GEOMETRY AND TOPOLOGY OF CAUSTICS — CAUSTICS '06
BANACH CENTER PUBLICATIONS, VOLUME 82
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2008

GLOBAL STRUCTURE OF HOLOMORPHIC WEBS ON SURFACES

VINCENT CAVALIER and DANIEL LEHMANN

Département des Sciences Mathématiques, CP 051, Université de Montpellier II
Place Eugène Bataillon, F-34095 Montpellier Cedex 5, France
E-mail: cavalier@math.univ-montp2.fr, lehmann@math.univ-montp2.fr

Abstract. The webs have been studied mainly locally, near regular points (see a short list of references on the topic in the bibliography). Let d be an integer ≥ 1. A d-web on an open set U of $\mathbb{C}^2$ is a differential equation $F(x,y,y')=0$ with $F(x,y,y')=\sum_{i=0}^{d} a_i(x,y)(y')^{d-i}$, where the coefficients a_i are holomorphic functions, a_0 being not identically zero. A regular point is a point (x,y) where the d roots in y' are distinct (near such a point, we have locally d foliations mutually transverse to each other, and *caustics* appear through the points which are not regular).

It happens that many concepts on local webs may be globalized, but not always in an obvious way, and under the condition that they do not depend on local coordinates. The aim of this paper is to make these facts precise and to define the tools necessary for a global study of webs on a holomorphic surface, and in particular on the complex projective plane $\mathbb{P}_2$. Moreover new concepts, inducing new problems, will appear, such as the dicriticality, the irreducibility or the quasi-smoothness, which have no interest locally near a regular point of the web.

1. Global definition of a web.

First of all, we homogenize the equation in the abstract, for allowing the contact elements to be "vertical" (this notion does not make sense by change of local coordinates), and write the differential equation $\varpi = 0$, where

$$\varpi = \sum_{i=0}^{d} a_i(x,y)(dx)^i(dy)^{d-i}$$

is now a homogeneous polynomial of degree d on U (removing also the condition $a_0 \not\equiv 0$). Moreover, if we multiply ϖ by a holomorphic non-vanishing function, we do not change the solutions of the differential equation. Hence, gluing together local webs defined as above, we get the following global definition.

2000 *Mathematics Subject Classification*: 14C21, 53A60.

Key words and phrases: global web, dicriticality, type, degree, indistinguishability, quasi-smoothness, Blaschke curvature, Chern curvature, abelian relation.

This paper is a summary of [CaLe], without proofs.

A d-web on a holomorphic surface M is the data of a holomorphic line bundle E on M and of a homogeneous polynomial $\varpi : S^d(TM) \to E$ of degree d on M with holomorphic coefficients in E, i.e. a holomorphic section of $S^d(T^*M) \otimes E$, TM denoting the complex tangent space to M, and $S^d(TM)$ its d-th symmetric power. Locally, once given local holomorphic coordinates (x, y) and a local holomorphic non-vanishing section σ_E on some open set U of M, the restriction of ϖ to U may be written $\varpi|_U = (\sum_{i=0}^d a_i(x,y)(dx)^i(dy)^{d-i}) \otimes \sigma_E$, with holomorphic coefficients a_i. Thus, the homogeneous polynomial $\sum_{i=0}^d a_i(x,y)(dx)^i(dy)^{d-i}$ may be written

$$\prod_{i=0}^{d}(r_i(x,y)\,dx + s_i(x,y)\,dy),$$

and the local "leaves" of the web are the curves, solutions of one of the differential equation $r_i(x,y)\,dx + s_i(x,y)\,dy = 0$. Moreover, we require that

- the germs of the coefficients a_i at each point are primes (all their common divisors $u(x,y)$ must be units in the ring of germs of functions, avoiding extra-solutions $u(x,y) = 0$),
- the discriminant set of the homogeneous polynomial $\sum_{i=0}^d a_i(x,y)(dx)^i(dy)^{d-i}$, i.e. the set where its resultant vanishes (still called "caustic"), be an analytic set of complex dimension at most 1 (off the caustic, the solutions of the d differential equations $r_i(x,y)\,dx + s_i(x,y)\,dy = 0$ must be mutually transversal).

These conditions and definitions depend neither on the choice of the local coordinates (x, y) nor on the local trivialisation σ_E.

2. Type and degree. The bundle E is called the *type* of the web. When $M = \mathbb{P}_2$, the d-webs of *degree* n are those for which $E = \mathcal{O}(n + 2d)$ (the $(n + 2d)$-th tensor power of the dual $\mathcal{O}(1)$ of the tautological bundle $\mathcal{O}(-1)$): they are the webs such that a generic straight line of $\mathbb{P}_2$ is tangent to some leaf of the web at n distinct points (see Section 6 below). In particular an *algebraic* d-web (web whose the leaves are the tangents to some algebraic envelope of class d) has degree 0: a generic straight line has in fact no chance to belong to such an envelope; the converse is also true:

THEOREM 2.1. *The webs of degree* 0 *are the algebraic webs.*

3. The contact manifold and the tautological contact form. Let $\widetilde{M}$ be the total space of the bundle $\mathbb{P}TM \xrightarrow{\pi} M$, projectivised of TM (for any point $m \in M$, the fibre $\widetilde{M}_m = \pi^{-1}(m)$ is the projective line $\mathbb{P}(T_mM)$ of the directions of lines in T_mM). A point $\widetilde{m}$ of $\widetilde{M}$ is called a *contact element* of M at $m = \pi(\widetilde{m})$. For any non-vanishing vector $v \in T_mM$, $[v] \in \widetilde{M}_m$ will denote the contact element generated by v.

Let (x, y) be local holomorphic coordinates on an open set U of M. We define local coordinates on the set U_x of the contact elements in $\pi^{-1}(U)$ which are different from $[(\frac{\partial}{\partial y})_m]$ in the following way: the point $[(\frac{\partial}{\partial x})_m + p(\frac{\partial}{\partial y})_m]$ has local coordinates x, y, p), (x, y) denoting the coordinates of m in U. [Observe that we get $[(\frac{\partial}{\partial y})_m]$ with the new coordinates $x' = y$ and $y' = x$].

We shall denote by L the tautological line bundle of $\mathbb{P}(TM)$: it is the sub-vector-bundle of $\pi^{-1}(TM)$ whose fibre at each point $[v] \in \widetilde{M}$ is the subspace of T_mM generated by the vector v, with $m = \pi([v])$. Let $\mathcal{L}$ be the quotient bundle

$$0 \to L \to \pi^{-1}(TM) \to \mathcal{L} \to 0.$$

We shall denote by $\mathcal{V}$ the sub-bundle of $T\widetilde{M}$ of vectors tangent to the fibres of $\pi : \widetilde{M} \to M$, hence the exact sequence of vector bundles $0 \to \mathcal{V} \to T\widetilde{M} \to \pi^{-1}(TM) \to 0$.

Let $\omega : T\widetilde{M} \to \mathcal{L}$ be the composition of the two projections $\pi^{-1}(TM) \to \mathcal{L}$ and $T\widetilde{M} \to \pi^{-1}(TM)$ above.

Theorem 3.1. *There exists a canonical isomorphism* $\mathcal{L} \cong L \otimes \mathcal{V}$ *such that, on the domain of local coordinates* (x, y, p), ω *reads* $(dy - p\,dx) \otimes ((\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}) \otimes \frac{\partial}{\partial p})$.

This 1-form ω with coefficients in $\mathcal{L}$ will be called the *tautological contact form.*

4. The surface W, the critical curve and the caustic. Let $\varpi : S^d(TM) \to E$ be a d-web on M. If we compose the map $L^d \to \pi^{-1}(S^d(TM))$ induced by the natural inclusion $L \to \pi^{-1}(TM))$ with the map $\pi^{-1}(\varpi) : \pi^{-1}(S^d(TM)) \to \pi^{-1}(E)$, we get a holomorphic map $L^d \to \pi^{-1}(E)$, i.e. a holomorphic section s_W of the line bundle $\check{L}^d \otimes \pi^{-1}(E)$. The zero set $W = (s_W)^{-1}(0)$ of this section is an analytic complex surface in $\widetilde{M}$.

Theorem 4.1. *The data of* W *is equivalent to that of* ϖ.

In fact, if ϖ is written locally $(\sum_{i=0}^{d} a_i(x, y)(dx)^i(dy)^{d-i}) \otimes \sigma_E$, W has local equation $F(x, y, p) = 0$ in $\widetilde{M}$, where $F(x, y, p) = \sum_{i=0}^{d} a_i(x, y)p^{d-i}$. Thus, it is sometimes the surface W that we shall call "the web".

We shall denote by W' the regular part of W, $\Sigma(W) = W \setminus W'$ its singular part, and $\pi_W : W \to M$ the restriction of π to W.

We shall say that the web is *smooth* if W has no singularity ($W = W'$). This implies that W is irreducible in $\widetilde{M}$. More generally, we shall say that the web is *quasi-smooth* if each irreducible component of W is smooth.

Let W_0 be the subset of points $\widetilde{m}$ in W' where the differential $\pi_{\widetilde{m}} : T_{\widetilde{m}}W' \to T_mM$ is an isomorphism, and denote by Γ_W its complement $W \setminus W_0$ in W (containing $\Sigma(W)$). We call Γ_W the *critical curve* of the web, and its projection $\pi(\Gamma_W)$ on M its *discriminant curve* or *caustic*. The *regular part* of the web is the projection $M_0 = \pi(W_0)$ of W_0, i.e. the set of points in M not belonging to the caustic (it is generally strictly smaller than the projection $\pi(W')$ of the regular part W' of W).

Lemma 4.2. *The critical curve* Γ_W *is a complex analytical set of complex dimension at most* 1. *The restriction to* W_0 *of the projection* $\pi_W : W \to M$ *is a d-fold covering over its image* M_0 $(= \pi(W_0))$.

Theorem 4.3. *The critical curve* Γ_W *is the zero-set* $(s_\Gamma)^{-1}(0)$ *of a holomorphic section* s_Γ *of the line bundle* $[\pi^{-1}E \otimes \mathcal{V}^* \otimes \check{L}^d]_{|W}$ *over* W, *locally defined in* $\widetilde{M}$ *by the equations* $F'_p = 0$ *and* $F = 0$.

Remark. Assuming M_0 to be connected, each connected component of W_0 is itself a covering of M_0, which is completely defined up to isomorphism by the data of a conjugation class of sub-group of the fundamental group $\pi_1(M_0)$. Hence, the family of these conjugation classes is an invariant of the web.

Theorem 4.4. *There exists a canonical holomorphic section* s_Σ *of the line bundle* $[\pi^{-1}(E)\otimes \check{L}^{d+1}]|_{\Gamma_W}$*, locally defined by the equations* $(F'_x + pF'_y = 0,\ F = 0,\ F'_p = 0)$*. Moreover, when* $M = \mathbb{P}_2$*,* s_Σ *has a natural extension to all of* W *(still denoted by* s_Σ*) which is a section of* $[\pi^{-1}(E)\otimes \check{L}^{d+1}]|_W$*, locally defined by the equations* $(F'_x + pF'_y = 0,$ $F = 0)$ *on an affine open set of* $\mathbb{P}_2$ *with affine coordinates* (x, y)*.*

Definition. We shall say that a web is *non-dicritical* (resp. *dicritical*) if the section s_Σ over Γ_W is not (resp. is) identically zero. More generally, if s_Σ vanishes on some irreducible component C of Γ_W, we shall say that the web is dicritical along C.

5. Canonical foliation $\widetilde{\mathcal{F}}$ on W'.

Let $\omega_W : TW' \to \mathcal{L}|_{W'}$ be the restriction of the contact form ω to the tangent space TW' to the regular part W' of W: it is necessarily integrable since W' has dimension 2, and so defines a holomorphic foliation $\widetilde{\mathcal{F}}$ on W'. A *leaf of the web in* M is the projection by π of any leaf of $\widetilde{\mathcal{F}}$ in W' or of its closure in W.

Theorem 5.1. *We can also define this foliation as a holomorphic morphism* $\ell : \mathcal{M} \to TW'$ *of a suitable line bundle* $\mathcal{M}$ *in* TW'*,*

- *locally defined by the vector field* $X_1 = F'_p\,(\frac{\partial}{\partial x} + p\,\frac{\partial}{\partial y}) - (F'_x + p\,F'_y)\,\frac{\partial}{\partial p}$ *in the case of a non-dicritical web, and* $\mathcal{M} = [\pi^{-1}E^*\otimes\mathcal{V}\otimes L^{d+1}]_{|W'}$ *in this case,*
- *locally defined by the vector field* $X_2 = \frac{\partial}{\partial\widetilde{x}} + p\,\frac{\partial}{\partial\widetilde{y}}$ *on* W_0*, and* $X_2 = \frac{\partial}{\partial x} + p\,\frac{\partial}{\partial y}$ *on* Γ_W *in the dicritical case, and* $\mathcal{M} = L_{|W'}$ *in this case.*

Proposition 5.2. *The projection* $\pi_W : W_0 \to M_0$ *of the covering maps locally* $\widetilde{\mathcal{F}}$ *on* d *distinct foliations* $\mathcal{F}_i$ $(1 \le i \le d)$*, mutually transversal on* M_0*.*

Remark. Note that these d foliations are distinguishable only locally, on an open set above which the previous covering is trivial. Globally, they may be undistinguishable. See Section 6 below.

Theorem 5.3.

(i) *If the web is non-dicritical,* $\Sigma(\widetilde{\mathcal{F}})$ *is equal to the zero-set* $s_\Sigma^{-1}(0)$ *of the section* s_Σ*.*
(ii) *If the web is dicritical,* $\widetilde{\mathcal{F}}$ *has no singularity on* W'*.*

Let C be an irreducible smooth compact component of Γ_W along which the web is non-dicritical. Let $\{m_\alpha\}$ be the set of (isolated) points of $C\cap\Sigma(\widetilde{\mathcal{F}})$. Near each point m_α, choose local coordinates (x, y, p) and a local trivialisation of E, hence a local equation $F = 0$ of W. Denote by ν_α the order at m_α of the restriction of $F'_x + pF'_y$ to C.

Theorem 5.4. *The following formula holds:*

$$\sum_\alpha \nu_\alpha = -(\pi^* c_1(E) + (d+1)c_1(L)) \frown [C],$$

in which the sum $\sum_\alpha \nu_\alpha$ *does not depend on the various choices above.*

This is a simple application of [CL].

6. Irreducibility of W and global indistinguishability of the local foliations.

Have in mind the case of an algebraic 2-web on $\mathbb{P}_2$ whose leaves are the straight lines belonging to some envelope of class 2: according to the fact that this envelope is a proper conic, or degenerates into two points, W is irreducible or has two irreducible components, and the two local foliations on M_0 are globally indistinguishable or distinguishable.

THEOREM 6.1.

(i) *If* $W_0 = W \setminus \Gamma_W$ *is connected, the web is irreducible.*
(ii) *Conversely, if the surface* W *is compact, connected and smooth* (*this last assumption implying in particular that the web is irreductible*), *the open set* $W_0 = W \setminus \Gamma_W$ *is also connected.*

COROLLARY 6.2. *Every web whose surface* W *is compact and connected may be decomposed into irreducible webs* $W = W_1 \cup W_2 \cup \cdots \cup W_r$. *The number of connected components of* W_0 *is exactly* r *if the web is quasi-smooth, and at least* r *in the general case.*

One says that a d-web W is *completely reducible* if $r = d$. Locally, near every point of M_0, a web is always completely reducible. An *open set of distinguishability* will be every open set U of M_0 such that the restriction $W_U = W \cap \pi^{-1}(U)$ of W to U is completely reducible (or equivalently such that the restriction of the covering $\pi_W : W_0 \to M_0$ to U is trivial).

The *space of leaves of* M_0, denoted by $W_0/\widetilde{\mathcal{F}_0}$, is the space of leaves of W_0 for the foliation $\widetilde{\mathcal{F}}_0$ induced by $\widetilde{\mathcal{F}}$ on W_0.

Let m be a point of M_0 and F_i and F_j two distinct germs of leaves of the web at m_0, respectively belonging to the local foliations $\mathcal{F}_i$ and $\mathcal{F}_j$ of the web. Let $\widetilde{m_i}$ and $\widetilde{m_j}$ be the lifts of m in W_0, such that the germs of leaves of $\widetilde{\mathcal{F}}$ at $\widetilde{m_i}$ and $\widetilde{m_j}$ map respectively onto F_i and F_j by π. We shall say that F_i and F_j are *globally indistinguishable* if the leaves of $\widetilde{\mathcal{F}}_0$ through $\widetilde{m_i}$ and $\widetilde{m_j}$ belong to the same connected component of $W_0/\widetilde{\mathcal{F}_0}$. If not, F_i and F_j will be called *globally distinguishable.*

THEOREM 6.3.

(i) *If* $W = W_1 \cup W_2 \cup \cdots \cup W_r$ *has* r *irreducible components, the space of leaves* $W_0/\widetilde{\mathcal{F}_0}$ *has at least* r *connected components, and exactly* r *if* M *is compact and the web quasi-smooth.*
(ii) *Two germs of leaves of the web at a point* m *of* M_0, F_i *and* F_j, *are globally distinguishable if and only if the corresponding points* $\widetilde{m_i}$ *and* $\widetilde{m_j}$ *in* W_0 *do not belong to the same connected component of* W_0.

7. Webs on $\mathbb{P}_2$.

Denote by (X, Y, Z) the homogeneous coordinates on $\mathbb{P}_2$, and (u, v, w) the homogeneous coordinates on the dual projective plane $\mathbb{P}'_2$ of projective straight lines in $\mathbb{P}_2$: the line of coordinates (u, v, w) is the line of equation $uX + vY + wZ = 0$ in $\mathbb{P}_2$.

LEMMA 7.1. *The manifold* $\widetilde{\mathbb{P}_2}$ *is naturally identified to the space of points* $([X, Y, Z], [u, v, w])$ *in* $\mathbb{P}_2 \times \mathbb{P}'_2$ *such that* $uX + vY + wZ = 0$: *a contact element is a pair given by a point in* $\mathbb{P}_2$ *and a line through this point. By this identification,* π *becomes the restriction of the first projection of* $\mathbb{P}_2 \times \mathbb{P}'_2$.

The spaces $\mathbb{P}_2$ and $\mathbb{P}'_2$ have completely symmetric roles: the second projection $\pi' : \widetilde{\mathbb{P}_2} \to \mathbb{P}'_2$ is also a space fibred by projective lines, with which the same constructions as with π can be done. Denote respectively $\mathcal{O}(-1)$ and $\mathcal{O}'(-1)$ the tautological line bundles of $\mathbb{P}_2$ and $\mathbb{P}'_2$, $\mathcal{O}(1)$ and $\mathcal{O}'(1)$ their dual. Let $\ell = \pi^{-1}(\mathcal{O}(-1))$ and $\ell' = {\pi'}^{-1}(\mathcal{O}'(-1))$.

Lemma 7.2.

(i) *The vector bundle* $\pi^{-1}(\bigwedge^2 T\mathbb{P}_2)$ *may be identified to* $\breve{\ell}^3$.
(ii) *The vector bundle* L *may be identified to the tensor product* $\ell' \otimes \breve{\ell}^2$.
(iii) *The vector bundle* $\mathcal{V}$ *is isomorphic to the tautological bundle* L' *of* $\widetilde{\mathbb{P}_2}$ (*identified to the projectivized bundle of* $T\mathbb{P}'_2$), *and* $\mathcal{L} = L \otimes L'$.

Let $H(X, Y, Z; u, v, w)$ be a polynomial in the variables X, Y, Z, u, v, w, homogeneous of degree n with respect to the variables (X, Y, Z), and homogeneous of degree d with respect to the variables (u, v, w). We call (n, d) the bi-degree of homogeneity of H. Let W be the surface of equations $(H = 0, uX + vY + wZ = 0)$ in $\widetilde{\mathbb{P}_2}$. Every polynomial $\overline{H}$ defining the same surface W has the same bi-degree. The integer n is in fact equal to the number of points at which a generic straight line $[u_0, v_0, w_0]$ of $\mathbb{P}_2$ meets the surface of equation $H(X, Y, Z; u_0, v_0, w_0) = 0$ in $\mathbb{P}_2$, i.e. is tangent to a solution of the differential equation $H(x, y, 1; y', -1, y - xy') = 0$ defined by W on the affine set $Z \neq 0$, with $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$. It has therefore a geometrical meaning not depending on the polynomial H. This is true in particular when W is a web on $\mathbb{P}_2$. From Lemma 7.2 we deduce

Proposition 7.3. *If* H *has bidegree* (n, d), *then* $E = \mathcal{O}(n + 2d)$.

The number n is then the *degree* of the web, such as defined in Section 2.

Then W has for equation $H(x, y, 1; p, -1, y - px) = 0$, whose left term is a polynomial of degree d with respect to p, with coefficients $a_i(x, y)$ polynomial of degree $\leq n + d$ with respect to (x, y). Conversely, any web on $\mathbb{P}_2$ may be defined by this procedure from a bi-homogeneous polynomial H. Identifying $\mathbb{C}^2$ to the affine open set $Z \neq 0$ of $\mathbb{P}_2$,

Theorem 7.4.

(i) *A web on* $\mathbb{P}_2$ *is completely defined by its restriction to* $\mathbb{C}^2$.
(ii) *A web of equation* $F(x, y, p) = 0$ *on* $\mathbb{C}^2$, *where* $F(x, y, p) = \sum_{i=0}^{d} a_i(x, y) p^{d-i}$, *may be extended as a web on all of* $\mathbb{P}_2$, *if and only if all coefficients* a_i *are polynomial in the natural coordinates* (x, y) *of* $\mathbb{C}^2$.

Denote respectively by $\xi = c_1(\breve{\ell})$, $\xi' = c_1(\breve{\ell}')$ and $\eta = c_1(\breve{L})$ the Chern classes of the bundles $\pi^{-1}(\mathcal{O}(1))$, $\breve{\pi}^{-1}(\breve{\mathcal{O}}(1))$ and $\breve{L}$.

Lemma 7.5. *The following formulae hold*:

(i) $\eta = \xi' - 2\xi$,
(ii) $H^*(\widetilde{\mathbb{P}_2}, \mathbb{Z}) = \mathbb{Z}[\xi, \xi']/(\xi^3, \xi'^3, \xi^2 + \xi'^2 - \xi\xi')$.

Definition. The *co-critical set* of the d-web W is the set Γ'_W defined in $\widetilde{\mathbb{P}_2}$ by the equations $vH'_X - uH'_Y = 0$, $wH'_Y - vH'_Z = 0$, $uH'_Z - wH'_X = 0$, and $H = 0$. Its restriction to the open set $Zv \neq 0$ is defined by the equations $(F'_x + pF'_y = 0,\ F = 0)$, or $(D'_x = 0,$

$D = 0$), where

$$F(x, y, p) = H(x, y, 1; p, -1, y - px) \ \text{ and } \ D(x, p, r) = H(x, r + px, 1; p, -1, r).$$

We can observe that the non-dicritical webs on $\mathbb{P}_2$ are precisely the *bi-webs*, i.e. the webs whose surface W defines also a web on $\mathbb{P}'_2$. In this case, the co-critical set Γ'_W is a curve, which is the critical curve of the n-web on $\mathbb{P}'_2$, while Γ_W is its co-critical curve; moreover, the intersection $\Gamma_W \cap \Gamma'_W$ is the singular set $\Sigma(\widetilde{\mathcal{F}})$ of the foliation $\widetilde{\mathcal{F}}$.

Theorem 7.6. *Every web W on $\mathbb{P}_2$ has a non-empty critical curve Γ_W. For every non-dicritical web on $\mathbb{P}_2$, the singular set $\Sigma(\widetilde{\mathcal{F}}) = \Gamma_W \cap \Gamma'_W$ of $\widetilde{\mathcal{F}}$ is non-empty* (*the intersection number $\Gamma_W . \Gamma'_W$ may be computed explicitly, using the residues ν_α of Theorem* 5.4).

Recall that a d-web on $\mathbb{P}_2$ is said to be *algebraic* if its leaves are the straight lines which belong to some algebraic envelope C of class d. If C is defined by the tangential equation $\Phi(u, v, w) = 0$ (where Φ denotes some homogeneous polynomial of degree d), the corresponding algebraic d-web is defined by $H(X, Y, Z; u, v, w) = \Phi(u, v, w)$, not depending on the variables X, Y, Z. The local equation of the surface W of the web, for $Zv \neq 0$, is written: $\Phi(p, -1, y - px) = 0$ with the affine coordinates defined above. (Up to multiplication by a scalar, H is well defined). Algebraic webs are then the webs of degree $n = 0$ (see Theorem 2.1).

Theorem 7.7. *For a web on $\mathbb{P}_2$ to be algebraic, it is necessary and sufficient that the identity $F'_x + pF'_y \equiv 0$ holds, with $F(x, y, p) = H(x, y, 1; p, -1, y - px)$, or equivalently the section $s_{\Gamma'}$ of the bundle $\pi^{-1}(E) \otimes \breve{L}^{d+1}|_W$ is identically 0.*

A web on $\mathbb{P}_2$ is said to be *linear* if all its leaves are straight lines of $\mathbb{P}_2$.

Theorem 7.8. *Every linear web globally defined on $\mathbb{P}_2$ is algebraic.*

Theorem 7.9. *A quasi-smooth web on $\mathbb{P}_2$ is algebraic, if and only if any of its irreducible components is dicritical.*

Remark. If W is not quasi-smooth, the web may be dicritical without being algebraic. Here is an example (with arbitrary scalar constants h and k):

$$\begin{aligned} H = u^3Z^3 - XZ^2u^2v + (X^2Z/3 - XZ^2 + kZ^3)uv^2 \\ - (X^3/27 - X^2Z/6 + kXZ^2/3 + hZ^3 + YZ^2)v^3. \end{aligned}$$

8. Background on 3-webs.

Let M_0 be the regular part of a 3-web on a surface M. Locally, near any point m of M_0, there are three mutually transversal foliations $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ with respective tangent bundles T_1, T_2, T_3. Let (i, j, k) be any permutation of $(1, 2, 3)$: the projection of T_i over T_j parallel to T_k defines a natural isomorphism Φ_{ij} from T_i onto T_j. However the bundles T_i may not be defined globally on all of M_0. We shall remedy this by the following construction. Let A' be the set of (non-ordered) triples $\{X_1, X_2, X_3\}$ of tangent vectors at a point m of M_0, such that

(i) $X_1 + X_2 + X_3 = 0$,
(ii) each X_i is tangent to one of the leaves of the web.

This set A' has a natural structure of holomorphic line bundle over M, locally isomorphic to any of the three T_i by the map $\Phi_i : \{X_i, X_j, X_k\} \mapsto X_i$ since the triple $\{X_i, X_j, X_k\}$ is completely determined by the data of any of the three vectors.

The connection of Blaschke. Locally, every T_i may be seen as the normal bundle to both $\mathcal{F}_j$ and $\mathcal{F}_k$. Since TM_0 is locally equal to $T_j \oplus T_k$, there exists on T_i a unique holomorphic connection ∇^i which is a Bott connection for both $\mathcal{F}_j$ and $\mathcal{F}_k$, such that ∇^i and ∇^j correspond to each other by the isomorphism Φ_{ij}. Since $\Phi_{ij} \circ \Phi_i = \Phi_j$, there exists a unique holomorphic connection ∇^b on A' corresponding to ∇^i by Φ_i. We shall call it the *connection of Blaschke*, its curvature K^b being the classical *Blaschke curvature.*

The connection of Chern. With the previous definitions, we can define a unique holomorphic connection ∇^c on TM_0, corresponding locally to $\nabla^j \oplus \nabla^k$ by the isomorphism $TM_0 \to T_j \oplus T_k$: this is *the connection of Chern*, whose curvature K^c is called the *Chern curvature.* It is easy to prove that the connection of Chern is also the unique holomorphic connection on TM_0, whose torsion vanishes and which preserves the web in the following sense: each local T_i is preserved by the covariant derivative of ∇^c.

Observe that ∇^c may be written $\begin{pmatrix} \Phi_j(\nabla^b) & 0 \\ O & \Phi_k(\nabla^b) \end{pmatrix}$, with respect to the local decomposition $TM_0 = T_j \oplus T_k$, hence $K^c = \begin{pmatrix} K^b & 0 \\ O & K^b \end{pmatrix}$.

Abelian relations. An *abelian relation for a 3-web* above an open set U of M_0 over which the three foliations $\mathcal{F}_i$ are distinguishable the data of three holomorphic closed 1-forms $\omega_1, \omega_2, \omega_3$ such that $\omega_1 + \omega_2 + \omega_3 = 0$ and $\operatorname{Ker} \omega_i$ is the tangent space T_i to $\mathcal{F}_i$ for each $i = 1, 2, 3$. This relation is said to be "non-trivial" if the ω_i's are not zero. For example, if there exists some coordinate system (x, y) and 3 affine functions $u_i(x, y)$ $(i = 1, 2, 3)$ which are respectively first integrals of the $\mathcal{F}_i$'s, there exist scalar constants a_i such that $a_1 u_1 + a_1 u_1 + a_1 u_1$ be constant: thus the family $\omega_i = a_i du_i$ defines a non-trivial abelian relation. It is easy to see that the set of abelian relations over a given U (or of germs at a point m of M_0) has a natural structure of vector space, whose dimension (called "the rank") is 0 or 1.

THEOREM 8.1 (Blaschke-Chern). *The following two assertions are equivalent*:

(i) *The Blaschke curvature vanishes.*
(ii) *There exists a non-trivial abelian relation near each point of* M_0.

This result follows from the fact that, when the Blaschke curvature vanishes, the connection of Chern which has simultaneously zero curvature and zero torsion, preserves therefore some locally affine structure on M_0; it is easy to deduce locally functions u_i, affine with respect to the previous affine structure, which are first integrals of the $\mathcal{F}_i$'s.

9. Abelian relations for arbitrary d.

Let W be a d-web on M, U be an open set in M_0, and $\widetilde{U}$ its pre-image $(\pi_W)^{-1}(U)$ in W_0. For any holomorphic section ξ of the dual $\check{\mathcal{L}}$ of $\mathcal{L}$ over $\widetilde{U}$, $\langle \xi, \omega_W \rangle$ is a scalar holomorphic 1-form.

DEFINITION. The *space of abelian relations over an open set* U *of* M_0 is the subspace $R(U)$ of holomorphic sections $\xi \in \Gamma(\widetilde{U}, \check{\mathcal{L}})$ such that

(i) $d < \xi, \omega_W >= 0$,
(ii) $⨍ < \xi, \omega_W >= 0$, where $⨍ : \bigwedge^* TW_0 \to \bigwedge^* TM_0$ denotes integration along the fibre of $\pi_W : W_0 \to M_0$ (in fact the finite sum of d terms, since π_W is a d-fold covering).
The dimension $r(U)$ of this vector-space is called the *rank* of W over U.

REMARK. Obviously, this definition also has a meaning for the germ of the web at a point $m \in M_0$.

LEMMA 9.1. *Let $A(U)$ (resp. $B(U)$) be the subspace of holomorphic sections $\xi \in \Gamma(\widetilde{U}, \breve{\mathcal{L}})$ satisfying only the condition* (ii) *(resp. the space of holomorphic* 2*-forms on $\widetilde{U}$ in the kernel of $⨍$). The pre-sheaves $U \mapsto A(U)$ and $U \mapsto B(U)$ are $\mathcal{O}_{M_0}$-locally free sheaves of respective rank $d-2$ and $d-1$.*

Let A and B be the corresponding holomorphic vector bundles over M_0. The map

$$\mathcal{D} : \xi \mapsto d\langle \xi, \omega_W \rangle$$

is then a linear differential operator of order 1, generalizing the local operator ρ in [H4] p. 437, and such that the abelian relations are the solutions of the equation $\mathcal{D}\xi = 0$.

Let R_k be the space of $(k+1)$-jets of solutions of the equation $\mathcal{D}\xi = 0$, i.e. the kernel of the k-th prolongation $D_k : J^{k+1}A \to J^k B$ of the morphism $D : J^1 A \to B$ defined by $\mathcal{D}$: it is a vector bundle over M_0. In [H4], Hénaut proved that R_{d-4} is a vector bundle of rank $(d-1)(d-2)/2$, and that the natural projection $\Psi : R_{d-3} \to R_{d-4}$ is an isomorphism.

We may interpret this isomorphism as defining a connection on the bundle $\mathcal{E} = R_{d-4}$ over M_0. In fact, R_{d-3} is equal to $J^{d-2}A \cap J^1 R_{d-4}$. Since $J^1 R_{d-4}$ is the space of elements of connection on R_{d-4} and since Ψ is linear, Ψ^{-1} is a connection on $\mathcal{E}$, the "connection of Hénaut", whose curvature generalized the Blaschke connection of the case $d = 3$ and is an obstruction for the web to have the maximal rank $(d-1)(d-2)/2$: the abelian relations are in fact the sections ξ of A such that $j^{d-3}\xi$ is a section of $\mathcal{E}$ with vanishing covariant derivative (and conversely, if a section of $\mathcal{E}$ with vanishing covariant derivative is the $d-3$-jet of some section ξ of A, ξ is an abelian relation).

REMARK. Of course, when $d = 3$, $A = \mathcal{E}$. Moreover, for any $d > 3$, any section of $\mathcal{E}$ with vanishing covariant derivative is effectively the $d-3$-jet of an abelian relation ξ when the rank of the web is maximal.

In the particular case $d = 3$, $\mathcal{E} = A$ coincide with the dual of the bundle A' defined in the previous section, and the connection of Hénaut coincide with the dual of the connection of Blaschke.

References

[BB] W. Blaschke and G. Bol, *Geometrie der Gewebe*, Springer, Berlin, 1938.

[CL] C. Camacho and D. Lehmann, *Residues of holomorphic foliations relative to a general submanifold*, Bull. London Math. Soc. 37 (2005), 435–445.

[CaLe] V. Cavalier and D. Lehmann, *Introduction à l'étude globale des tissus sur une surface holomorphe*, Ann. Inst. Fourier (Grenoble) 57 (2007), 1095–1133.

[Ce] D. Cerveau, *Théorèmes de type Fuchs pour les tissus feuilletés*, in: Complex Analytic Methods in Dynamical Systems (IMPA, January 1992), Astérisque 222 (1994), 49–92.

[C] S. S. Chern, *Abzählungen für Gewebe*, Abh. Hamburg 11 (1936), 163–170.

[H1] A. Hénaut, *Sur la linéarisation des tissus de* $\mathbb{C}^2$, Topology 32 (1993), 531–542.

[H2] A. Hénaut, *Caractérisation des tissus de* $\mathbb{C}^2$ *dont le rang est maximal et qui sont linéarisables*, Compositio Math. 94 (1994), 247–268.

[H3] A. Hénaut, *Sur la courbure de Blaschke et le rang des tissus de* $\mathbb{C}^2$, Natur. Sci. Rep. Ochanomizu Univ. 51 (2000), 11–25.

[H4] A. Hénaut, *On planar web geometry through abelian relations and connections*, Ann. of Math. (2) 159 (2004), 425–445.

[H5] A. Hénaut, *Analytic web geometry*, in: Web Theory and Related Topics (Toulouse, 1996), J. Grifone and E. Salem (eds.), World Sci., River Edge, NJ, 2001, 6–47.

[Ng] P. Nagy, *Webs and curvature*, in: Web Theory and Related Topics (Toulouse, 1996), J. Grifone and E. Salem (eds.), World Sci., River Edge, NJ, 2001, 48–91.

[N] I. Nakai, *Curvature of curvilinear* 4*-webs and pencils of one forms: variation on a theorem of Poincaré, Mayrhofer and Reidemeister*, Comment. Math. Helv. 73 (1998), 177–205.

[P] H. Poincaré, *Sur les surfaces de translation et les fonctions abéliennes*, Bull. Soc. Math. France 29 (1901), 61–86.

Added in proof. We have recently been informed that:

1) Global webs on the projective plane have already been defined in

J. Yartey, *Number of singularities of a generic web on the complex projective plane*, J. Dynamical Control Systems 11 (2005), 281–296.

2) In [H4], A. Hénaut recovered independently, and with another terminology, results of

A. Pantazi, *Sur la détermination du rang d'un tissu plan*, C. R. Acad. Sci. Roumanie 2 (1938), 108–111.