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REACHABLE SETS FOR A CLASS OF CONTACT SUB-LORENTZIAN METRICS ON $\mathbb{R}^3$, AND NULL NON-SMOOTH GEODESICS

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Abstract. We compute future timelike and nonspacelike reachable sets from the origin for a class of contact sub-Lorentzian metrics on $\mathbb{R}^3$. Then we construct non-smooth (and therefore non-Hamiltonian) null geodesics for these metrics. As a consequence we deduce that the sub-Lorentzian distance from the origin is continuous at points belonging to the boundary of the reachable set.

1. Introduction and statement of the results. A *sub-Lorentzian structure* (or *metric*) on $\mathbb{R}^3$ is a couple (H, g), where H is, by definition, a rank 2 bracket generating distribution on $\mathbb{R}^3$, and g is a Lorentzian metric on H. Since our considerations are local, one can assume that all objects are defined in a suitably small neighbourhood of the origin. The simplest example of a sub-Lorentzian metric on $\mathbb{R}^3$, i.e. the Heisenberg case, was studied in papers [5], [6]. Among other things reachable sets $I^+(0, U)$, $J^+(0, U)$ from 0 were computed for this metric, where U is a normal neighbourhood of 0. As a consequence, we proved continuity of the Heisenberg sub-Lorentzian distance from the origin at points of the boundary $\partial J^+(0, U)\backslash\partial U$. Note that such a distance is, in general, upper semi-continuous. On the other hand, every time-oriented sub-Lorentzian structure on $\mathbb{R}^3$ (or rather a germ at the origin of such a structure) can be transformed to a normal form depending on two smooth functions φ and ψ of three variables x, y, z (see Theorem 3.1 below, and [4] for more details).

In this paper we study a class of contact time-oriented sub-Lorentzian structures which admit a normal form with ψ depending only on z-variable. Our aim is to generalize

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above-mentioned results obtained in the Heisenberg case to this more general class of sub-Lorentzian structures on $\mathbb{R}^3$.

To be more precise, in Section 2 we present a review of basic notions and facts concerning the sub-Lorentzian geometry. In Section 3 we compute future timelike and nonspacelike reachable sets from the origin for a class of sub-Lorentzian structures described above—Theorem 3.2. Using this, in Section 4, we construct non-smooth maximizing geodesics. These geodesics are null, unique, have exactly one corner point and, which is obvious, are contained in the boundary of the (timelike) reachable set—Theorem 4.1. Section 5 contains some final remarks. In particular we show the sub-Lorentzian distance from the origin is continuous on the set $\partial J^+(0,U)\backslash\partial U$.

2. Basic notions and facts of sub-Lorentzian geometry. All details and proofs may be found in [3], [6].

A *sub-Lorentzian manifold* is a triple (M,H,g), where M is a smooth connected manifold of dimension $n+1$, H is a smooth bracket generating distribution on M of constant rank $k+1$, and g is a Lorentzian metric on H. The couple (H,g) is called a *sub-Lorentzian metric* on M.

By a *horizontal* or *admissible* curve we mean an absolutely continuous curve $\gamma : [a,b]\to M$ with square integrable derivative such that $\dot\gamma(t)\in H_{\gamma(t)}$ a.e. on $[a,b]$. Bracket generating hypothesis guarantees that any two points in M can be joined by a horizontal curve (Rashevsky–Chow's theorem).

From now on we assume all curves, vectors and vector fields to be horizontal.

The metric g on H allows us to distinguish some classes of vectors: a vector v is called *timelike* if $g(v,v)<0$, is called *nonspacelike* if $g(v,v)\le 0$, and is *null* if $g(v,v)=0$ but $v\neq 0$.

By a *time orientation* of (M,H,g) we mean a continuous timelike vector field on M. We suppose our (M,H,g) to be time-oriented by a field X. Time orientation divides all nonspacelike vectors into two classes. Namely, a nonspacelike $v\in H_p$ is said to be *future directed* (resp. *past directed*) if $g(v,X(p))<0$ (resp. $g(v,X(p))>0$).

Throughout this paper we are going to use the following abbreviations: *f.d.* stands for 'future directed', *t.f.d.* for 'timelike future directed' and *nspc.f.d.* for 'nonspacelike future directed'.

For a nspc.f.d. curve $\gamma:[a,b]\to M$ let us define its *length* to be

$$L(\gamma)=\int_a^b |g(\dot\gamma,\dot\gamma)|^{1/2}\,dt.$$

Fix an open set $U\subset M$; a nspc.f.d. $\gamma:[a,b]\to U$ is called a *U-maximizer* if γ is longest among all nspc.f.d. curves contained in U and joining $\gamma(a)$ to $\gamma(b)$. Such a γ is called a *U-geodesic* if for each $t\in(a,b)$ (resp. $t=a$ or $t=b$) there is an $\varepsilon>0$ such that the restriction $\gamma|_{[t-\varepsilon,t+\varepsilon]}$ (resp. $\gamma|_{[a,a+\varepsilon]}$ or $\gamma|_{[b-\varepsilon,b]}$) is a U-maximizer. By a *unique U-maximizer* we mean such a nspc.f.d. curve $\gamma:[a,b]\to U$ that for each $t_1,t_2\in[a,b]$ with $t_1<t_2$ the restriction $\gamma|[t_1,t_2]$ is the unique U-maximizer between $\gamma(t_1)$ and $\gamma(t_2)$.

Let $\varphi:U\to\mathbb{R}$ be a smooth function defined on an open set $U\subset M$. By the *horizontal gradient* of the function φ we mean the vector field denoted by $\nabla_H\varphi$ and defined by the condition $(\partial_v \varphi)(p) = g(v, \nabla_H \varphi(p))$ for any $p \in U$ and $v \in H_p$. It is a simple matter to verify that if $\nabla_H \varphi$ is t.f.d. on U, then φ is decreasing along nspc.f.d. curves in U. Indeed, if $\gamma : [a,b] \to U$ is nspc.f.d. then $(\varphi(\gamma(t)))^{\dot{}} = g(\dot{\gamma}(t), \nabla_H \varphi(\gamma(t))) < 0$ a.e. on $[a,b]$. Similarly, if $\nabla_H \varphi$ is null f.d. then φ is non-increasing along nspc.f.d. curves in U (because in this case $g(\dot{\gamma}(t), \nabla_H \varphi(\gamma(t))) \leq 0$ a.e. on $[a,b]$).

Let U be an open set in M and fix a $p_0 \in U$. By $I^+(p_0, U)$ we denote the *future timelike reachable set from* p_0, which is defined to be the set of all points in U that can be reached from p_0 by a t.f.d. curve contained in U. Similarly, $J^+(p_0, U)$ is the *future nonspacelike reachable set from* p_0 which is defined as the set of all points in U that can be reached from p_0 by a nspc.f.d. curve contained in U. We will abbreviate $I^+(p_0, M)$ (resp. $J^+(p_0, M)$) to $I^+(p_0)$ (resp. to $J^+(p_0)$). Since we do not consider past reachable sets, we will simply speak about timelike (resp. nonspacelike) reachable sets.

For a general (open) set U, U-maximizers joining two given points may not exist. However, if U is a *normal neighbourhood* of a point $p_0 \in M$ (see [3]), then for every $p \in J^+(p_0, U)$ there exists a U-maximizer joining p_0 to p (as it follows from an easy adaptation of proposition 5.3 [3] for a nonspacelike case).

Let U be a normal neighbourhood of p_0. Now we can say a little more about reachable sets from p_0; it turns out that $J^+(p_0, U)$ is a closed subset with respect to U and $J^+(p_0, U) = \mathrm{cl}_U(I^+(p_0, U))$, where cl_U stands for the closure with respect to U. Note that $I^+(p_0, U)$ need not be open (see [6] and compare it with properties of reachable sets in the Lorentzian case [2], [7]).

By $\mathcal{H}$ we will denote the *geodesic Hamiltonian* associated with the sub-Lorentzian metric (H, g). Locally it can be defined as follows. Let $X_0, X_1, \ldots, X_k$ be an orthonormal frame for H defined on an open set U with X_0 timelike; then

$$\mathcal{H}(x, \lambda) = -\frac{1}{2}\langle \lambda, X_0(x) \rangle^2 + \frac{1}{2}\sum_{j=1}^{k} \langle \lambda, X_j(x) \rangle^2 \tag{2.1}$$

on $T^*M|U$. By $\overrightarrow{\mathcal{H}}$ we denote the corresponding Hamiltonian vector field on T^*M, and Φ_s stands for its flow. A curve $\gamma : [\alpha, \beta] \to U$ is called a *Hamiltonian geodesic* if it is of the form $\gamma(s) = \pi \circ \Phi_s(\lambda)$, where $\pi : T^*M \to M$ is the canonical projection; in such a case the curve $[\alpha, \beta] \ni s \to \Phi_s(\lambda)$ is called *a Hamiltonian lift* of γ. Note that every Hamiltonian geodesic is smooth and has constant causal character, i.e. it is everywhere either timelike or null. Clearly, in the Lorentzian case all geodesics are Hamiltonian.

Finally, let D_q be the set of all such covectors $\lambda \in T_p^*M$ that the curve $s \to \Phi_s(\lambda)$ is defined on $[0,1]$. The smooth mapping

$$\exp_p : D_p \to M, \quad \exp_p(\lambda) = \pi \circ \Phi_1(\lambda)$$

is called the *exponential mapping* (*with the pole at* p).

3. Reachable sets

3.1. *Normal form.* From now on we will have $M = \mathbb{R}^3$. Let (H, g) be a time-oriented sub-Lorentzian structure defined near the origin in $\mathbb{R}^3$. The following theorem holds.

Theorem 3.1 ([4]). *There are coordinates x, y, z defined near zero in which (H, g) admits an orthonormal frame in the following normal form*

$$\begin{aligned} X &= \frac{\partial}{\partial x} + y\varphi(x,y,z)\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + \frac{1}{2}y(1+\psi(x,y,z))\frac{\partial}{\partial z}, \\ Y &= \frac{\partial}{\partial y} - x\varphi(x,y,z)\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) - \frac{1}{2}x(1+\psi(x,y,z))\frac{\partial}{\partial z}, \end{aligned} \tag{3.1}$$

with the time orientation X, where φ and ψ are smooth functions defined in a neighbourhood of zero. Moreover, if H is contact then we can additionally suppose that

$$\varphi(0,0,z) = \psi(0,0,z) = \frac{\partial \psi}{\partial x}(0,0,z) = \frac{\partial \psi}{\partial y}(0,0,z) = 0. \tag{3.2}$$

In case $\varphi = \psi = 0$ we obtain the Heisenberg sub-Lorentzian metric which is described in more details in [5], [6]. In particular one knows reachable sets in this case ([6] Theorem 2.1):

$$I^+(0) = \{-x^2 + y^2 + 4|z| < 0,\ x > 0\}$$

and

$$J^+(0) = \{-x^2 + y^2 + 4|z| \le 0,\ x \ge 0\};$$

moreover,

$$I^+(0,U) = I^+(0) \cap U, \qquad J^+(0,U) = J^+(0) \cap U$$

for any normal neighbourhood U of 0.

Using Theorem 3.1 one easily derives the corollary below.

Corollary 3.1. *For any sub-Lorentzian metric defined on a neighbourhood of a given point $p \in \mathbb{R}^3$ there are exactly two null f.d. Hamiltonian geodesics starting from p.*

Proof. Indeed, these are half-lines $\{y = \pm x,\ z = 0,\ x > 0\}$ in coordinates given by Theorem 3.1 (cf. [4]). ■

3.2. *Reachable sets in case $\psi = \psi(z)$, $\psi(0) = 0$.* In this section we generalize results concerning reachable sets obtained in [6] for the Heisenberg sub-Lorentzian metric.

Consider a time-oriented sub-Lorentzian structure (H, g) defined near the origin in $\mathbb{R}^3$ by $H = \mathrm{Span}\{X, Y\}$. We suppose that X, Y is an orthonormal basis for (H, g) given in the normal form (3.1) with a time orientation X, where φ is arbitrary, ψ depends only on z, and $\psi(0) = 0$. Observe, at the beginning, that the equation for horizontal curves takes the form

$$((y^2 - x^2)\varphi + 1)\,dz - \frac{1}{2}(1+\psi)(y\,dx - x\,dy) = 0. \tag{3.3}$$

Let us note that although we do not assume (3.2), nevertheless our structure is still contact, provided V is a sufficiently small neighbourhood of 0. Indeed, if we denote by ω the left-hand side of (3.3), then $H = \ker \omega$ and

$$(\omega \wedge d\omega)(0) = dx \wedge dy \wedge dz.$$

In the sequel we suppose that V is an open ball centered at zero and of radius $r_0 > 0$, where r_0 is chosen so small that the following relations are satisfied on $\overline{V}$:

(i) ω is a contact form;
(ii) $1 + \psi \neq 0$;

(iii) $|y^2\varphi| < 1$;
(iv) $-(1+y^2\varphi)^2 + x^2y^2\varphi^2 < 0$.

The last two assumptions will soon become clear.

For a real number α let us define a function $\eta_\alpha : V \to \mathbb{R}$ by the formula

$$\eta_\alpha(x,y,z) = -x^2 + y^2 + \alpha\left|\int_0^z \frac{d\zeta}{1+\psi(\zeta)}\right|.$$

One readily verifies that

$$\nabla_H \eta_\alpha = \bigl(2x - \tfrac{1}{2}\alpha y\bigr)X + \bigl(2y - \tfrac{1}{2}\alpha x\bigr)Y$$

on $V \cap \{z > 0\}$, and

$$\nabla_H \eta_\alpha = \bigl(2x + \tfrac{1}{2}\alpha y\bigr)X + \bigl(2y + \tfrac{1}{2}\alpha x\bigr)Y$$

on $V \cap \{z < 0\}$. We define a subset Γ_α of V as

$$\Gamma_\alpha = \{\eta_\alpha < 0,\ x > 0\}.$$

Clearly $\nabla_H \eta_\alpha$ is t.f.d. for $0 \le \alpha < 4$ and is null f.d. for $\alpha = 4$ on the set $\Gamma_0 \cap \{z \neq 0\}$. Consequently, for every α, $0 \le \alpha \le 4$, η_α is non-increasing (resp. decreasing) along nspc.f.d. (resp. t.f.d.) curves contained in $\Gamma_0 \cap \{z \neq 0\}$. Moreover, since $\eta_{\alpha|\{z=0\}} = \eta_{0|\{z=0\}}$, η_α is decreasing along nspc.f.d. curves contained in $\Gamma_0 \cap \{z = 0\}$.

At the same time let us observe that $I^+(0,V) \subset \Gamma_0$ and $J^+(0,V) \subset \overline{\Gamma_0}$. To see this it is enough to look at the fields X, Y restricted to $\partial\Gamma_0$, and to note that any nspc.f.d. curve which projects onto the set $\{y = x,\ z = 0\}$ (resp. onto $\{y = -x,\ z = 0\}$) coincides with $\{y = x,\ z = 0\}$ (resp. $\{y = -x,\ z = 0\}$); this last assertion follows from (3.1).

Now we will show that

$$I^+(0,V) = \Gamma_4. \tag{3.4}$$

First let us notice that

$$I^+(0,V) \cap \{z = 0\} = \Gamma_4 \cap \{z = 0\}. \tag{3.5}$$

It is clear because the curves $y = ax$, $x > 0$, $z = 0$, $-1 < a < 1$, are t.f.d. and fill the whole of $\Gamma_4 \cap \{z = 0\}$.

To prove "$\subset$" in (3.4) take a $p \in I^+(0,V)$. There exists a t.f.d. curve $\gamma : [0,T] \to V$ with $\gamma(0) = 0$, $\gamma(T) = p$. As was already mentioned the function $t \to \eta_4(\gamma(t))$ is decreasing, so $\eta_4(p) = \eta_4(\gamma(T)) < 0$.

In order to prove the reverse inclusion fix a $p = (x_0, y_0, z_0) \in \Gamma_4$. By (3.5) it suffices to consider the case $z_0 \neq 0$. Suppose $z_0 > 0$ (the case $z_0 < 0$ is similar). Since $\eta_4(p) < 0$, by continuity there exists an α, $0 < \alpha < 4$, such that $\eta_{16/\alpha}(p) < 0$. Now, let us write out equations for the trajectory $\gamma(t) = (x(t), y(t), z(t))$ of $\nabla_H \eta_\alpha$:

$$\begin{cases} \dot{x} = 2x - \frac{1}{2}\alpha y + \frac{1}{2}\alpha y(x^2 - y^2)\varphi(x,y,z), \\ \dot{y} = -\frac{1}{2}\alpha x + 2y + \frac{1}{2}\alpha x(x^2 - y^2)\varphi(x,y,z), \\ \dot{z} = \frac{1}{4}\alpha(1+\psi(z))(x^2 - y^2), \end{cases} \tag{3.6}$$

with initial conditions

$$x(0) = x_0, \qquad y(0) = y_0, \qquad z(0) = z_0. \tag{3.7}$$

We want to solve the equation $z(t)=0$. First let us notice that the value x^2-y^2 remains positive along γ. Indeed, using (3.6) we have $(x^2-y^2)^{\dot{}}=4(x^2-y^2)$ along γ, which gives

$$x^2(t)-y^2(t)=(x_0^2-y_0^2)e^{4t}>0 \tag{3.8}$$

for any real number t. At the same time $x(t)$ decreases as t decreases because assumptions (iii) and (iv) guarantee that the horizontal gradient of the function $(x,y,z)\to x$ is timelike past directed. Thus (3.8) implies that $x(t)$ preserves positive sign and $|y(t)|$ decreases together with t. Further, the third equation in (3.6) together with (3.8) yield that $z(t)$ decreases together with t. Summing up, $\gamma(t)$ stays in Γ_4 for $t<0$.

Now, let

$$\bar{t}=\frac{1}{4}\ln\frac{\eta_{16/\alpha}(p)}{\eta_0(p)};$$

clearly $\bar{t}<0$. Since $\dot{z}=\frac{1}{4}\alpha(1+\psi(z))(x^2-y^2)>0$, the last equation in (3.6) can be rewritten as

$$\frac{4}{\alpha}\int_{z_0}^{z}\frac{d\zeta}{1+\psi(\zeta)}=\int_0^t(x^2(s)-y^2(s))ds. \tag{3.9}$$

Inserting $t=\bar{t}$ and $z=z(\bar{t})$ into (3.9), and recalling (3.8) we finally obtain

$$\int_{z_0}^{z(\bar{t})}\frac{d\zeta}{1+\psi(\zeta)}=\int_{z_0}^{0}\frac{d\zeta}{1+\psi(\zeta)},$$

which gives $z(\bar{t})=0$ (cf. assumption (ii)). It means that the trajectory, say, σ of $-\nabla_H\eta_\alpha$, $\sigma(0)=p$, joins p to a point in $\{z=0\}\cap\Gamma_4=\{z=0\}\cap I^+(0,V)$. Such a σ is obviously timelike past directed. Reversing time in σ results in $p\in I^+(0,V)$.

Now, if U is a normal neighbourhood of 0, $\overline{U}\subset V$, then the same reasoning as in ([6]) leads to the equality $I^+(0,U)=I^+(0,V)\cap U$. Finally, recall that $J^+(0,U)=\mathrm{cl}_U(I^+(0,U))$ (cf. Section 2). In this way we finish the proof of the following

THEOREM 3.2. *Let (H,g) be such a contact time-oriented sub-Lorentzian metric defined near the origin in $\mathbb{R}^3$ that there exist coordinates (x,y,z) in which (H,g) admits an orthonormal frame X,Y in the normal form (3.1) with ψ depending only on z and satisfying $\psi(0)=0$. Then, for every sufficiently small normal neighbourhood U of the origin,*

$$I^+(0,U)=\left\{-x^2+y^2+4\left|\int_0^z\frac{d\zeta}{1+\psi(\zeta)}\right|<0,\ x>0\right\}\cap U$$

and

$$J^+(0,U)=\left\{-x^2+y^2+4\left|\int_0^z\frac{d\zeta}{1+\psi(\zeta)}\right|\le 0,\ x\ge 0\right\}\cap U.$$

4. Construction of non-smooth geodesics. Again we work with the sub-Lorentzian structure given by the normal form (3.1), where ψ depends only on z, $\psi(0)=0$. We assume we are in a sufficiently small normal neighbourhood U of the zero. The aim of this section is to construct null non-smooth maximizers. The construction is based on four observations.

Let

$$\tilde{\partial}J^+(0,U)=\partial J^+(0,U)\backslash\partial U.$$

First of all observe that no nspc.f.d. curve initiating in the interior of $J^+(0,U)$ can reach the boundary $\tilde{\partial} J^+(0,U)$. This is well-known in the geometric control theory, and in our case can be deduced as follows. Take a nspc.f.d. γ defined on $[a,b]$, such that $\gamma(a) \in I^+(0,U)$. Then $\eta_4(\gamma(a)) < 0$, and since $t \to \eta_4(\gamma(t))$ is non-increasing, $\eta_4(\gamma(t)) < 0$ for every $t \in [a,b]$. Consequently, if $p \in \tilde{\partial} J^+(0,U)$, then each nspc.f.d. curve joining 0 to p must be entirely contained in $\tilde{\partial} J^+(0,U)$.

The further three facts we shall need are enclosed in lemmas below (Lemma 4.3 holds for general contact sub-Lorentzian metrics on $\mathbb{R}^3$).

Lemma 4.1. *For each* $p \in \tilde{\partial} J^+(0,U) \cap \{z \neq 0\}$

$$\dim T_p(\tilde{\partial} J^+(0,U)) \cap H_p = 1; \tag{4.1}$$

more precisely, for any such p

$$T_p(\tilde{\partial} J^+(0,U)) \cap H_p = \mathrm{Span}(\nabla_H \eta_4(p)).$$

Proof. Since $\nabla_H \eta_4$ is a null field, it is tangent to level surfaces of η_4. It is thus sufficient to show (4.1). Let $p = (x_0, y_0, z_0) \in \tilde{\partial} J^+(0,U) \cap \{z \neq 0\}$; take for instance $z_0 > 0$. Now $v \in T_p(\tilde{\partial} J^+(0,U)) \cap H_p$, $v = (v_1, v_2, v_3)$, if and only if

$$\begin{cases} -2x_0 v_1 + 2y_0 v_2 + \frac{4}{1+\psi(z_0)} v_3 = 0, \\ (1+\psi(z_0)) y_0 v_1 - (1+\psi(z_0)) x_0 v_2 - 2[(y_0^2 - x_0^2)\varphi(p) + 1] v_3 = 0. \end{cases} \tag{4.2}$$

Clearly, for any p as above, the matrix of the linear system (4.2) has rank 2. ∎

In particular, Lemma 4.1 implies that for any $p \in \tilde{\partial} J^+(0,U) \cap \{z > 0\}$ (resp. $p \in \tilde{\partial} J^+(0,U) \cap \{z < 0\}$) there exists exactly one nspc.f.d. curve passing through p and contained in $\tilde{\partial} J^+(0,U) \cap \{z > 0\}$ (resp. in $\tilde{\partial} J(0,U) \cap \{z < 0\}$), namely a null f.d. curve which is equal, up to a change of parameter, to the corresponding trajectory of the field $\nabla_H \eta_4$.

Lemma 4.2. *Let* $p \in \tilde{\partial} J^+(0,U) \cap \{z \neq 0\}$, $p = (x_0, y_0, z_0)$, *and denote by* σ *the trajectory of* $\nabla_H \eta_4$ *with initial condition* $\sigma(0) = p$. *Then the limit* $p_\infty = \lim_{t \to -\infty} \sigma(t)$ *exists, where*

$$p_\infty \in \{y = x,\ z = 0,\ x > 0\} \tag{4.3}$$

in case $z_0 > 0$ *and*

$$p_\infty \in \{y = -x,\ z = 0,\ x > 0\} \tag{4.4}$$

in case $z_0 < 0$.

Proof. The limit p_∞ exists since σ, being a smooth null curve, can be reparameterized so as to satisfy (4.7) or (4.8) below. Suppose $z_0 > 0$. Then $\sigma(t) = (x(t), y(t), z(t))$ is a solution to the system

$$\begin{cases} \dot{x} = 2x - 2y + 2y(x^2 - y^2)\varphi(x,y,z), \\ \dot{y} = -2x + 2y + 2x(x^2 - y^2)\varphi(x,y,z), \\ \dot{z} = (1+\psi(z))(x^2 - y^2). \end{cases} \tag{4.5}$$

As in the proof of Theorem 3.2 one makes sure that $x^2(t) - y^2(t) = (x_0^2 - y_0^2)e^{4t}$ and

$\dot{z}(t) > 0$ for every t. Rewrite the last equation in (4.5) in the form

$$\int_{z_0}^{z} \frac{d\zeta}{1+\psi(\zeta)} = \int_0^t (x^2(s) - y^2(s))ds = \frac{1}{4}(x_0^2 - y_0^2)(e^{4t} - 1). \tag{4.6}$$

Let $z_\infty = \lim_{t\to-\infty} z(t)$. Letting $t \to -\infty$ in (4.6) we obtain

$$\int_{z_0}^{z_\infty} \frac{d\zeta}{1+\psi(\zeta)} = -\frac{1}{4}(x_0^2 - y_0^2),$$

which gives

$$4\int_0^{z_\infty} \frac{d\zeta}{1+\psi(\zeta)} = -x_0^2 + y_0^2 + 4\int_0^{z_0} \frac{d\zeta}{1+\psi(\zeta)} = \eta_4(p) = 0.$$

This yields $z_\infty = 0$ and (4.3) is true.

In the similar way one shows (4.4). ∎

Lemma 4.3. *Every smooth null curve which contains a segment of the line $\{y = x, z = 0\}$ (resp. of the line $\{y = -x,\ z = 0\}$) coincides with (a segment of) $\{y = x,\ z = 0\}$ (resp. $\{y = -x,\ z = 0\}$). Moreover there are only two null f.d. and smooth curves starting from the origin.*

Proof. Every smooth null f.d. curve, up to a change of parameter, is a trajectory either of the field (i) $X+Y$ or (ii) $X-Y$. Let γ be a smooth null curve that contains a segment of the line $\{y = x, z = 0\}$. Then the case (i) holds and γ is a solution to the system

$$\begin{cases} \dot{x} = 1 + y(y-x)\varphi(x,y,z), \\ \dot{y} = 1 + x(y-x)\varphi(x,y,z), \\ \dot{z} = \frac{1}{2}(1+\psi(z))(y-x). \end{cases} \tag{4.7}$$

(4.7) implies that $(y-x)^{\cdot} = -(y-x)^2\varphi(x,y,z)$ along γ, and our assertion follows.

Next, suppose that γ contains a segment of $\{y = -x,\ z = 0\}$. Then (ii) holds and γ is a solution to the system

$$\begin{cases} \dot{x} = 1 + y(y+x)\varphi(x,y,z), \\ \dot{y} = -1 + x(y+x)\varphi(x,y,z), \\ \dot{z} = \frac{1}{2}(1+\psi(z))(y+x). \end{cases} \tag{4.8}$$

(4.8) gives $(\dot{y+x}) = (y+x)^2\varphi(x,y,z)$ along γ, and again our assertion follows.

The last part is now obvious. ∎

Now, for a given $p = (x_0, y_0, z_0) \in \tilde{\partial}J^+(0,U) \cap \{z \neq 0\}$, we are in a position to construct a null f.d. curve connecting 0 to p. As it follows from Lemma 4.1 and the remark coming after it, such a curve is a unique U-maximizer. It is also not smooth according to Lemma 4.3.

Suppose that p is as above and $z_0 > 0$. First we issue the trajectory σ_p of the field $-\nabla_H\eta_4$ from p. By Lemma 4.2 σ_p tends to a point p_∞ of the form $p_\infty = (a,a,0)$, $a > 0$, as t goes to ∞. Changing parameterization of σ_p, we reach p_∞ in a finite time, say T. Next, p_∞ can be joined to zero by the segment of the half-line $\{y = x,\ z = 0,\ x > 0\}$ parameterized as $t \to (a+T-t, a+T-t, 0)$, $T \le t \le T+a$. After time reversal we obtain a null f.d. curve joining 0 to p.

In the similar manner we construct the unique null U-maximizer joining 0 to p in case $z_0 < 0$. In this case $p_\infty = (a, -a, 0)$ and we use the curve $t \to (a + T - t, -a - T + t, 0)$, $T \le t \le T + a$, to reach 0.

On the other hand we have Corollary 3.1. Taking all these facts together, we obtain

Theorem 4.1. *Suppose that (H, g) is a sub-Lorentzian structure as in Theorem 3.2, and U is a sufficiently small normal neighbourhood of the origin. Then, for any $p = (x_0, y_0, z_0) \in \tilde{\partial} J^+(0, U)$, there exists a unique U-maximizer γ_p joining 0 to p. Every such γ_p is null and is contained in the boundary $\tilde{\partial} J^+(0, U)$ of the reachable set from 0. In case $z_0 = 0$, γ_p is a segment of one of the two null f.d. Hamiltonian geodesics starting from the origin. In case $z_0 \ne 0$, γ_p is not smooth with exactly one corner point.*

To give explicit example of a non-smooth maximizer consider the Heisenberg sub-Lorentzian metric (i.e. the one for which φ and ψ vanish identically). As we already know, in this case $J^+(0) = \{-x^2 + y^2 + 4|z| \le 0,\ x \ge 0\}$. If $p = (x_0, y_0, z_0) \in \partial J^+(0)$ with, say, $z_0 > 0$, then

$$\gamma_p(t) = \begin{cases} (t, t, 0) \text{ for } 0 \le t \le \frac{1}{2}(x_0 + y_0) \\ (t, x_0 + y_0 - t, \frac{1}{2}(x_0 + y_0)t - \frac{1}{4}(x_0 + y_0)^2) \text{ for } \frac{1}{2}(x_0 + y_0) < t \le x_0 \end{cases}$$

is a null non-smooth maximizer joining 0 to p.

5. Final remarks. In this section we present some remarks and corollaries. Let (H, g) be a sub-Lorentzian structure as in Theorem 3.2, and let U be a normal neighbourhood of 0.

First of all let us notice that the set $J^+(0, U)$ is not the image under the exponential mapping $\exp_0$, as it is the case in the Lorentzian geometry.

Next, let $f[U]$ be a sub-Lorentzian distance from 0. $f[U]$ is defined as follows: $f[U](p) = \sup L(\gamma)$ where the supremum is taken over all nspc.f.d. curves $\gamma : [0, T] \to U$, $\gamma(0) = 0$, $\gamma(T) = p$, in case $p \in J^+(0, U)$, and $f[U](p) = 0$ otherwise. Since, in our case, the boundary $\tilde{\partial} J^+(0, U)$ is formed by null f.d. curves, we see that

$$f[U]_{|\tilde{\partial} J^+(0,U)} = 0.$$

Moreover, the arguments similar to those in [6] show that

(i) $f[U]$ is continuous at every point of $\tilde{\partial} J^+(0, U)$;
(ii) if U_1 is such a normal neighbourhood of 0 that $U_1 \subset U$ then $f[U_1]$ and $f[U]$ coincide on $J^+(0, U_1)$.

Finally, let us observe that all nspc.f.d. curves that are contained in U can be obtained as solutions to the affine in control system

$$\dot{q} = X(q) + uY(q) \tag{5.1}$$

with a scalar input u, $|u| \le 1$. Here X, Y is an orthonormal basis for (H, g) defined on U with a time orientation X, and controls are supposed to be measurable and bounded. Thus the existence of null non-smooth geometrically optimal curves is not surprising (cf. [1]) but without knowing the boundary of reachable sets it would be difficult to determine the number of switching times along each such curve.

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