Abstract. We continue the paper [Ts] on the boundedness of polynomials in the Volterra operator. This provides new ways of constructing power-bounded operators. It seems interesting to point out that a similar procedure applies to the operators satisfying the Ritt resolvent condition: compare Theorem 5 and Theorem 9 below.

1. Preliminaries. An operator $A$ is called power-bounded if

$$\sup_{n \geq 0} \|A^n\| < \infty.$$  

Denote by $V$ the classical Volterra operator

$$(Vf)(x) = \int_0^x f(s)ds, \quad 0 < x < 1, \text{ on } L^p(0,1), \quad 1 \leq p \leq \infty.$$  

The more general Riemann–Liouville integral operator of fractional order $\alpha > 0$ is defined by

$$(J^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} f(s)ds, \quad 0 < x < 1, \text{ on } L^p(0,1), \quad 1 \leq p \leq \infty,$$

where $\Gamma$ is the Euler gamma function. In particular, $V = J^1$.  

2000 Mathematics Subject Classification: 47A10, 47A35, 47G10.

The paper is in final form and no version of it will be published elsewhere.
Recall that the Ritt condition for the resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$ of a bounded operator $A$ on a Banach space is

$$\|R(\lambda, A)\| \leq \frac{\text{const}}{|\lambda - 1|}, \quad |\lambda| > 1,$$

which is equivalent to a geometric condition much stronger than the power boundedness of $A$, namely,

$$\sup_{n \geq 0} n\|A^n - A^{n+1}\| < \infty$$

has to be added to the power boundedness of $A$, see [NaZe], [Ne]. Examples are the operators $I - J^\alpha$ with $0 < \alpha < 1$, see [Ly]. In particular, the geometric characterization in terms of the behaviour of the powers gives easily the following:

**Proposition 1.** Let $A$ and $B$ be two commuting Ritt operators. Then their product $AB$ is also a Ritt operator. □

If the operator $A$ is merely power-bounded, then the weaker Kreiss condition

$$\|R(\lambda, A)\| \leq \frac{\text{const}}{|\lambda - 1|}, \quad |\lambda| > 1,$$

holds, but not conversely in general.

The behaviour of the consecutive powers has been studied in [Ly], [Ne] and [ToZe]. We shall need the following simple facts (see [Ts]):

**Proposition 2.** Let $A$ and $B$ be two commuting power-bounded operators on a Banach space, $0 \leq t \leq 1$. Then the convex combination $tA + (1 - t)B$ is a power-bounded operator. □

**Proposition 3.** Let $\sigma(Q) = \{0\}$. If $I - Q$ satisfies the Ritt condition, then so does $I - tQ$ for $t \geq 0$. Consequently, $(1 - t)I + t(I - Q)^2$ is a Ritt operator for $t \geq 0$. □

**2. The results**

**Lemma 4.** The resolvent for $aV + bV^2$ ($a$ and $b$ constants) is

$$(R(\lambda, aV + bV^2)f)(x) = \frac{f(x)}{\lambda} + \frac{1}{\sqrt{a^2 + 4b\lambda}} \left( \frac{a + \sqrt{a^2 + 4b\lambda}}{2\lambda} \right)^2 \int_0^x e^{\frac{a + \sqrt{a^2 + 4b\lambda}}{2\lambda}(x-s)} f(s)ds$$

$$- \frac{1}{\sqrt{a^2 + 4b\lambda}} \left( \frac{a - \sqrt{a^2 + 4b\lambda}}{2\lambda} \right)^2 \int_0^x e^{\frac{a - \sqrt{a^2 + 4b\lambda}}{2\lambda}(x-s)} f(s)ds,$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ and $\sigma(aV + bV^2) = \{0\}$.

**Proof.** Let $C^\infty(0,1)$ be the space of infinitely differentiable functions on $(0,1)$. If $f \in C^\infty(0,1)$, then the equation

$$((\lambda I - aV - bV^2)g)(t) = f(t)$$

is equivalent to the differential equation

$$\lambda g''(t) - ag'(t) - bg(t) = f''(t),$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ and $\sigma(aV + bV^2) = \{0\}$. □
which is satisfied by
\[
g(x) = (R(\lambda, I - aV - bV^2)f)(x) = \frac{f(x)}{\lambda} + \frac{1}{\sqrt{a^2 + 4b\lambda}} \left( \frac{a + \sqrt{a^2 + 4b\lambda}}{2\lambda} \right)^2 \int_0^x e^{-\frac{a + \sqrt{a^2 + 4b\lambda}}{2\lambda}(x-s)} f(s)ds
\]
\[
- \frac{1}{\sqrt{a^2 + 4b\lambda}} \left( \frac{a - \sqrt{a^2 + 4b\lambda}}{2\lambda} \right)^2 \int_0^x e^{-\frac{a - \sqrt{a^2 + 4b\lambda}}{2\lambda}(x-s)} f(s)ds.
\]
Note that \(C^\infty(0,1)\) is dense in \(L^p(0,1)\) (1 \(\leq p \leq \infty\). □

**Theorem 5.** The operator \(I - aV + bV^2\) is power-bounded on \(L^2(0,1)\) for \(a > 0\) and \(b \geq 0\) (and also for \(a = b = 0\)).

**Proof.** Case \(0 \leq b \leq a^2/4\). We can write
\[
I - aV + bV^2 = \left( I - \frac{a - \sqrt{a^2 - 4b}}{2} V \right) \left( I - \frac{a + \sqrt{a^2 - 4b}}{2} V \right),
\]
and use [Ts, Theorem 1].

Case \(b > a^2/4\). Note that \((I - \frac{at}{2} V)^2\) is power-bounded for each \(t > 0\), by [Ts, Theorem 1]. It then follows from Proposition 1 that
\[
(1 - \lambda)I + \lambda \left( I - atV + \frac{a^2t^2}{4} V^2 \right) = I - \lambda atV + \frac{\lambda a^2t^2}{4} V^2
\]
is power-bounded for \(0 < \lambda < 1\). So, \(t = 1/\lambda\) with \(t = 4b/a^2 > 1\) proves the claim. □

**Proposition 6.** The operator \(I - aV + zV^2\) \((z \in \mathbb{C})\) is not power-bounded on \(L^2(0,1)\), for \(a < 0\), and also for \(a > 0\) and \(z \in \mathbb{C} \setminus [0,\infty)\), or \(a = 0\) and \(z \neq 0\).

**Proof.** Using Lemma 4 we obtain
\[
-(R(\lambda, I - aV - zV^2)f)(x) = (R(1 - \lambda, aV + zV^2)f)(x) = \frac{f(x)}{1 - \lambda} + \frac{1}{\sqrt{a^2 + 4z(1 - \lambda)}} \left( \frac{a + \sqrt{a^2 + 4z(1 - \lambda)}}{2(1 - \lambda)} \right)^2 \int_0^x e^{-\frac{a + \sqrt{a^2 + 4z(1 - \lambda)}}{2(1 - \lambda)}(x-s)} f(s)ds
\]
\[
- \frac{1}{\sqrt{a^2 + 4z(1 - \lambda)}} \left( \frac{a - \sqrt{a^2 + 4z(1 - \lambda)}}{2(1 - \lambda)} \right)^2 \int_0^x e^{-\frac{a - \sqrt{a^2 + 4z(1 - \lambda)}}{2(1 - \lambda)}(x-s)} f(s)ds
\]
where \(\lambda \neq 1\). Analyzing the behaviour of these expressions as \(\lambda \to 1_+\), we see that the resolvent \(R(\lambda, I - aV - zV^2)\) does not satisfy the Kreiss condition on \(L^2(0,1)\). See also [Ts, Theorem 3]. □

**Theorem 7.** Let \(m \geq 1\) be fixed. The operator
\[
L_m(V) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k \frac{V^k}{k!}
\]
is power-bounded on \(L^2(0,1)\).

**Proof.** Recall that the zeros of the Laguerre polynomials \(L_m(\cdot)\) are real, positive and simple (see [MaOb, p. 84] or [Sz, p. 122]). Suppose that \(a_1, a_2, \ldots, a_m\) are the zeros of the Laguerre polynomial \(L_m\). We can write
Proposition 1.

Proof. Case $0 \leq b \leq 1/4$. We can write

$$I - V^{1/2} + bV = \left( I - \frac{1}{2} \sqrt{1 - 4b} V^{1/2} \right) \left( I - \frac{1}{2} \sqrt{1 - 4b} V^{1/2} \right),$$

and use Proposition 3. Note that $V^{1/2} = \mathcal{J}^{1/2}$, hence $I - V^{1/2}$ is a Ritt operator.

Case $b > 1/4$. It follows from Proposition 2, and from the power boundedness of $(I - \frac{1}{2} V^{1/2})^2$, $t > 0$ (see Proposition 3), that

$$(1 - \lambda)I + \lambda \left( I - tV^{1/2} + \frac{t^2}{4} V \right) = I - \lambda tV^{1/2} + \frac{\lambda t^2}{4} V$$

is power-bounded for $0 < \lambda < 1$. So, $\lambda = 1/t$ with $t = 4b > 1$ proves the claim.

Case $b < 0$. It follows from Proposition 2, the power boundedness of $I - aV^{1/2}$ ($a > 0$, see Proposition 3) and $I - tV$ ($t > 0$, [Ts, Theorem 1]) that

$$(1 - \lambda)(I - aV^{1/2}) + \lambda(I - tV) = I - a(1 - \lambda)V^{1/2} - \lambda tV$$

is power-bounded for $0 < \lambda < 1$. We choose $a = 1/(1 - \lambda)$, with $0 < \lambda = -b/t < 1$, which is possible for a sufficiently large $t > 0$. The proof is complete.

Theorem 9. Let $\sigma(Q) = \{0\}$. If $I - Q$ is a Ritt operator, then so is the operator $I - aQ + bQ^2$ for $a > 0$ and $b \geq 0$ (and also for $a = b = 0$).

Proof. If $a^2 \geq 4b \geq 0$, we can write

$$I - aQ + bQ^2 = \left( I - \frac{a - \sqrt{a^2 - 4b}}{2} Q \right) \left( I - \frac{a + \sqrt{a^2 - 4b}}{2} Q \right),$$

where both the factors are Ritt operators, by Proposition 3, hence so is their product, by Proposition 1.

Suppose that $0 < a^2 < 4b$. Let $0 < s < 1$ and $t > 0$. By Proposition 3,

$$(1 - s)I + s \left( 1 - \frac{at}{2} Q \right)^2 = I - astQ + \frac{a^2 st^2}{4} Q^2$$

is a Ritt operator. Choosing $s = 1/t$ with $t = 4b/a^2 > 1$, we get the result.

Proposition 10 ([Al]). Let $\sigma(Q) = \{0\}$. If the operators $I - Q$ and $I + Q$ are power-bounded, then $Q = 0$.

Proof. We can write

$$Q = Q \left( \frac{I - Q + I + Q}{2} \right)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (I - Q)^{n-k} Q(I + Q)^{k}.$$
Observe that, for large $n$, either $(I - Q)^{n-k}Q$ or $Q(I + Q)^k$ is small, by [Es, Theorem 9.1], while the remaining operator powers (actually both $(I + Q)^k$ and $(I - Q)^{n-k}$) are bounded, by assumption. It follows that $Q = 0$. ■

**Remarks**

**Remark 11.** Let

$$M_n(T) = \frac{I + T + \ldots + T^{n-1}}{n}.$$ 

The operator $I - V$ is not power-bounded on $L^1(0, 1)$ ($\|(I - V)^n\|$ is of order $n^{1/4}$), but $\|M_n(I - V)\|$ is bounded; see ([Hi], [ToZe]). It can be shown that $\|M_n(I - tV)\|$ is bounded, with respect to $n$, for each fixed $t > 0$. Indeed, an argument similar to that for Proposition 3 (see [Ts, Proposition 2]) shows that the resolvent of the operator $I - tV$, for a fixed $t > 0$, remains uniformly Abel bounded on the half-line $\lambda > 1$, which is equivalent to the Cesàro boundedness of $I - tV$ (see [MoSaZe, Theorem 3.1]). Thus, we see one more advantage of the resolvent characterizations of various geometric properties of the powers.

**Remark 12.** Observe that the power-boundedness in Theorem 8 for $b < 0$ is due to the fact that the operator $I - V^{1/2}$ satisfies the Ritt condition (which makes it possible to use Proposition 3).

**Remark 13.** In Theorem 5, for $a > 0$ and $b > a^2/4$, the operator is a product of two operators of the form $I - zV$, with $z \notin \mathbb{R}$, that are not power-bounded by [Ts, Theorem 1]. Nevertheless their product is power-bounded.

**Remark 14.** Let $\sigma(Q) = \{0\}$. Suppose that the operators $I - Q$ and $I - Q^2$ are power-bounded. Does it follow that $I - Q + tQ^2$ is power-bounded for $t \in \mathbb{R}$? This would be a generalization of Theorem 8. What about the operators in Theorem 9, for other values of $a$ and $b$?

**Remark 15.** Let $m$ be fixed. Observe that the operator $L_m(J^\alpha)$, for $0 < \alpha < 1$, satisfies the Ritt condition on $L^p(0, 1)$, for $1 \leq p \leq \infty$, by [Ly], Propositions 1 and 3, and the proof of Theorem 7, but not for $\alpha = 1$ and $m = 1$. However, by Theorem 7 and [Es, Theorem 9.1] we know that

$$\lim_{k \to \infty} \|L_m(V)^k - L_m(V)^{k+1}\| = 0.$$ 

What is the rate of this convergence? Does it depend on $m$?

**Remark 16.** Suppose that $A$ satisfies the Kreiss condition. Does it follow that also $A^2$ is a Kreiss operator?

**Acknowledgments.** The authors are grateful to Professor Mostafa Mbekhta for interesting discussions, and the University of Lille 1 for support and hospitality.
References


