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AFFINELY INVARIANT SYMMETRY SETS

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Abstract. The classical medial axis and symmetry set of a smooth simple plane curve M, depending as they do on circles bitangent to M, are invariant under euclidean transformations. This article surveys the various ways in which the construction has been adapted to be invariant under affine transformations. They include affine *distance* and *area* constructions, and also the 'centre symmetry set' which generalizes central symmetry. A connexion is also made with the tricentre set of a convex plane curve, which is the set of points which are the centres of three chords.

1. Introduction. For a smooth closed curve in $\mathbb{R}^2$ (resp. a smooth closed surface in $\mathbb{R}^3$), M, the *symmetry set* is the closure of the set of centres of circles (resp. spheres) tangent at more than one point to M. The *medial axis* of M is the subset of the symmetry set for which the circle (resp. sphere) is *maximal* in the sense that its radius equals the absolute minimum distance from its centre to M. Both these constructions, which are essentially euclidean because of the use of spheres, have been the subject of extensive investigation both in the mathematical and in the computer vision literature. See for example [3, 8, 25, 26] for some different viewpoints. They also have generalizations to higher dimensions, the most obvious being when M is a hypersurface in $\mathbb{R}^n$. Together with the focal set, the symmetry set forms the 'full bifurcation set' of the family of distance-squared functions on M, parametrized by the points of the ambient euclidean space.

There are also a number of constructions which depend only on the affine structure of M, that is they depend not on distance and angle but on affine concepts such as parallelness, midpoint or equality of areas. The motivation for seeking and investigating such constructions lies in the wish to detect symmetry when a scene is presented only after distortion by an affine transformation, such as viewing 'at an angle'.

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In this article I shall survey these affinely invariant constructions and give some details and examples of each, with special emphasis on interesting lines of investigation which have so far not been followed up. Nearly all the examples will be in $\mathbb{R}^2$, with occasional mention of $\mathbb{R}^3$.

There are classical constructions in affine differential geometry which can be used to imitate the above metric definition of the symmetry set of a plane curve; these give rise to the 'affine distance symmetry set' (ADSS) which is described in §2. It is also possible to imitate a dual construction of the symmetry set as an envelope of lines and this gives rise to the 'affine envelope symmetry set' (AESS); see §3. Both the ADSS and the AESS have close connexions with conics, as the natural analogue of circles. A number of constructions use pairs of points for which the tangent lines (or planes) to M are parallel: the 'centre symmetry set' (CSS) measures the extent to which M is centrally symmetric, and the singular points of the 'affine equidistants' sweep out the CSS in much the same way that the focal set of a curve or surface M is swept out by the singularities of the euclidean parallels, or offsets, of M.

An entirely different approach is given by an area-based symmetry set, the *affine area symmetry set* (AASS), which was introduced for plane curves in [20] as a construction which is robust to small perturbations of the boundary shape: an objection to constructions which depend on the calculation of higher derivatives of a curve or surface is that these quantities are highly sensitive to noise or to perturbations of M. The AASS, like the euclidean symmetry set or the ADSS, is defined for a plane curve M, at least for a strictly convex curve (see §4) by means of a family of functions on M parametrized by an open set of points of the plane. The bifurcation set of this family consists of the AASS together with the midpoints of chords joining parallel tangent pairs, which is one of the affine equidistants mentioned above. This set also has interesting connexions with the boundary of the 'tricentre set', that is the boundary of the set of points inside M which are the midpoints of *three* chords. See §4.

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2. The affine distance symmetry set (ADSS) of a plane curve. This is a direct analogue of the euclidean symmetry set, but uses a different family of functions. The euclidean distance between points is not of course an affine invariant, but, *given a curve M in $\mathbb{R}^2$*, we can, following Izumiya [16], define an *affine distance function f*. Suppose first that M has no inflexions (points of zero curvature), and use a parametrization γ by 'affine arclength s', which means that the determinant $|\gamma', \gamma''| = 1$ for all parameter values, $'$ standing for d/ds, or, later, $\partial/\partial s$. Then we define

$$f : \mathbb{R}^2 \times \mathbb{R}, \quad f(\mathbf{x}, s) = |\mathbf{x} - \gamma(s), \gamma'(s)|.$$

(If γ is a closed curve then the second factor $\mathbb{R}$ should be replaced by the circle S^1.) In the special case when γ is a central conic (an ellipse or hyperbola), and $\mathbf{x}$ is its centre, this function is constant; its value is called the *affine radius* of the conic.

Using this family for a general curve γ without inflexions, we can imitate the definitions in the euclidean case. Proofs of the results below are in [9, 10].

(a) The *affine normal* at $\gamma(s)$ is the set of points $\mathbf{x}$ in the plane for which $f'(\mathbf{x}, s) = 0$. This is equivalent to saying that the affine normal is the set of points $\gamma(s) + t\gamma''(s)$ for $t \in \mathbb{R}$. The affine normal is also the locus of points $\mathbf{x}$ which are centres of conics having 4-point contact with γ at $\gamma(s)$.

(b) The *affine centre of curvature* of γ at $\gamma(s)$ is the unique point $\mathbf{x}$ for which $f'(\mathbf{x}, s) = f''(\mathbf{x}, s) = 0$. This is the point

$$\gamma(s) + \tfrac{1}{\mu}\gamma''(s),$$

where the *affine radius of curvature* is $\mu(s) = |\gamma''(s), \gamma'''(s)|$. This is a finite point unless $\gamma''' = 0$, which is called an *affine inflexion*. It is the centre of the unique conic having at least 5-point contact with γ at $\gamma(s)$.

(c) The *affine evolute* of γ is the envelope of the affine normals; it is also the set of affine centres of curvature. It has cusps at the affine centre of curvature at points where the unique 5-point contact conic has 6-point contact (sextactic points).

(d) The ADSS is the set of points $\mathbf{x} \in \mathbb{R}^2$ for which there exist two distinct parameter values s_1, s_2, with

$$(1) \qquad f(\mathbf{x}, s_1) = f(\mathbf{x}, s_2) \text{ and } f'(\mathbf{x}, s_1) = 0, \; f'(\mathbf{x}, s_2) = 0,$$

together with limit points of this set.

The definition (d) is exactly analogous to the euclidean requirement that the euclidean distance function should have extrema at two points of γ, meaning that $\mathbf{x}$ lies on two euclidean normals to γ, and that the euclidean distances of $\mathbf{x}$ from these two curve points should be equal. Put together, these mean in the euclidean case that there is a *circle* tangent to γ at the two points. In the affine case, where circles are replaced by conics which have a greater freedom, we cannot deduce that there is a single conic which is tangent to γ at $\gamma(s_1), \gamma(s_2)$. We can deduce that there are *two conics* with the same centre $\mathbf{x}$ and the same affine radius, having *four-point* contact with γ at $\gamma(s_1), \gamma(s_2)$.

The condition on s_1, s_2 that there should exist an $\mathbf{x}$ as in (1) is

$$(2) \qquad |\gamma(s_1) - \gamma(s_2), \;\; \gamma''(s_1) - \gamma''(s_2)| = 0, \;\; \text{or} \;\; |\gamma_1 - \gamma_2, \;\; \gamma_1'' - \gamma_2''| = 0 \text{ for short}$$

which can be called the *pre-ADSS* condition.

Because the ADSS is based upon a distance, there is the possibility of defining a corresponding *medial axis*, the ADMA, by insisting that the extrema in (1) are absolute minima. Indeed we can equally require them to be absolute maxima since it is not so clear in this case which is more appropriate. The conic which has 4-point contact at $\gamma(s_1)$ and $\gamma(s_2)$ could be a hyperbola, and there is therefore no concept of the conic being 'inside'

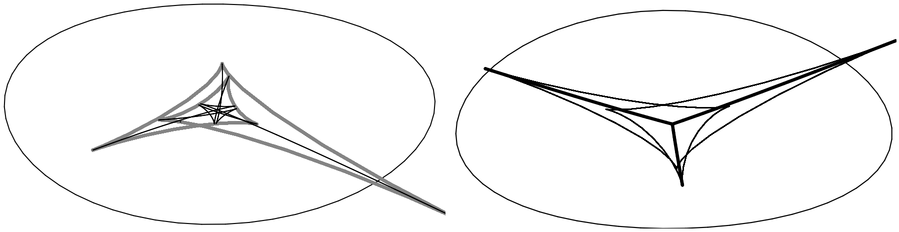

Fig. 1. Left: an affine distance symmetry set (ADSS), with the affine evolute shown in grey. Notice the complicated structure of the ADSS, similar to that of the euclidean symmetry set. Right: an affine distance medial axis (ADMA), drawn heavily with this time the affine evolute a thinner line. The ADMA has three basic almost straight branches. Although this does not in theory guarantee three partial affine axes of symmetry, in practice it appears that this is 'almost' the case.

or 'outside' the curve γ. This is in contrast with the euclidean symmetry set, where a bitangent circle can be required to be entirely inside a curve γ. Note that every point of a smooth closed convex curve γ contributes to the ADMA in the same way that every point of a closed curve contributes to the euclidean medial axis. Illustrations of an ADSS and ADMA are shown in Figure 1.

The ADSS, and consequently the ADMA, has many of the properties one would expect from the analogy with the euclidean case.

(e) The ADSS (or ADMA) has endpoints in the cusps of the affine evolute. These are A_3 points of the affine distance function f, where the second and third derivatives vanish as well as the first. The above conics, and the contact points $\gamma(s_1), \gamma(s_2)$, have coincided.
(f) The ADSS (but not the ADMA) has cusps which lie on the affine evolute, and occur when one of the conics has 5-point contact. These are A_1A_2 points. The generic structure of the ADMA is, as for the euclidean medial axis, that of a tree with branches at which three arcs meet.

The ADSS has one unfortunate feature: if it is straight we cannot guarantee that the curve γ is symmetric in an affine sense. An *affine reflexion* is an affine map from the plane to itself, of order two (equal to its inverse) and preserving a line ℓ. A curve which is, as a whole, taken to itself by an affine reflexion is called *affine symmetric* with ℓ as an *axis of affine symmetry.* In the euclidean case, if the euclidean symmetry set of two arcs of a curve is a straight line ℓ, we can deduce that those arcs are symmetric in the usual euclidean sense with ℓ as axis of symmetry. However the obvious analogy, that if the ADSS of two arcs is straight, then the arcs are symmetric by an affine reflexion, is not true; there is an example in [9]. However in many examples of closed curves it does appear that the ADSS has 'almost straight branches' and that, as measured by the affine envelope symmetry set (the subject of the next section) there is approximately an affine symmetry preserving part of the curve. The exact relationship between affine symmetry and the ADSS is not well understood.

Another problem with the ADSS is that it does not appear to be possible to generate it in a simple way by a *flow*. The euclidean symmetry set of γ can be thought of as the locus of self-intersections of the euclidean parallels (offsets) of γ, and these are generated by a flow along the normals of γ at unit speed. There does not appear to be any analogous way of generating the ADSS. (Added in proof: See [22] for a solution.)

Strictly speaking the existence of the ADSS depends on γ being free from inflexions (a convex curve). However, inflexions are themselves affinely invariant points of a curve, and we can segment a curve by means of the inflexions before treating each pair of inflexion-free arcs as above when calculating the ADSS. We mention here two significant facts about the ADSS of a curve with inflexions (see [7]). Firstly, in a generic 1-parameter family of such curves, the local transitions on the euclidean symmetry set were listed in [4], and certain transitions, in principle possible for bifurcation sets in the plane, were excluded for geometrical reasons. However, all the 'missing' transitions occur in the context of the ADSS. Secondly, consider singularities of the ADSS arising directly from inflexions: for example, at the point where an inflexional tangent meets the curve again, the ADSS exhibits a singularity of the highly degenerate type (t^5, t^6). In a 1-parameter family of curves, two such singularities can cancel one another and leave a smooth ADSS branch. Because the family of affine distance functions has infinite values at such point, a systematic treatment appears to need V. I. Arnold's theory of 'fractions' [1] (analogous to meromorphic functions), but there is no published work on the ADSS from this point of view.

Note on reconstruction. Suppose we are given a smooth arc S, parametrized by $\gamma(s)$ where s is euclidean arclength, and a smooth function r defined on S such that $r'(s) < 1$ for all s close to some s_0. We can 'reconstruct' a pair of arcs M_1, M_2 with symmetry set S by taking the envelope of circles, centred at the $\gamma(s)$ and of radius $r(s)$; the condition $r'(s) < 1$ ensures that the envelope is real. It is not clear what is needed besides a smooth arc S in order to 'reconstruct' two arcs with ADSS equal to S. A variant of this problem is the following. Given S and one arc M_1 can we reconstruct a second arc M_2 such that the two M_i have ADSS equal to S? (In the euclidean case we merely need the normals to M_1 to meet S; this immediately gives the radii and centres of the bitangent circles.) In fact, using the interpretation of the ADSS in terms of contact with conics, it is shown in [15] that:

(g) we can choose a general point $\mathbf{p}_2 \in \mathbb{R}^2$ and a point $\mathbf{p}_1 \in M_1$ and then find a smooth curve M_2 through $\mathbf{p}_2$ such that S is the ADSS of M_1 and M_2 and $\mathbf{p}_1, \mathbf{p}_2$ are corresponding points.

There is an analogous construction for the ADSS of a surface in $\mathbb{R}^3$, based also on an affine distance function (see for example [2, p. 110]). However, there does not appear to be any simple relationship with quadric surfaces, and the properties of this ADSS have not as yet been investigated. There is a good deal of information on the affine focal surface, which forms the remainder of the full bifurcation set of the family of affine distance functions, in [5].

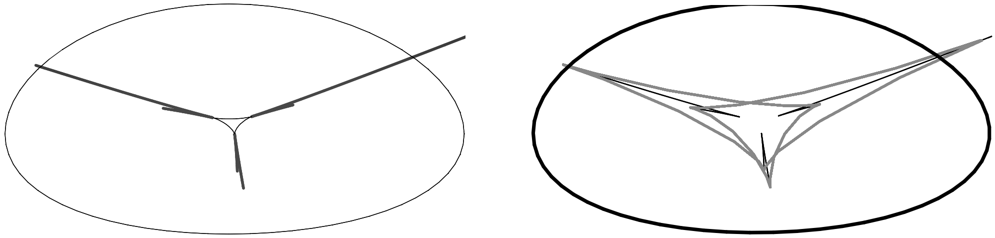

Fig. 2. Left: the affine envelope symmetry set (AESS, heavy line) of a convex curve, together with (thin line) an 'affine equidistant' §5, also called the 'area evolute' §4. As can be seen these fit together nicely with cusps facing one another. This is investigated in [15]. Right: an AESS together with the affine evolute, the latter in grey. The three almost straight branches of the AESS *do* indicate three partial axes of affine symmetry.

3. The affine envelope symmetry set (AESS) of a plane curve. The construction of the AESS is dual to that of the ADSS. In the euclidean case this dual construction yields *the same* symmetry set as the usual construction, but in the affine setting it yields a different symmetry set. The details of the following can be found in [10] and [21].

The AESS can be defined as an envelope of lines, as follows. Associate to a pair of parameter points s_1, s_2 on the curve M with parametrization γ the line ℓ which passes through the intersection of the tangent lines to γ at the points $\gamma(s_1), \gamma(s_2)$ and through the midpoint of the segment joining $\gamma(s_1)$ and $\gamma(s_2)$. The obvious variant of this applies when the tangents are parallel: ℓ is parallel to the tangents and through the midpoint of the segment. In the limit as $s_2 \to s_1$ the line ℓ in fact approaches the affine normal of γ at the point $\gamma(s_1) = \gamma(s_2)$ (see (a) in §2).

For a closed curve parametrized by $\gamma : S^1 \to \mathbb{R}^2$ this defines a mapping m from $S^1 \times S^1$ to the set of lines in $\mathbb{R}^2$, that is the dual plane. Locally this is simply a mapping $\mathbb{R}^2 \to \mathbb{R}^2$ and it turns out that (s_1, s_2) is a critical point if and only if there is a conic having 3-point contact with γ at $\gamma(s_1)$ and $\gamma(s_2)$. The locus of centres of such conics is called the AESS and is the envelope of the lines ℓ for which (s_1, s_2) is a critical point of m. That is, the critical locus of m defines the AESS as an envelope. The envelope of affine normals arises as part of this envelope (the limiting case $s_1 = s_2$) but we exclude this from the AESS, just as the evolute is excluded from the symmetry set in the euclidean case.

An example of an AESS is shown in Figure 2. The AESS has the following properties:

(a) The AESS has cusps at the centres of conics where the contact is 4-point at one of the two contact points (and usually 3-point at the other).
(b) The AESS has endpoints in the cusps of the affine evolute (exactly as for the ADSS).
(c) If the AESS has a straight segment, then the corresponding arcs of the curve γ are affine symmetric with that segment as affine axis (defined as in §2).
(d) The condition for s_1, s_2 to give a point of the AESS is (with affine arclength parametrization)

$$|\gamma(s_1) - \gamma(s_2), \quad \gamma'(s_1) + \gamma'(s_2)| = 0;$$

this equation therefore determines the *pre-AESS*.

Two more remarks on the AESS:

(e) There is no obvious concept of an 'Affine Envelope Medial Axis', that is a medial axis which is a significantly simpler subset of the AESS but nevertheless captures all of the curve γ. In view of (c) above it would be very interesting to discover such a medial axis.

(f) It would be a pleasant surprise if the essential structure of the AESS were preserved by *projective* transformations. (This possibility was suggested to the author as an interesting question by Prof W. Thurston.)

Unfortunately, it is been shown [21] that the proposal in (f) fails both for the AESS and the ADSS. For the AESS the problem lies partly in (e): the AESS generically has cusps and there is no obvious way to restrict it so that these are removed. The number of cusps can be changed by projective transformations, which can produce a swallowtail transition on the AESS. The motivation for (f) is that the essential structure of the euclidean symmetry set is preserved by Möbius transformations of the plane, that is 'fractional linear' transformations ($z \mapsto (az+b)/(cz+d),\ z \in \mathbb{C},\ ad-bc \neq 0$), which actually map the plane extended by a point at infinity to itself. These preserve circles, and contact between curves, so bitangent circles are taken to bitangent circles. On the other hand *centres* of circles are not preserved so a Möbius transformation does not actually take the symmetry set (resp. medial axis) of a curve to the symmetry set (resp. medial axis) of the transformed curve. But the 'graph structure' of the symmetry set—the branches, the endpoints and the triple crossings, and also the cusp structure—is preserved. At the medial axis level, where the underlying combinatorial structure is a tree, the branches, the Y-junctions and the endpoint structure are all preserved by Möbius transformations so that the medial axis is taken to another which is isomorphic as a graph.

Note on reconstruction. The analogue for the AESS of (g) in §2 is the following (see [15]).

(g) We can choose a general point $\mathbf{p}_2 \in \mathbb{R}^2$ and then find a smooth curve M_2 through $\mathbf{p}_2$ such that S is the AESS of M_1 and M_2.

In this case we do not have the freedom to choose a corresponding point $\mathbf{p}_1$ on M_1. It is unclear whether there is another simple 'reconstruction' result along the lines 'given S and some other information, analogous to the radius function in the euclidean case, there are unique arcs M_1, M_2 having S for their AESS'.

It is not clear whether there is a natural definition of the AESS for a surface M in $\mathbb{R}^3$. For example, given a pair of points of M we can consider the perpendicular bisector plane for the chord joining them. This gives us a map $M \times M \to G$ where G is the 3-dimensional grassmannian of planes in $\mathbb{R}^3$. So we cannot imitate the construction given above in $\mathbb{R}^2$.

4. The affine area symmetry set of a plane curve. Let M be a simple closed plane curve. We shall usually take M to be smooth, with a parametrization γ, but it can equally well be piecewise smooth, as in Figure 3. We shall assume here that M is strictly convex; some results on the non-convex case are in [20, 19]. Our purpose here is to point out the similarity of definition with that of the other distance-based symmetry sets and to link it to the affine equidistants which arise in §5.

For each point $\mathbf{x}$ in the interior of M and each point $\gamma(t)$ on M, let $A(t, \mathbf{x})$ be the area of the region inside M cut off by, and to the right of, the chord from $\gamma(t)$ through $\mathbf{x}$. See Figure 3. We can regard A as a two-parameter family of functions defined on M. Using the basic formula

$$A(t, \mathbf{x}) = \tfrac{1}{2} \int_t^u |\gamma(s) - \mathbf{x},\ \gamma'(s)| ds,$$

where $|\ ,\ |$ denotes the determinant of a 2×2 matrix, and $u(t, \mathbf{x})$ is the other end of the chord, we can derive the following ([15, 20]).

(a) $A'(t, \mathbf{x}) = 0$ (where $'$ means $\partial/\partial t$) if and only if $\mathbf{x}$ is the midpoint of this chord.
(b) $A' = A'' = 0$ at $(t, \mathbf{x})$ if and only if in addition the tangents to M at the endpoints of this chord are parallel.

The *area evolute* of M is the set of points $\mathbf{x}$ for which there exists t with $A' = A'' = 0$ at $(t, \mathbf{x})$. This is the locus of midpoints of chords having parallel tangents at their endpoints; it can also be obtained as the envelope of the lines halfway between pairs of parallel tangents. Note that if M is described anticlockwise, that is with positive curvature, the area evolute is a curve with cusps also described with positive curvature, making the 'inside' of the area evolute on the right of the curve, as with the euclidean evolute of a plane curve without inflexions.

The area evolute has also been called the *midpoint parallel tangent locus* and the *anti-symmetry set*, see [15, 18]). It is also an example of an *affine equidistant* as in §5.

(c) $A' = A'' = A''' = 0$ at $(t, \mathbf{x})$, so that the area evolute is singular, with generically an ordinary cusp, if and only if, in addition to the conditions of (a) and (b), the point $\mathbf{x}$ is also on the AESS (§3), that is, there is a conic having 3-point contact at the two endpoints of the chord. The AESS also has a cusp at such a point. See Figure 2, left. The condition for an ordinary cusp on the area evolute can also be expressed in a less obviously affinely independent way by saying that the (euclidean) curvatures of M are the same at the two endpoints but their derivatives with respect to (euclidean) arclength are not.

The affine area symmetry set (AASS) is then defined as the remaining part of the full bifurcation set of the family A, that is the set of points $\mathbf{x}$ for which there exist distinct t_1, t_2 such that

$$A(t_1, \mathbf{x}) = A(t_2, \mathbf{x}) \text{ and } A' = 0 \text{ at } (t_1, \mathbf{x}) \text{ and } (t_2, \mathbf{x}), \tag{3}$$

together with limit points of this set.

(d) The tangent to the AASS is parallel to the line joining $\gamma(t_1)$ to $\gamma(t_2)$, or equally well to the line joining the other ends $\gamma(u_1), \gamma(u_2)$ of the two chords which cut off equal areas inside M. Since the two chords have the same midpoint, their four endpoints are at the vertices of a parallelogram.

Because we are dealing again with a family of functions we can define a corresponding *affine area medial axis* (AAMA) by requiring that the extrema of A in (3) are absolute minima. Two examples of an AASS are given in Figure 3. Because they are based on an area construction, this symmetry set and medial axis are very robust to small deformations in the curve M, and to noisy data. See [20].

A possible extension to curves with inflexions is discussed in [20]; naturally we have to decide how to measure the area of a region cut off by a chord meeting the curve at more than just its endpoints. A different approach, where we essentially count enclosed areas on one side of the chord as positive and on the other side as negative, has been investigated in [19] but no definitive treatment exists. The affine equidistants for surfaces, that is midpoints of chords joining surface points where the tangent planes are parallel, are studied in $\mathbb{R}^3$ in [27]. There is no work on the extension of the AASS itself to surfaces in $\mathbb{R}^3$, though in this case the extension, using volumes of regions cut off by planes, is a natural one. It can be shown that, if the volume cut off inside a strictly convex surface M by planes through a fixed interior point $\mathbf{x}$ is an extremum for a plane Π, then $\mathbf{x}$ is the centre of mass of the intersection of M and Π, in analogy with (a) above.

We conclude this section by describing briefly an interesting connexion between the area evolute and the boundary, B say, of the set T of *tricentres* inside a strictly convex closed curve M, where by a tricentre we mean the common midpoint of *three* chords of M. The existence of tricentres is proved in [23]. Indeed it is shown that there is a 2-parameter family of tricentres, since two of the angles between the three chords can be specified; thus tricentres can be expected to form a 2-dimensional region T. Boundary points of T occur when two or more of the chords tend to coincidence. When two chords with a common midpoint tend to coincidence the tangents at their ends become parallel—compare (a) and (b) above—and when three chords with a common endpoint tend to coincidence the curvatures at the ends of the chord also become equal, as in (c) above. In either case B will be contained in the area evolute of M as defined above.

(e) In fact (at any rate for a generic M), T is the boundary of the region 'enclosed' by the area evolute, and we indicate below why this is so. An example is shown in Figure 3(vii).
(f) As above we use a parametrization γ of M. For a generic M, if six distinct parameter values $(t_i, u_i),\ i = 1, 2, 3$ give the endpoints of three distinct chords of M with a common midpoint m_0, then nearby parameter values fill out an open neighbourhood of m_0 in the plane.

Assume that the six parameters are distinct. Also consider the three tangent pairs at the ends of the three chords and assume that at least one of these pairs is not parallel (a generic assumption). The result (f) can be established using the map $F : \mathbb{R}^6 \to \mathbb{R}^4$ given locally by

$$F(t_1, u_1, t_2, u_2, t_3, u_3) = (\gamma(t_1) + \gamma(u_1) - \gamma(t_2) - \gamma(u_2),\ \ \gamma(t_1) + \gamma(u_1) - \gamma(t_3) - \gamma(u_3)),$$

together with the midpoint map $m(t_1, u_1, t_2, u_2, t_3, u_3) = \frac{1}{2}(\gamma(t_1) + \gamma(u_1))$. Thus $m(F^{-1}(\mathbf{0}))$ is locally the set of tricentres. It is easy to check that $F^{-1}(\mathbf{0})$ is locally a smooth 2-manifold and m restricted to this set is a local diffeomorphism.

But use of F when the t_i and u_i are not distinct is hampered by the presence of 'unwanted' solutions to $F = \mathbf{0}$. These include $t_2 = t_3,\ u_2 = u_3$ where the chords with

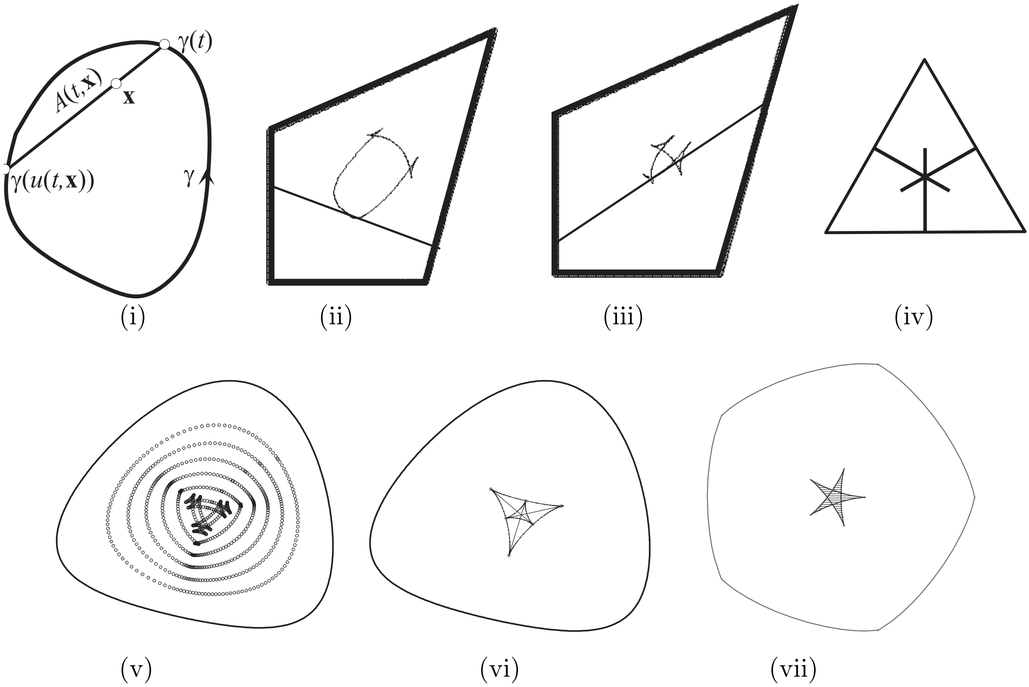

Fig. 3. See §4 for details. (i) The area $A(t, \mathbf{x})$ is cut off inside the oval γ by the chord through $\gamma(t)$ and $\mathbf{x}$. The other end of the chord is at $\gamma(u)$ where, for a strictly convex curve, u is a function of t and $\mathbf{x}$. The envelope of chords cutting off a fixed area is an *area parallel* of γ; this is also the locus of midpoints of such chords; (ii) and (iii) show two area parallels of a quadrilateral, with four and six cusps respectively, and one of the corresponding chords in each case. (iv): The self-intersections of the area parallels for a triangle, which make up the affine area symmetry set (AASS). (v): some area parallels for a rounded triangle; (vi): the outer 3-cusped curve is the area evolute, the inner 3-cusped curve is the area parallel corresponding to half the area, and the three nearly straight lines are the AASS, which is the locus of self-intersections of the parallels. The 'area medial axis' will have the same three arms but extending only as far as the triple junction. (vii): the shaded region 'inside' the area evolute is also the set of points which are centres of three chords of the curve.

endpoints t_1, u_1 and t_2, u_2 have a common midpoint, and indeed $t_1 = t_2 = t_3,\ u_1 = u_2 = u_3$. Both of these give 2-dimensional families of solutions and they prevent $F^{-1}(\mathbf{0})$ from being smooth at the points we need to discuss. Thus it will be difficult to restrict m to the 'wanted' part of $F = \mathbf{0}$ and find conditions for the result to be a fold or a cusp map. A different approach is outlined below.

Suppose that the tangents at the ends of one chord are parallel. Without loss of generality, the chord is along the x-axis, the tangents are parallel to the y-axis and the centre of the chord is at the origin $\mathbf{O}$. The curve is then given in polar coordinates by say $r = r(\theta)$. The function $R(\theta) = r(\theta + \pi) - r(\theta)$ is zero precisely when $\mathbf{O}$ is the midpoint of the chord at slope θ and since $R(\theta + \pi) = -R(\theta)$ it follows that R is determined by its values on $0 \le \theta < \pi$. Furthermore $R(0) = R'(0) = 0$ and double zeros of R correspond with chords having parallel tangents at their ends, while triple zeros require in addition that the curvatures at the two ends are equal. Note that R must have an odd number of zeros in $0 \leq \theta < \pi$; consequently if it has exactly a double zero at $\theta = 0$ then there is another zero which makes the origin the centre of three chords, two of them having coincided with the x-axis. The origin is then locally in the set B, that is so far as points of M near to $(\pm r(0), 0)$ are concerned. The area evolute passes through the origin tangent to the y-axis since this is the line halfway between the parallel tangents at $(\pm r(0), 0)$.

Examining the function R we find the following.

1. If R has exactly a double zero at $\theta = 0$ then the points near the origin which are tricentres lie on the same side of the area evolute as the 'interior' of the area evolute (that is, on the right of this curve);
2. If R has exactly a triple zero at $\theta = 0$ then the area evolute has a cusp at the origin (with cuspidal tangent the y-axis) and the points near the origin which are tricentres lie within this cusp.

Thus, locally, the interior of the area evolute is the same as the set of tricentres. Globally different smooth branches may overlay one another but the set of tricentres is then, as claimed in (e) above, the interior of the area evolute in the natural sense illustrated in Figure 3(vii).

It would be interesting to study the situation for curves with inflexions. The result of [23] is not known to hold then. In addition, it is not clear whether there are sensible generalizations to surfaces in $\mathbb{R}^3$.

5. The centre symmetry set and affine equidistants. Consider a simple closed smooth plane curve M and for each pair of distinct points at which the tangents are parallel draw the straight line through those points. The *envelope* of these lines is called the *centre symmetry set* (CSS) of M; it was first defined in a different way in [17]. For a centrally symmetric curve the CSS is a point, namely the centre of symmetry. A particularly simple case occurs for a *curve M of constant width*, for then it is well-known that all chords joining pairs $\mathbf{p}_1, \mathbf{p}_2$ of parallel-tangent points are in fact normal to M at each end-point $\mathbf{p}_1, \mathbf{p}_2$. In this case the CSS is doubly covered by the evolute of M. An example is drawn in Figure 4, left.

From the chords joining parallel tangent pairs $\mathbf{p}_1, \mathbf{p}_2$ we can construct also a family of *affine equidistants* by fixing a real number λ and taking the locus of points $(1-\lambda)\mathbf{p}_1+\lambda\mathbf{p}_2$. Of course λ and $1 - \lambda$ give the same locus, so that $\lambda = \frac{1}{2}$ is special. This is the locus of midpoints of parallel tangent pairs and has arisen above (§4) as the area evolute of M, that is as the bifurcation set of the area family A (the remaining part of the full bifurcation set being the AASS). See Figure 4, right.

There has been extensive work on the local structure of the CSS both for plane curves and for surfaces in $\mathbb{R}^3$; see [6, 12, 13, 14, 28]. For a curve M in $\mathbb{R}^2$ the CSS is necessarily real (Figure 4), and has one point for each pair of parallel tangent points of M. For a surface in $\mathbb{R}^3$, the CSS is the envelope of a 2-parameter family of lines, hence may not be everywhere real: corresponding to each pair of parallel tangent points of M there may be 2, 1 or 0 real CSS points. Hence there are generically 'special curves' on M over which there is a single CSS point. This contrasts with typical euclidean constructions

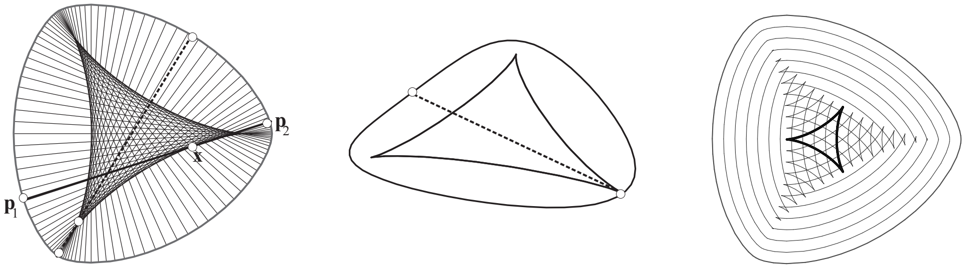

Fig. 4. Left: a curve of constant width with its chords joining parallel tangent pairs drawn; these are automatically binormals to the curve. The CSS is the envelope of these chords, hence is doubly covered by the evolute. The chord $\mathbf{p}_1\mathbf{p}_2$ is tangent to the CSS at $\mathbf{x}$ and the ratio of distances $\mathbf{p}_1\mathbf{x}$ to $\mathbf{x}\mathbf{p}_2$ is equal to the ratio of curvatures $\kappa(\mathbf{p}_2)$ to $\kappa(\mathbf{p}_1)$. The other chord drawn, tangent at a cusp on the CSS, has the ratio of curvatures at the endpoints of the chord an extremum. Centre: the CSS of a more general oval, again having an odd number of cusps (a generic phenomenon). The chord drawn, tangent at a cusp on the CSS, has one end at a degenerate inflexion—in a family of curves, this one is about to become non-convex—and the other end at the cusp point itself. Right: a family of affine equidistants of the same curve as in the left figure, with the special equidistant $\lambda = \frac{1}{2}$, also known as the 'area evolute' (§4), drawn heavily. The cusps on the equidistants trace out the CSS.

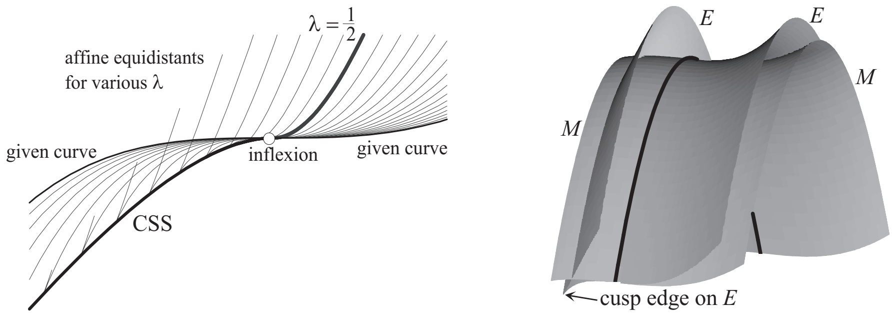

Fig. 5. Left: the given curve has an inflexion and the CSS of this curve is tangent to this inflexion and has an endpoint there. The affine equidistants for $\lambda \neq \frac{1}{2}$ have an inflexion at the same point and with the same tangent, and have a cusp which, as λ varies, traces out the CSS. The special equidistant for $\lambda = \frac{1}{2}$ (dark line) has an endpoint like the CSS. Right: one of the $\lambda \neq \frac{1}{2}$ equidistants E for a surface M with a parabolic curve (the dark line). The intersection of M and E is 'inflexional' along the parabolic curve and transverse along another curve, and E has a cuspidal edge. As λ varies the cuspidal edge sweeps out the CSS.

such as the euclidean focal set, where the two focal points generically coincide only over isolated umbilics of M. The CSS situation is similar to that for the 'affine focal surface', mentioned also in §2 above: there are generically 'special' curves in the hyperbolic region of M where the two affine principal curvatures coincide. There are many results about the latter curves in [5]: for example, they meet the parabolic curve only at hyperbolic and certain elliptic cusps of Gauss, and then two special curve branches are tangent to the parabolic curve. Many other results concerning cusps of Gauss ('godrons') are in the work of R. Uribe-Vargas [24].

One of the most interesting cases of the CSS and of the associated equidistants is the purely local case which in $\mathbb{R}^2$ results from a curve with an inflexion $\mathbf{p}$ and in $\mathbb{R}^3$ from a surface with a parabolic point $\mathbf{p}$. (See [13, §5].) In either case there are parallel tangent pairs for which both points are arbitrarily close to $\mathbf{p}$. Figure 5 shows the curve case. The surface case for a generic parabolic point or a cusp of Gauss is essentially a product of this picture with a line perpendicular to the plane of the diagram: the equidistant $\lambda = \frac{1}{2}$ clings to the parabolic curve and the nearby equidistants have double contact along this curve and cuspidal edges which trace out the CSS. This, and investigations of families of surfaces for which the parabolic curve undergoes a transition, are in [11, 27].

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