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POLYNOMIALS IN THE VOLTERRA AND RITT OPERATORS

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Abstract. We continue the paper [Ts] on the boundedness of polynomials in the Volterra operator. This provides new ways of constructing power-bounded operators. It seems interesting to point out that a similar procedure applies to the operators satisfying the Ritt resolvent condition: compare Theorem 5 and Theorem 9 below.

1. Preliminaries. An operator A is called *power-bounded* if

$$\sup_{n\geq 0} \|A^n\| < \infty.$$

Denote by V the classical *Volterra operator*

$$(Vf)(x) = \int_0^x f(s)ds, \quad 0 < x < 1, \text{ on } L^p(0,1),\ 1 \leq p \leq \infty.$$

The more general *Riemann–Liouville integral operator* of fractional order $\alpha > 0$ is defined by

$$(J^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)ds, \quad 0 < x < 1, \quad \text{on } L^p(0,1),\ 1 \leq p \leq \infty,$$

where Γ is the Euler gamma function. In particular, $V = J^1$.

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Recall that the *Ritt condition* for the resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$ of a bounded operator A on a Banach space is

$$\|R(\lambda, A)\| \le \frac{\text{const}}{|\lambda - 1|}, \qquad |\lambda| > 1,$$

which is equivalent to a geometric condition much stronger than the power boundedness of A, namely,

$$\sup_{n\ge 0} n\|A^n - A^{n+1}\| < \infty$$

has to be added to the power boundedness of A, see [NaZe], [Ne]. Examples are the operators $I - J^\alpha$ with $0 < \alpha < 1$, see [Ly]. In particular, the geometric characterization in terms of the behaviour of the powers gives easily the following:

PROPOSITION 1. *Let A and B be two commuting Ritt operators. Then their product AB is also a Ritt operator.* ■

If the operator A is merely power-bounded, then the weaker *Kreiss condition*

$$\|R(\lambda, A)\| \le \frac{\text{const}}{|\lambda| - 1}, \qquad |\lambda| > 1,$$

holds, but not conversely in general.

The behaviour of the consecutive powers has been studied in [Ly], [Ne] and [ToZe]. We shall need the following simple facts (see [Ts]):

PROPOSITION 2. *Let A and B be two commuting power-bounded operators on a Banach space, $0 \le t \le 1$. Then the convex combination $tA + (1-t)B$ is a power-bounded operator.* ■

PROPOSITION 3. *Let $\sigma(Q) = \{0\}$. If $I - Q$ satisfies the Ritt condition, then so does $I - tQ$ for $t \ge 0$. Consequently, $(1-t)I + t(I-Q)^2$ is a Ritt operator for $t \ge 0$.* ■

2. The results

LEMMA 4. *The resolvent for $aV + bV^2$ (a and b constants) is*

$$\begin{aligned}(R(\lambda, aV + bV^2)f)(x) &= \frac{f(x)}{\lambda} \\ &\quad + \frac{1}{\sqrt{a^2+4b\lambda}}\left(\frac{a+\sqrt{a^2+4b\lambda}}{2\lambda}\right)^2 \int_0^x e^{\frac{a+\sqrt{a^2+4b\lambda}}{2\lambda}(x-s)} f(s)\,ds \\ &\quad - \frac{1}{\sqrt{a^2+4b\lambda}}\left(\frac{a-\sqrt{a^2+4b\lambda}}{2\lambda}\right)^2 \int_0^x e^{\frac{a-\sqrt{a^2+4b\lambda}}{2\lambda}(x-s)} f(s)\,ds,\end{aligned}$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ and $\sigma(aV + bV^2) = \{0\}$.

Proof. Let $C^\infty(0,1)$ be the space of infinitely differentiable functions on $(0,1)$. If $f \in C^\infty(0,1)$, then the equation

$$((\lambda I - aV - bV^2)g)(t) = f(t)$$

is equivalent to the differential equation

$$\lambda g''(t) - a g'(t) - b g(t) = f''(t),$$

which is satisfied by

$$\begin{aligned} g(x) &= (R(\lambda, I - aV - bV^2)f)(x) \\ &= \frac{f(x)}{\lambda} + \frac{1}{\sqrt{a^2+4b\lambda}}\left(\frac{a+\sqrt{a^2+4b\lambda}}{2\lambda}\right)^2 \int_0^x e^{\frac{a+\sqrt{a^2+4b\lambda}}{2\lambda}(x-s)} f(s)ds \\ &\quad - \frac{1}{\sqrt{a^2+4b\lambda}}\left(\frac{a-\sqrt{a^2+4b\lambda}}{2\lambda}\right)^2 \int_0^x e^{\frac{a-\sqrt{a^2+4b\lambda}}{2\lambda}(x-s)} f(s)ds. \end{aligned}$$

Note that $C^\infty(0,1)$ is dense in $L^p(0,1)$ $(1 \le p \le \infty)$. ∎

THEOREM 5. *The operator $I - aV + bV^2$ is power-bounded on $L^2(0,1)$ for $a > 0$ and $b \ge 0$ (and also for $a = b = 0$).*

Proof. Case $0 \le b \le a^2/4$. We can write

$$I - aV + bV^2 = \left(I - \frac{a-\sqrt{a^2-4b}}{2}V\right)\left(I - \frac{a+\sqrt{a^2-4b}}{2}V\right),$$

and use [Ts, Theorem 1].

Case $b > a^2/4$. Note that $(I - \frac{at}{2}V)^2$ is power-bounded for each $t > 0$, by [Ts, Theorem 1]. It then follows from Proposition 1 that

$$(1-\lambda)I + \lambda\left(I - atV + \frac{a^2t^2}{4}V^2\right) = I - \lambda atV + \frac{\lambda a^2 t^2}{4}V^2$$

is power-bounded for $0 < \lambda < 1$. So, $t = 1/\lambda$ with $t = 4b/a^2 > 1$ proves the claim. ∎

PROPOSITION 6. *The operator $I - aV + zV^2$ $(z \in \mathbb{C})$ is not power-bounded on $L^2(0,1)$, for $a < 0$, and also for $a > 0$ and $z \in \mathbb{C} \setminus [0,\infty)$, or $a = 0$ and $z \ne 0$.*

Proof. Using Lemma 4 we obtain

$$\begin{aligned} -(R(\lambda, I - aV - zV^2)f)(x) &= (R(1-\lambda, aV + zV^2)f)(x) \\ &= \frac{f(x)}{1-\lambda} + \frac{1}{\sqrt{a^2+4z(1-\lambda)}}\left(\frac{a+\sqrt{a^2+4z(1-\lambda)}}{2(1-\lambda)}\right)^2 \int_0^x e^{\frac{a+\sqrt{a^2+4z(1-\lambda)}}{2(1-\lambda)}(x-s)} f(s)ds \\ &\quad - \frac{1}{\sqrt{a^2+4z(1-\lambda)}}\left(\frac{a-\sqrt{a^2+4z(1-\lambda)}}{2(1-\lambda)}\right)^2 \int_0^x e^{\frac{a-\sqrt{a^2+4z(1-\lambda)}}{2(1-\lambda)}(x-s)} f(s)ds \end{aligned}$$

where $\lambda \ne 1$. Analyzing the behaviour of these expressions as $\lambda \to 1_+$, we see that the resolvent $R(\lambda, I - aV - zV^2)$ does not satisfy the Kreiss condition on $L^2(0,1)$. See also [Ts, Theorem 3]. ∎

THEOREM 7. *Let $m \ge 1$ be fixed. The operator*

$$L_m(V) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k \frac{V^k}{k!}$$

is power-bounded on $L^2(0,1)$.

Proof. Recall that the zeros of the *Laguerre polynomials* $L_m(\cdot)$ are real, positive and simple (see [MaOb, p. 84] or [Sz, p. 122]). Suppose that $a_1, a_2, \ldots, a_m$ are the zeros of the Laguerre polynomial L_m. We can write

$$\begin{aligned} m!L_m(V) &= (a_1 - V)(a_2 - V)\dots(a_m - V) \\ &= \Big(I - \frac{1}{a_1}V\Big)\Big(I - \frac{1}{a_2}V\Big)\dots\Big(I - \frac{1}{a_m}V\Big)\prod_{i=1}^{m} a_i. \end{aligned}$$

It is clear that $\prod_{i=1}^m a_i = m!$. Hence $L_m(V)$ is power-bounded by [Ts, Theorem 1]. ∎

THEOREM 8. *The operator $I - V^{1/2} + bV$ is power-bounded on $L^2(0,1)$, for $b \in \mathbb{R}$.*

Proof. Case $0 \le b \le 1/4$. We can write

$$I - V^{1/2} + bV = \Big(I - \frac{1 + \sqrt{1-4b}}{2}V^{1/2}\Big)\Big(I - \frac{1 - \sqrt{1-4b}}{2}V^{1/2}\Big),$$

and use Proposition 3. Note that $V^{1/2} = J^{1/2}$, hence $I - V^{1/2}$ is a Ritt operator.

Case $b > 1/4$. It follows from Proposition 2, and from the power boundedness of $(I - \frac{t}{2}V^{1/2})^2$, $t > 0$ (see Proposition 3), that

$$(1-\lambda)I + \lambda\Big(I - tV^{1/2} + \frac{t^2}{4}V\Big) = I - \lambda t V^{1/2} + \frac{\lambda t^2}{4}V$$

is power-bounded for $0 < \lambda < 1$. So, $\lambda = 1/t$ with $t = 4b > 1$ proves the claim.

Case $b < 0$. It follows from Proposition 2, the power boundedness of $I - aV^{1/2}$ ($a > 0$, see Proposition 3) and $I - tV$ ($t > 0$, [Ts, Theorem 1]) that

$$(1-\lambda)(I - aV^{1/2}) + \lambda(I - tV) = I - a(1-\lambda)V^{1/2} - \lambda t V$$

is power-bounded for $0 < \lambda < 1$. We choose $a = 1/(1-\lambda)$, with $0 < \lambda = -b/t < 1$, which is possible for a sufficiently large $t > 0$. The proof is complete. ∎

THEOREM 9. *Let $\sigma(Q) = \{0\}$. If $I - Q$ is a Ritt operator, then so is the operator $I - aQ + bQ^2$ for $a > 0$ and $b \ge 0$ (and also for $a = b = 0$).*

Proof. If $a^2 \ge 4b \ge 0$, we can write

$$I - aQ + bQ^2 = \Big(I - \frac{a - \sqrt{a^2-4b}}{2}Q\Big)\Big(I - \frac{a + \sqrt{a^2-4b}}{2}Q\Big),$$

where both the factors are Ritt operators, by Proposition 3, hence so is their product, by Proposition 1.

Suppose that $0 < a^2 < 4b$. Let $0 < s < 1$ and $t > 0$. By Proposition 3,

$$(1-s)I + s\Big(1 - \frac{at}{2}Q\Big)^2 = I - astQ + \frac{a^2st^2}{4}Q^2$$

is a Ritt operator. Choosing $s = 1/t$ with $t = 4b/a^2 > 1$, we get the result. ∎

PROPOSITION 10 ([Al]). *Let $\sigma(Q) = \{0\}$. If the operators $I - Q$ and $I + Q$ are power-bounded, then $Q = 0$.*

Proof. We can write

$$Q = Q\Big(\frac{I - Q + I + Q}{2}\Big)^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(I-Q)^{n-k}Q(I+Q)^k.$$

Observe that, for large n, either $(I-Q)^{n-k}Q$ or $Q(I+Q)^k$ is small, by [Es, Theorem 9.1], while the remaining operator powers (actually both $(I+Q)^k$ and $(I-Q)^{n-k}$) are bounded, by assumption. It follows that $Q = 0$. ∎

Remarks

REMARK 11. Let

$$M_n(T) = \frac{I + T + \ldots + T^{n-1}}{n}.$$

The operator $I - V$ is not power-bounded on $L^1(0,1)$ ($\|(I - V)^n\|$ is of order $n^{1/4}$), but $\|M_n(I - V)\|$ is bounded; see ([Hi], [ToZe]). It can be shown that $\|M_n(I - tV)\|$ is bounded, with respect to n, for each fixed $t > 0$. Indeed, an argument similar to that for Proposition 3 (see [Ts, Proposition 2]) shows that the resolvent of the operator $I - tV$, for a fixed $t > 0$, remains uniformly Abel bounded on the half-line $\lambda > 1$, which is equivalent to the Cesàro boundedness of $I - tV$ (see [MoSaZe, Theorem 3.1]). Thus, we see one more advantage of the resolvent characterizations of various geometric properties of the powers.

REMARK 12. Observe that the power-boundedness in Theorem 8 for $b < 0$ is due to the fact that the operator $I - V^{1/2}$ satisfies the Ritt condition (which makes it possible to use Proposition 3).

REMARK 13. In Theorem 5, for $a > 0$ and $b > a^2/4$, the operator is a product of two operators of the form $I - zV$, with $z \notin \mathbb{R}$, that are not power-bounded by [Ts, Theorem 1]. Nevertheless their product is power-bounded.

REMARK 14. Let $\sigma(Q) = \{0\}$. Suppose that the operators $I - Q$ and $I - Q^2$ are power-bounded. Does it follow that $I - Q + tQ^2$ is power-bounded for $t \in \mathbb{R}$? This would be a generalization of Theorem 8. What about the operators in Theorem 9, for other values of a and b?

REMARK 15. Let m be fixed. Observe that the operator $L_m(J^\alpha)$, for $0 < \alpha < 1$, satisfies the Ritt condition on $L^p(0,1)$, for $1 \le p \le \infty$, by [Ly], Propositions 1 and 3, and the proof of Theorem 7, but not for $\alpha = 1$ and $m = 1$. However, by Theorem 7 and [Es, Theorem 9.1] we know that

$$\lim_{k\to\infty} \|L_m(V)^k - L_m(V)^{k+1}\| = 0.$$

What is the rate of this convergence? Does it depend on m?

REMARK 16. Suppose that A satisfies the Kreiss condition. Does it follow that also A^2 is a Kreiss operator?

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