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ON A-CONVEX AND lm-CONVEX ALGEBRA STRUCTURES OF A LOCALLY CONVEX SPACE

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Abstract. Given a locally convex space (V, Γ), we find (all) the multiplications π on V (associative or not) such that the algebra $A \equiv (V, \pi, \Gamma)$ becomes (i) A-convex, (ii) lm-convex.

0. Preliminaries. Given any seminorm $p \in sem(V) := \{q : V \to \mathbb{R}_+ \text{ a seminorm}\}$ on a $\mathbb{C}$-vector space V, we define (see also [6])

(0.1) $$\Lambda(p, T) := \{\lambda \geq 0 : p \circ T \leq \lambda p\} \subseteq \mathbb{R}_+, \quad T \in L(V) \equiv L(V, V),$$

(0.2) $$\mathcal{L}_p(V) := \{T \in L(V) : \Lambda(p, T) \neq \emptyset\} \subseteq L(V),$$

(0.3) $$\tilde{p} : \mathcal{L}_p(V) \to \mathbb{R}_+ : \tilde{p}(T) := \inf \Lambda(p, T),$$

(0.4) $$\mathcal{L}_{\tilde{p}}(V) \equiv (\mathcal{L}_p(V), \tilde{p}), \quad \tilde{\Gamma} \equiv \{\tilde{p} : p \in \Gamma\}, \quad \emptyset \neq \Gamma \subseteq sem(V),$$

(0.5) $$\mathcal{L}_\Gamma(V) \equiv \bigcap_{p \in \Gamma} \mathcal{L}_p(V),$$

(0.6) $$\mathcal{L}_{\tilde{\Gamma}}(V) \equiv (\mathcal{L}_\Gamma(V), \tilde{\Gamma}).$$

Obviously $\mathcal{L}_{\tilde{p}}(V)$ is a seminormed algebra while $\mathcal{L}_{\tilde{\Gamma}}(V)$ is an lm-convex (locally multiplicatively convex) algebra which, in particular, is (i) Hausdorff (ii) σ-complete if (V, Γ) is (see also [2], [7]).

In this context, the algebra $\mathcal{L}_{\tilde{\Gamma}}(V)$, as above, whose elements might also be called "Γ-uniformly bounded" endomorphisms of the locally convex space (V, Γ), has been studied in [7], [8]. The same algebra has been considered by E. A. Michael [3], the relevant study being "geometrical" (via a local basis), in contradistinction with ours (ibid.) where the treatment is algebraic ("arithmetical", viz. through appropriate "operator seminorms").

A seminorm p of an algebra A is

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(i) *left A-convex* iff for each $\omega \in A$ there exists a $\lambda_\omega > 0 : p(\omega\theta) \leq \lambda_\omega p(\theta)$ for all $\theta \in A$.

(ii) *locally multiplicatively convex* (lmc) or algebra-seminorm iff there exists $\lambda > 0$ satisfying $p(\omega\theta) \leq \lambda p(\omega)p(\theta)$, $\omega, \theta \in A$.

(By a seminorm p of a vector space V we mean a $p : V \to \mathbb{R}_+$ such that $p(\omega + \theta) \leq p(\omega) + p(\theta)$, $p(\lambda\omega) = |\lambda|p(\omega)$ for all $\lambda \in \mathbb{C}$, $\omega, \theta \in V$.)

On the other hand, we know that there exists a vector-space isomorphism $\pi \to l$ between the vector spaces $\mathcal{B}(V) \equiv \mathcal{B}(V \times V, V) := \{\pi : V \times V \to V \text{ bilinear}\}$ and $L(V, L(V))$ given by the relation

$$l(\omega)(\theta) \equiv \pi(\omega, \theta),\ \omega, \theta \in V, \tag{0.7}$$

or else by

$$l(\omega)(\theta) \equiv \pi(\theta, \omega),\ \omega, \theta \in V. \tag{0.8}$$

We call $l \in L(V, L(V))$ in (0.7) the *left regular representation* (lRR) of π while the l in (0.8) the *right* RR of π (rRR of π). In this last case we put the symbol r instead of l.

1. Main results. In the following V denotes a $\mathbb{C}$-vector space (we also can then consider V as a space over the real field $\mathbb{R}$).

Now we obtain the following:

PROPOSITION 1.1. *For each seminorm $p \in sem(V)$ a multiplication $\pi \equiv l$ on V makes p:*

(a) *an A-convex seminorm iff*

$$l \in L(V, \mathcal{L}_p(V)) \subseteq L(V, L(V)), \tag{1.1}$$

(b) *an algebra-seminorm iff*

$$l \in \mathcal{L}_p(V, L(V)) \equiv \mathcal{L}((V, p), \mathcal{L}_{\tilde{p}}(V)). \tag{1.2}$$

For each family $\emptyset \neq \Gamma \subseteq sem(V)$ of seminorms on V a multiplication $\pi \equiv l$ on V (associative or not) makes Γ:

(c) *an A-convex family of seminorms iff*

$$l \in L(V, \mathcal{L}_\Gamma(V)) = \bigcap_{p \in \Gamma} L(V, \mathcal{L}_p(V)), \tag{1.3}$$

(d) *a family of algebra-seminorms iff*

$$l \in \mathcal{L}_\Gamma(V, L(V)) \equiv \bigcap_{p \in \Gamma} \mathcal{L}_p(V, L(V)). \tag{1.4}$$

Proof. (a) Let p be an A-convex seminorm on $(V, \pi \equiv l)$. Then for arbitrary $\omega \in V$ there exists $\lambda_\omega > 0$ such that $p(\omega\theta) \leq \lambda_\omega p(\theta)$, $\theta \in V$, $p \circ l(\omega) \leq \lambda_\omega p$. Thus, $\lambda_\omega \in \Lambda(p, l(\omega)) \neq \emptyset$, $l(\omega) \in \mathcal{L}_p(V)$, ($\omega$ arbitrary), $l \in L(V, \mathcal{L}_p(V))$.

Conversely, let $l \in L(V, \mathcal{L}_p(V))$. Then $l(\omega) \in \mathcal{L}_p(V)$, $\omega \in V$ so that $\Lambda(l(\omega), p) \neq \emptyset$. Namely $p(\omega\theta) \equiv p(l(\omega)(\theta)) \leq \lambda p(\theta)$, $\theta \in V$ for all $\lambda \in \Lambda(l(\omega), p) \neq \emptyset$. In other words p becomes A-convex, on $(V, \pi \equiv l)$ $(\omega\theta \equiv \pi(\omega, \theta) \equiv l(\omega)(\theta)$, $\omega, \theta \in V)$.

(b) Let now p be an algebra-seminorm on (V, π). Then $p(\omega\theta) \leq \lambda p(\omega)p(\theta)$, $\omega, \theta \in V$ (for some $\lambda > 0$), so that $p(l(\omega)(\theta)) \leq \lambda p(\omega)p(\theta)$, $\lambda p(\omega) \in \Lambda(l(\omega), p) \neq \emptyset$, $\omega \in V$.

Thus $\tilde{p}(l(\omega)) := \inf \Lambda(p, l(\omega)) \leq \lambda p(\omega)$, $\omega \in V$ which means that $l \in \mathcal{L}_p(V, L(V)) \equiv \mathcal{L}((V,p), \mathcal{L}_{\tilde{p}}(V))$. Below $\hat{p}(l)$ stands for the bound of each $l \in \mathcal{L}((V,p), (\mathcal{L}_p(V), \tilde{p}))$.

Conversely, if $l \in \mathcal{L}_p(V, L(V))$ there exists $\lambda > 0$ such that $\tilde{p}(l(\omega)) \leq \lambda p(\omega)$, $\omega \in V$. Thus $\lambda p(\omega) \in \Lambda(p, l(\omega))$, so that $p(\omega\theta) \equiv p((l(\omega)(\theta)) = (p \circ l(\omega))(\theta) \leq \lambda p(\omega) p(\theta)$, $\omega, \theta \in V$.

(c), (d) are easy consequences of (a), (b) respectively. ∎

Remark 1.2. (a) We write $AC(V,p)$ (respectively: $AC(V,\Gamma)$) for the set (space) of all multiplications (associative or not) of V making p (resp. Γ) A-convex. Thus Proposition 1.1 implies:

$$AC(V,p) = L(V, \mathcal{L}_p(V)), \; p \in \Gamma \tag{1.5}$$

$$AC(V,\Gamma) = L(V, \mathcal{L}_\Gamma(V)) = \bigcap_{p\in\Gamma} L(V, \mathcal{L}_p(V)), \; \emptyset \neq \Gamma \subseteq semV. \tag{1.6}$$

Similarly: For the subspaces $lmc(V,p)$, $lmc(V,\Gamma)$ of $L(V, L(V))$ making p, Γ (respectively) algebra-seminorms we get

$$lmc(V,p) = \mathcal{L}_p(V, L(V)) \subseteq AC(V,p), \; p \in \Gamma \tag{1.7}$$

$$lmc(V,\Gamma) = \mathcal{L}_\Gamma(V, L(V)) \equiv \bigcap_{p\in\Gamma} \mathcal{L}_p(V, L(V)) \subseteq AC(V,\Gamma), \emptyset \neq \Gamma \subseteq sem(V). \tag{1.8}$$

(b) Now we remark that $\mathbb{C} \cdot I \subseteq \mathcal{L}_\Gamma(V)$ for each family $\emptyset \neq \Gamma \subseteq sem(V)$ so that we get

$$V^+ \equiv L(V, \mathbb{C}) \cong L(V, \mathbb{C} \cdot I) \subseteq L(V, \mathcal{L}_\Gamma(V)) \equiv AC(V, \Gamma). \tag{1.9}$$

By this last relation we see that:

Corollary 1.3. *The dimension of the space $AC(V,\Gamma)$ of all multiplications $\pi \equiv l$ of V, making (V, l, Γ) an A-convex algebra, is at least the* $\dim V^+$ *of the algebraic dual V^+ of V.*

(c) For each A-convex seminorm $p \in sem(A)$ of an algebra A, the lRR l of A defines a seminorm $\dot{p} \equiv \tilde{p} \circ l \in sem(V)$ which is, in particular, an algebra seminorm if the algebra A is associative. Thus we obtain:

Corollary 1.4. *For each $l \in L(V, \mathcal{L}_p(V))$ the seminorm $\dot{p} \equiv \tilde{p} \circ l \in sem(V)$ satisfies the relation:*

$$p(\omega\theta) \leq \dot{p}(\omega) p(\theta), \; \omega, \theta \in V. \tag{1.10}$$

If in particular (V, l) is associative then $\dot{p}$ is an algebra seminorm of $A \equiv (V, l \equiv \pi)$ (which we call the lm-convex seminorm of (V, l) corresponding to p).

(d) For the space $lmc(V,\Gamma) \equiv \bigcap_{p\in\Gamma} \mathcal{L}_p(V, L(V))$ we see that it is getting smaller while Γ is getting larger (see (1.8)). In other words we observe that:

(d-1) For $\emptyset \neq \Gamma \subseteq \Delta \subseteq sem(V) : \tau_\Gamma = \tau_\Delta$ (the locally convex topology defined on V by each $\emptyset \neq \Delta \subseteq sem(V)$) we can have $\mathcal{L}_\Gamma(V) \supseteq \mathcal{L}_\Delta(V) \neq \mathcal{L}_\Gamma(V)$ (see also [8]).

(d-2) If we take the topology τ on V to be the largest one, it is possible to get $\mathcal{L}_\Gamma(V, L(V)) = (0)$ (see (1.4)).

Thus arises a question:

QUESTION 1.5. Is it possible to obtain a family $\Delta \subseteq sem(V)$ in the (non-empty!) set:

$$sem(\tau) := \{\Delta \subseteq sem(V) : \tau_\Delta = \tau\} \neq \emptyset \tag{1.11}$$

such that $lmc(V, \Delta) \neq (0)$ while $lmc(V, \Gamma_{\max}(\tau)) = (0)$? (Here $\Gamma_{\max}(\tau) \equiv \bigcup_{\Gamma \in sem(\tau)} \Gamma$).

(e) Let $0 \neq f \in V^+$. By (1.9) we define $l \equiv l_f$ by the relation:

$$l \equiv l_f : V \to L(V) : l_f(\omega) \equiv f(\omega) \cdot I,\ f \in V^+. \tag{1.12}$$

Then we obtain an associative algebra (V, l_f) (in which f is the unique character). Moreover we put:

$$q_f : V \to \mathbb{R}_+ : q_f(\omega) \equiv |f(\omega)|,\ f \in V^+,\ \omega \in V,\ q_f \in sem(V). \tag{1.13}$$

Thus, we obtain the following:

COROLLARY 1.6. *For each $p \in sem(V)$, $0 \neq f \in V^+$ $(l_f(\omega) \equiv f(\omega)I)$ we obtain:*

$$\dot{p} = \dot{q}_f = q_f. \tag{1.14}$$

Proof. $p(\omega\theta) \equiv p(l_f(\omega)(\theta)) = p(f(\omega)\theta) = |f(\omega)|p(\theta) = q_f(\omega)p(\theta)$. ∎

(f) By the above we see that several questions arise regarding the use of the algebra $\mathcal{L}_{\tilde{p}}(V)$. In the present paper we further give some realizations of $AC(V, \Gamma)$ and $lmc(V, \Gamma)$.

2. Some concrete examples

PROPOSITION 2.1. *Let $\varepsilon^\delta := \{\varepsilon^\delta_\kappa : \kappa \in K\} \subseteq V^+$ be the dual set of an arbitrary base $\varepsilon = \{\varepsilon_\kappa : \kappa \in K\}$ of a $\mathbb{C}$-vector space V:*

$$\varepsilon^\delta_\kappa(\varepsilon_\lambda) := \delta_{\kappa\lambda} \equiv \begin{cases} 1, & \kappa = \lambda, \\ 0, & \kappa \neq \lambda,\ \kappa, \lambda \in K. \end{cases} \tag{2.1}$$

Then it is well known that for each $\omega \in V$ there exists a $K_\omega \subseteq K$ finite such that $\omega = \sum_{\kappa \in K_\omega} \varepsilon^\delta_\kappa(\omega)\varepsilon_\kappa$. Put:

$$\omega = \sum_{\kappa \in K_\omega} \varepsilon^\delta_\kappa(\omega)\varepsilon_\kappa \equiv \sum_{\kappa \in K} \varepsilon^\delta_\kappa(\omega)\varepsilon_\kappa,\ \omega \in V. \tag{2.2}$$

Then for arbitrary $\emptyset \neq \Gamma \subseteq sem(V)$ we get:

$$AC(V, \Gamma) = L(V, \mathcal{L}_\Gamma(V)) \cong l(K, \mathcal{L}_\Gamma(V)) \equiv \mathcal{L}_\Gamma(V)^K. \tag{2.3}$$

Proof. For each $F \in l(K, \mathcal{L}_\Gamma(V))$ we define $F' \in L(V, \mathcal{L}_\Gamma(V))$ by

$$F'(\varepsilon_\kappa) := F(\kappa),\ \kappa \in K. \tag{2.4}$$

Then the correspondence $F \to F'$ is a vector-space isomorphism. For example,

$$F' = G' \Leftrightarrow F(\kappa) = F'(\varepsilon_\kappa) = G'(\varepsilon_\kappa) = G(\kappa),\ \kappa \in K. \ ∎$$

Let now $l_\infty(K, \mathbb{R}_+)$ stand for the subspace of all bounded elements in $l(K, \mathbb{R}_+) \equiv \mathbb{R}^K_+$,

$$f \in l_\infty(K, \mathbb{R}) \Leftrightarrow \sup_K |f(\kappa)| < +\infty. \tag{2.5}$$

Also we put (for each $f \in l_\infty(K, \mathbb{R}_+)$, $\varepsilon \equiv \{\varepsilon_\kappa : \kappa \in K\} \subseteq V$ base)

$$q_\kappa : V \to \mathbb{R}_+ : q_\kappa(\omega) := |\varepsilon^\delta_\kappa(\omega)|,\ \omega \in V,\ \kappa \in K \tag{2.6}$$

$$p(\omega) \equiv \sum_{\kappa \in K_\omega} f(\kappa) q_\kappa(\omega) \equiv \sum_{\kappa \in K} f(\kappa) q_\kappa(\omega), \ \omega \in V. \tag{2.7}$$

Thus we obtain the following:

PROPOSITION 2.2. *Let* $f \in l_\infty(K, \mathbb{R}_+)$, $\varepsilon, \varepsilon^\delta, p$ *as above. Then*

$$lmc(V,p) \equiv \mathcal{L}_p(V, L(V)) \subseteq l_\infty(K, \mathcal{L}_p(V)). \tag{2.8}$$

If, moreover, $f(\kappa) \geq 1$, $\kappa \in K$ *then*

$$lmc(V,p) \cong l_\infty(K, \mathcal{L}_p(V)). \tag{2.9}$$

Proof. Let $l \in lmc(V,p)$ and we put $l'(\kappa) := l(\varepsilon_\kappa)$, $\kappa \in K$. Then $\tilde{p}(l'(\kappa)) = \tilde{p}(l(\varepsilon_\kappa) \leq \hat{p}(l)p(\varepsilon_\kappa) = \hat{p}(l)f(\kappa) \leq \hat{p}(l) \sup_K f(\kappa) \equiv \hat{p}(l) \cdot \|f\|_\infty < +\infty$. In other words: $\sup_{\mu \in K} \tilde{p}(l'(\kappa)) \leq \hat{p}(l) \cdot \|f\|_\infty < +\infty \Leftrightarrow l' \in l_\infty(K, \mathcal{L}_p(V))$ so that we have proved (2.8). Let in particular $f(\kappa) \geq 1$, $\kappa \in K$. Then for arbitrary $l' \in l_\infty(K, \mathcal{L}_p(V))$ we define $l \in \mathcal{L}((V,p), \mathcal{L}_{\tilde{p}}(V))$ by the relation

$$l(\varepsilon_\kappa) := l'(\kappa), \ \kappa \in K. \tag{2.10}$$

(Thus for the unique linear extension of l on V we get: $l(\omega) = \sum_{\kappa \in K_\omega} \varepsilon_\kappa^\delta(\omega) l'(\kappa)$, $\omega \in V$.) Hence:

$$\begin{aligned}\tilde{p}(l(\omega)) &\leq \sum_{\mu \in K_\omega} q_\mu(\omega) \tilde{p}(l'(\mu)) \leq \sup_K \tilde{p}(l'(\mu)) \cdot \sum_{K_\omega} q_\mu(\omega) \\ &\leq \|l'\|_\infty \cdot \sum_K f(\mu) q_\mu(\omega) = \|l'\|_\infty p(\omega), \ \omega \in V.\end{aligned}$$

Thus $l \in \mathcal{L}_p(V, L(V)) \equiv \mathcal{L}((V,p), \mathcal{L}_{\tilde{p}}(V))$, while $\hat{p}(l) \leq \|l'\|_\infty$. The proof of (2.9) is now easy (see also Proposition 2.1). ∎

REMARK 2.3. In [1] we find an interesting theory about the positive cone $sem(V)$. Also in [4], [5] there exists a rich general theory about lmc-algebras (with or without an involution $*$). On the other hand the famous Vidav-Palmer Theorem can get an easier form (if one uses the results above, see also [9]). In [2] we can also find some basic results on algebra seminorms (in particular for barrelled and m-barrelled seminorms). Finally we see that the A-convex structures on a locally convex space (V, Γ) constitute a "large" space (while the lm-convex structures can constitute the zero space (0) if the family Γ is sufficiently large; see also [8]). Thus the question 1.5 arises. Professor W. Żelazko posed similar questions at the Będlewo 2003 conference.

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