole and T. W. Gamelin [3] introduced the notion of tightness related with the space \( A^B_{wc} \). A precise connection between \( A^B_b \) and \( A^B_{wc} \) is not known (cf. [9]).

2. Hankel type operators and Bourgain algebras of \( G \)-disc algebras. Let \( A \subset B \) be two uniform algebras on a compact Hausdorff set \( X \) and let \( A^B_b \) be the Bourgain algebra of \( A \) with respect to \( B \).

**Proposition 1.** If the range \( S_f(A) = \pi_A(fA) \) of the Hankel type operator \( S_f \) for an \( f \in B \) is finite dimensional then \( f \in A^B_b \).

**Proof.** If \( \{ \varphi_n \}_n \) is a weakly null sequence in \( A \) then \( \{ f \varphi_n \}_n \) is weakly null in \( B \), and therefore \( \{ \pi_A(\varphi_n) \}_n \) is a weakly null sequence in \( \pi_A(fA) \subset B/A \). Hence \( \{ \pi_A(\varphi_n) \}_n \subset c^\| \| (\pi_A(fA)) \subset c^\| \| (B/A) \), since \( \pi_A(fA) \) is finite dimensional. Consequently \( f \in A^B_b \). □

The range of the completely continuous operator \( S_f \) need not be finite-dimensional. The following example is due to S. Saccone.

**Example 1.** Let \( A = A(T) \) be the disc algebra on the unit circle \( T \) and let \( B = C(T) \). Consider the function

\[
f(z) = \sum_{k=1}^\infty \frac{1}{k^2 z^k}.
\]

Since \( f \in C(T) \), it certainly belongs to \( A^B_b \). We claim that the range of the Hankel type operator of \( f \) is infinite dimensional. Indeed, let \( c_n = \| z^n f + A \|_{B/A} \), and let \( g_n(z) = (1/c_n) z^n \). Clearly, \( g_n \in A \), and \( \| g_n f + A \|_{B/A} = 1 \). To see that \( \pi_A(fA) \) is not finite dimensional it is enough to show that \( g_n f + A \) converges weakly to zero in \( B/A \).

The value of the \((-m)\)th Fourier coefficient of the function \( g_n f \) is

\[
\int_T g_n(z) f(z) \overline{z^m} \, dz = \frac{1}{c_n(m + n)^2}, \quad m, n \geq 1.
\]

Hence,

\[
c_n \geq \| z^n f \|_{L^2/H^2} = \sqrt{\sum_{k=1}^\infty \frac{1}{(n + k)^4}},
\]

thus

\[
1 \leq \frac{1}{n^2 c_n} \leq \frac{1}{\sqrt{n^4 \sum_{k=1}^\infty \frac{1}{(n + k)^4}}}.
\]

Furthermore,

\[
\sum_{k=1}^\infty \frac{1}{(n + k)^4} \geq \int_{n+1}^\infty \frac{1}{x^4} \, dx = \frac{1}{3(n + 1)^3},
\]
Therefore, and every by Proposition 1 the Hankel type operator to (\( S \)) precede nitely many predecessors in \( S \). By Proposition 2. Since \( S \) is generated linearly by \( P \), the set \( \chi S \setminus S \). The uniform algebra on \( G \) generated linearly by the semigroup \( S \) will be denoted by \( A_S \). Functions in \( A_S \) are called \( S \)-functions on \( G \) ([4, Ch. VII], [5], [6, Ch. II]).

**Proposition 2.** Any character \( \chi \in \hat{G} \) for which \( \mathcal{P}_\chi \) is finite belongs to \( (A_S)_b^{C(G)} \).

**Proof.** Note that the characters on \( G \) are linearly independent in \( C(G) \). Since the algebra \( A_S \) is generated linearly by \( S \subset C(G) \), the sets \( \mathcal{P}_\chi \) and \( \pi_{A_S}(\mathcal{P}_\chi) \) have the same cardinality. Therefore,

\[
\dim (S_\chi(A_S)) = \dim (\pi_{A_S}(\chi A_S)) = \text{card} (\pi_{A_S}(\mathcal{P}_\chi)) = \text{card} (\mathcal{P}_\chi) < \infty.
\]

By Proposition 1 the Hankel type operator \( S_\chi \) is completely continuous. Hence \( \chi \) belongs to \( (A_S)_b^{C(G)} \) as claimed.

Note that for any \( \chi \in S \) the set \( \mathcal{P}_\chi \) has the same cardinality as \( \chi \mathcal{P}_\chi = S \setminus \chi S = \{ \gamma \in S : \gamma \notin \chi S \} \), which is the set of all predecessors of \( \chi \) in \( S \), i.e. of all elements \( \gamma \) in \( S \) which precede \( \chi \) with respect to the ordering on \( \hat{G} \) determined by \( S \). If, in addition, \( S - S = \hat{G} \) and every \( \chi \in S \) has finitely many predecessors in \( S \) then every character \( \chi \in \hat{G} \) has finitely many predecessors in \( S \). As it follows from Proposition 2, then \( (A_S)_b^{C(G)} = C(G) \), and therefore the corresponding algebra \( A_S \) possesses the Dunford-Pettis property.

**Corollary 1.** If \( \chi \in S \) be such that \( S \setminus \{1_S\} \subset \chi S \), then \( \bar{\chi} \in (A_S)_b^{C(G)} \).

**Proof.** Since \( \chi \mathcal{P}_\chi = S \setminus \chi S = (\{1_S\} \cup (S \setminus \{1_S\})) \setminus \chi S \subset ((\{1_S\} \cup \chi S) \setminus \chi S \setminus \{1_S\}) \), we obtain that \( \mathcal{P}_\chi = \{\bar{\chi}\} \). Hence \( \bar{\chi} \in (A_S)_b^{C(G)} \) by Proposition 2.

**Corollary 2.** If \( A_S \) is a maximal algebra and the set \( \mathcal{P}_\chi \) is finite for some character \( \chi \in \hat{G} \setminus S \), then \( (A_S)_b^{C(G)} = C(G) \).

**Proof.** Indeed, \( \chi \in (A_S)_b^{C(G)} \) by Proposition 2. Since \( \chi \notin S \), then \( \chi \notin A_S \) and consequently \( (A_S)_b^{C(G)} = C(G) \) by the maximality of \( A \).
Example 2. If $H$ is a finite group, $G = (H \oplus \mathbb{Z})^c$ and $S \cong H \oplus \mathbb{Z}_+$, then $(A_S)_b^{C(G)} = C(G)$.

Indeed, for every character $\chi_{(h, n)} \in \hat{G}$, where $h \in H$ and $n \in \mathbb{Z}$, we have

$$\text{card}(\mathcal{P}_{\chi_{(h, n)}}) = \text{card}((h, n)(H \oplus \mathbb{Z}_+) \setminus H \oplus \mathbb{Z}_+) =$$

$$\text{card}((hH \oplus (n + \mathbb{Z}_+)) \setminus H \oplus \mathbb{Z}_+) = \text{card}((H \oplus (n + \mathbb{Z}_+)) \setminus H \oplus \mathbb{Z}_+) =$$

$$\text{card}(H \oplus (n + \mathbb{Z}_+), \mathbb{Z}_+) = \text{card} H + n < \infty.$$

By Proposition 2 we see that $\chi_{(h, n)} \in (A_S)_b^{C(G)}$ for every $h \in H$ and $n \in \mathbb{Z}$. Consequently $\hat{G} = H \oplus \mathbb{Z} \subset (A_S)_b^{C(G)}$, wherefrom $(A_S)_b^{C(G)} = C(G)$. $\blacksquare$

In the sequel we will assume that $S \cup (-S) = \hat{G} = \{\chi^n\}_{n \in \Gamma}$, for some subgroup $\Gamma \subset \mathbb{R}$ that is dense in $\mathbb{R}$, and that $S \cong \Gamma_+ = \Gamma \cap [0, \infty)$. In this case $A_S$ is called the $G$-disc algebra (or, the big disc algebra), and the elements of $A_S$ are called also generalized analytic functions on $G$. The following theorem identifies the algebra $(A_S)_b^{C(G)}$ for some $G$-disc algebras.

Theorem 1. If $G$ is a compact abelian group whose dual group $\hat{G} \cong \Gamma$ is dense in $\mathbb{R}$ and is divisible by an integer $n \in \Gamma$, then the Bourgain algebra $(A_{\Gamma_+})_b^{C(G)}$ of the $G$-disc algebra $A_{\Gamma_+}$ coincides with $A_{\Gamma_+}$.

Without loss of generality we can assume that $1 \in \Gamma_+$, thus $1/n \in \Gamma_+$, i.e. $\chi^{1/n} \in \hat{G}$. Clearly, $\Gamma_+$ is a subset of $(A_{\Gamma_+})_b^{C(G)}$. First we will prove two preliminary lemmas.

Lemma 1. The sequence of real valued functions $\varphi_n(x) = \left|\frac{1 + e^{i \frac{\pi}{n}}}{2}\right|^{2n}$ converges pointwise to 1 as $n \to \infty$ for every $x \in \mathbb{R}$.

Proof. Fix an $x \in \mathbb{R}$. Since $e^{i \frac{\pi}{n}} \neq -1$ for $n$ big enough, we have

$$\varphi_n(x) = \left|\frac{1 + e^{i \frac{\pi}{n}}}{2}\right|^2 n = \left(\frac{2 + 2 \cos \frac{\pi}{n}}{4}\right)^n = \cos \frac{2n x}{2n} \to 1$$

as $n \to \infty$. $\blacksquare$

Note that the convergence in Lemma 1 is not uniform since, say, $\varphi_n(x) = 0$ if $x = \pi n$ for any integer $n$.

Lemma 2. Under the setting of Theorem 1 the functions $\psi_n(g) = \left|\frac{1 + \chi^{1/n}(g)}{2}\right|^{2n}$ converge pointwise to 1 as $n \to \infty$ for every $g \in G$.

Proof. Let $j_e : \mathbb{R} \to G$ be the standard embedding of the real line onto a dense subgroup of $G$ such that $j_e(0) = e$ (cf. [4, Ch. VII], [6, Ch. II]). Then $\chi^{1/n}(j_e(x)) = e^{i \frac{\pi}{n}}$ and $\psi_n(j_e(x)) = \varphi_n(x)$ for every real $x$. Hence $\varphi_n(x) \to 1$ as $n \to \infty$ by Lemma 1.

Consider the following neighborhood $U$ of $e : U = (\chi^1)^{-1}\{e^{it}, -\pi/4 < t < \pi/4\} \subset G$. Note that if $\sqrt[n]{\cdot}$ is the principal value of the $n$-th root considered on the set $\{e^{it}, -\pi/4 < t < \pi/4\}$, then $\chi^{1/n}(h) = \sqrt[n]{\chi^1(h)}$ on $U$. For a given $g \in G$ there is a $h_g \in U$ such that $g = j_{h_g}(x)$ for some $x \in \mathbb{R}$, where $j_h = hj_e$ is the standard dense
embedding of $\mathbb{R}$ into $G$ with $j_h(0) = h$. Hence $\chi^\frac{1}{n}(h_g) = e^{i\frac{s}{n}}$ if $\chi^1(h_g) = e^{is}$ for some $s$, $-\pi/4 < s < \pi/4$, and therefore,

$$
\psi_n(g) = \psi_n(j_h(x)) = \left| \frac{1 + \chi^\frac{1}{n}(j_h(x))}{2} \right|^{2n} = \left| \frac{1 + \chi^\frac{1}{n}(h_g)\chi^\frac{1}{n}(j_h(x))}{2} \right|^{2n} = \left| 1 + e^{i\frac{s+x}{n}} \right|^{2n}.
$$

Consequently, by Lemma 1, $\psi_n(g) = \varphi_n(s + x) \to 1$ as $n \to \infty$. ■

The remark after Lemma 1 indicates that the convergence in Lemma 2 might not be uniform.

**Proof of Theorem 1.** Suppose that $\chi^3 \in (A_{F+})_b^{C(G)}$, and consider the sequence $\xi_n(g) = \psi_n(g) - 1$, where $\psi_n$ is the function in Lemma 2. The sequence \{\chi^1\xi_n\}_n converges pointwise to 0 on the compact group $G$, and therefore it is weakly null in $A_{F+}$. Since $\chi^3 \in (A_{F+})_b^{C(G)}$, there are functions $h_n \in A_{F+}$ such that $\|\chi^3\chi^1\xi_n - h_n\| < 1/n$ for every $n$, where $\| \cdot \|$ is the sup norm on $G$. By integrating over Ker $(\chi^\frac{1}{n})$, if necessary, we can assume that $h_n = q_n(\chi^\frac{1}{n})$ for some polynomial $q_n$. Since

$$(\chi^1\psi_n)(g) = (\chi^\frac{1}{n}(g))^n \left( \frac{1 + \chi^\frac{1}{n}(g)}{2} \right)^n \left( \frac{1 + \chi^\frac{1}{n}(g)}{2} \right)^n = p_n(\chi^\frac{1}{n}(g)),$$

where $p_n$ is the polynomial $p_n(z) = (1 + z)^{2n}$, we have that $\chi^1\psi_n \in A_{F+}$, and therefore, $\xi_n \in A_{F+}$ too. For $j = 2n$ the $j$-th Cesàro mean

$$
\sigma_j^{p_n} = \frac{S_0 + S_1 + \cdots + S_j}{j + 1}
$$

of $p_n$, where $S_k$ is the $k$-th partial sum of $p_n$, becomes

$$
\sigma_{2n}^{p_n}(z) = \frac{1}{4n(2n + 1)} \sum_{k=0}^{2n} (2n - k + 1) \binom{2n}{k} z^k.
$$

Hence

$$
4^n(2n + 1)\sigma_{2n}^{p_n}(z) = \sum_{k=0}^{2n} \binom{2n}{k} z^k + \sum_{k=0}^{2n-1} (2n - k) \binom{2n}{k} z^k = (1 + z)^{2n} + 2n(1 + z)^{2n-1} = (2n + 1 + z)(1 + z)^{2n-1}.
$$

Now

$$
\|\chi^3\chi^1\xi_n - h_n\| = \max_{g \in G} |(\chi^3\chi^1\xi_n)(g) - h_n(g)|
$$

$$
= \max_{g \in G} |(\chi^1\xi_n)(g) - (\chi^3h_n)(g)| = \max_{g \in G} |(\chi^1\psi_n)(g) - \chi^1(g) - (\chi^3(g)h_n)(g)|
$$

$$
= \max_{g \in G} |p_n(\chi^\frac{1}{n}(g)) - \chi^1(g) - (\chi^\frac{1}{n}(g))^3 q_n(\chi^\frac{1}{n}(g))|
$$

$$
= \max_{z \in \mathbb{T}} |p_n(z) - z^n - z^{3n} q_n(z)|.
$$

Note that $\sigma_{2n}^{p_n}(z) - z^n = \sigma_{2n}^{p_n}(z) - z^n - z^{3n} q_n(z)$ because the Cesàro mean $\sigma_{2n}$ depends only on the first $2n$ terms of the Taylor series. Since $\max_{z \in \mathbb{T}} |\sigma_n(z)| \leq \max_{z \in \mathbb{T}} |f(z)|$
holds for every $f \in A(\mathbb{T})$, we obtain
\[
\max_{z \in \mathbb{T}} |\sigma_{2n}^p(z) - z^n(z)| = \max_{z \in \mathbb{T}} |\sigma_{2n}^p(z) - z^n - z^{3n}q_n(z)(z)| \leq \\
\max_{z \in \mathbb{T}} |p_n(z) - z^n - z^{3n}q_n(z)| = \|\chi^3 \chi^1 \xi_n - h_n\| < 1/n,
\]
i.e. $\|\sigma_{2n}^p(z) - z^n\| \to 0$ as $n \to \infty$. However, $\sigma_{2n}^p(z) - z^n(z) = \sigma_{2n}^p(z) - z^n(n+1)/(2n+1)$, and thus $\sigma_{2n}^p(z) - z^n(-1) \to 1/2$ as $n \to \infty$ for odd $n$, contrary to $\|\sigma_{2n}^p(z) - z^n\| \to 0$. Hence $\|\chi^3 \chi^1 \xi_n - h_n\| \not\to 0$ for any $h_n \in A_{R_+}$, and therefore $\chi^3 \not\in (A_{R_+})_b^{C(G)}$. The maximality of $A_{R_+}$ implies that $(A_{R_+})_b^{C(G)} = A_{R_+}$, as desired.

Thanks are due to J. Cima, K. Stroethoff and S. Saccone for stimulating discussions.

References