GEOMETRY AND TOPOLOGY OF CAUSTICS — CAUSTICS '06
BANACH CENTER PUBLICATIONS, VOLUME 82
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2008

REAL DEFORMATIONS AND INVARIANTS OF MAP-GERMS

J. H. RIEGER

Institut für Mathematik, Universität Halle
D-06099 Halle (Saale), Germany
E-mail: rieger@mathematik.uni-halle.de

M. A. S. RUAS and R. WIK ATIQUE

ICMC, Universidade de São Paulo
13560-970 São Carlos, SP, Brazil
E-mails: maasruas@icmc.usp.br, rwik@icmc.usp.br

Abstract. A stable deformation f^t of a real map-germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is said to be an *M-deformation* if all isolated stable (local and multi-local) singularities of its complexification $f^t_{\mathbb{C}}$ are real. A related notion is that of a good real perturbation f^t of f (studied e.g. by Mond and his coworkers) for which the homology of the image (for $n < p$) or discriminant (for $n \geq p$) of f^t coincides with that of $f^t_{\mathbb{C}}$. The class of map germs having an M-deformation is, in some sense, much larger than the one having a good real perturbation. We show that all singular map-germs of minimal corank (i.e. of corank $\max(n-p+1,1)$) and $\mathcal{A}_e$-codimension 1 have an M-deformation. More generally, there is the question whether all $\mathcal{A}$-simple singular map-germs of minimal corank have an M-deformation. The answer is "yes" for the following three dimension ranges (n,p): $n \geq p$, $p \geq 2n$ and $p = n+1$, $n \neq 4$. We describe some new techniques for obtaining these results, which lead to simpler proofs and also to new results in the dimension range $n+2 \leq p \leq 2n-1$.

1. Introduction. In the theory of singularities of analytic mappings, a stable perturbation of an unstable germ plays a similar role to the Milnor fibre in the theory of isolated hypersurface singularities. The study of special points and the topology of the discriminant of such perturbations has led to important results of singularity theory in the last 20 years. For complex germs $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, $n \geq p-1$, the discriminant (or image, for $p > n$) of a stable perturbation has the homotopy type of a wedge of $p-1-$spheres

2000 *Mathematics Subject Classification*: Primary 58K60, 58K65; Secondary 32S30.
Key words and phrases: stable invariant, deformation, germ.
The paper is in final form and no version of it will be published elsewhere.

(see [5], [14]). Over the reals, the topology of the discriminant is more complicated and few results are known so far. Therefore, a natural question is the existence of a real perturbation which shows aspects of the topology of its complexification. One problem is the existence of real deformations of map-germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, for which the maximal numbers of isolated stable singular points are simultaneously present in the discriminant, which are called *M-deformations*. There is also the notion of *good real perturbation* due to Marar and Mond, for which the homology of the discriminant of a stable perturbation of a given germ coincides with that of its complexification. This is analogous to that of an M-variety $X_{\mathbb{R}}$ in real algebraic geometry for which the sum of the Betti numbers is the same as the corresponding sum of its complexification $X_{\mathbb{C}}$.

In [17], the first and second authors show that all $\mathcal{A}$-simple singularities of map-germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, where $n \geq p$, of minimal corank (i.e. of corank $n-p+1$) have an M-deformation. The proof of this result is based on the property that all $\mathcal{A}$-simple singularities f of minimal corank can be deformed into a germ of lower codimension whose 0-stable invariants differ from those of f by at most one – one can then inductively split off real stable singular points from 0 one by one.

The case $n < p$ is not completely understood yet. Results are known for some pairs (n, p). For plane curve-germs, the classical result by A'Campo [1] and Gusein-Zade [7] shows that they always have M-deformations, i.e. deformations with δ real double-points (notice that the δ-number is the only 0-stable invariant in this case). In this case the concept of a good real perturbation coincides with that of an M-deformation. As in the case of plane curves, when $p = 2n$, the only 0-stable invariant appearing is the number of double points. In [10], Klotz *et al.* classify $\mathcal{A}$-simple germs $f : \mathbb{R}^n \to \mathbb{R}^{2n}$ and show that they have M-deformations and this is equivalent to the existence of a good real perturbation.

When $p = n+1$, we show in [18] that all $\mathcal{A}$-simple corank-1 germs from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$, where $n \neq 4$, have an M-deformation. We also show that in dimensions $(4, 5)$ the open $\mathcal{A}$-orbit in A_3 is $\mathcal{A}$-simple and consists of germs that do not have an M-deformation and also do not have a good real perturbation. This was the first example of an $\mathcal{A}$-simple singular germ of minimal corank without an M-deformation (more examples will be constructed in Section 7 of the present paper). The proofs are based on new techniques for detecting positive $\mathcal{A}$-modality.

The class of map-germs having an M-deformation is larger than the one having a good real perturbation in at least three respects. Firstly, for specific pairs of dimensions (n, p) where both notions have been studied for $\mathcal{A}$-simple singular germs of minimal corank (except for $p = 2n$ where both notions coincide, see above). For example, for the pair $(2, 3)$ there is only one series of $\mathcal{A}$-simple corank-1 mono-germs having good real perturbations [12], but all such $\mathcal{A}$-simple corank-1 germs have M-deformations. Secondly, good real perturbations are known to exist for some real representative of each $\mathcal{A}_e$-codimension 1 orbit of minimal corank map-germs from $\mathbb{C}^n$ to $\mathbb{C}^p$, see [3], [8], [9] and [15]. In the present paper we show that all representatives of each $\mathcal{A}_e$-codimension 1 orbit of minimal corank map-germs from $\mathbb{R}^n$ to $\mathbb{R}^p$ have an M-deformation. (Notice that there are real $\mathcal{A}_e$-codimension 1 orbits, e.g. the beak-to-beak map of the plane, without a good real perturbation). Finally, all "low multiplicity" germs (to be defined below) have an M-deformation. For $n = p = 2$ germs of local multiplicity three have "low multiplicity" — and, for example, $f = (x, y^3 + x^3 y)$ does not have a good real perturbation.

In this paper, we address the question of the existence of M-deformations, reviewing known results and discussing methods and techniques which have recently shown to be useful to solve them. Using these techniques, we obtain new results on the existence of M-deformations for $\mathcal{A}$-simple map-germs in dimensions $(n, 2n-1)$ and we show that all corank-1, $\mathcal{A}_e$-codimension 1 map-germs have an M-deformation (complementing the results in [3] and [9]). The new techniques avoid partial classifications of $\mathcal{A}$-simple orbits, and we hope that they allow us to complete the determination of the *M-nice dimensions* (n, p) (i.e., those (n, p) for which all $\mathcal{A}$-simple singular germs $\mathbb{R}^n \to \mathbb{R}^p$ of minimal corank have an M-deformation) in forthcoming work.

The following theorem summarizes what is presently known on M-deformations (some statements are new and some were known before, see the remark following the statement of the theorem). For standard notation on $\mathcal{A}$- and $\mathcal{K}$-equivalence of map germs (such as the $\mathcal{A}_e$-codimension $\operatorname{cod}(\mathcal{A}_e, f)$ and the local multiplicity $m_f(0)$ of a map-germ f) we refer the reader to our earlier papers on M-deformations [17, 10, 18].

THEOREM 1.1. **A.** *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be an $\mathcal{A}$-simple, singular map-germ of minimal corank. Then*

1. If $\operatorname{cod}(\mathcal{A}_e, f) = 1$ then f has an M-deformation (also note that $\operatorname{cod}(\mathcal{A}_e, f) = 1$ and singular of minimal corank implies $\mathcal{A}$-simple).

2. If $m_f(0) \leq n/(p - n + 1) + 1$ (we say that such an f has "low multiplicity") then f has an M-deformation.

3. For all pairs of dimensions (n, p) with $p \leq n$ and $p \geq 2n - 1$ (and any $n \geq 1$) and for $p = n + 1$, $n \neq 4$, all f as above have an M-deformation.

B. *Suppose $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, with $n + 1 \leq p \leq 2n - 2$, $\mathcal{A}$-simple and singular of minimal corank and not of "low multiplicity". Then for dimensions $(4, 5)$ and also for $(n, p) = (4 + 3k, 5 + 4k)$, all $k \geq 1$, the open $\mathcal{A}$-orbit in A_3 (which is not of "low multiplicity") does not have an M-deformation.*

REMARK 1.2. The statements A.1, A.2, A.3 for $p = 2n - 1$ and B for $k \geq 1$ are new. The remaining results can be found in the following articles: A.3 for $n \geq p$ is in [17], A.3 for $p = 2n$ is in [1, 7] for $n = 1$ and in [10] for $n \geq 2$ (and note that for $p > 2n$ there are no 0-stable invariants, hence all germs have a trivial M-deformation), and finally A.3 for $p = n + 1$ and B for $k = 0$ are from [18].

The contents of the remaining sections of the present paper are as follows (all singular map-germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ considered here will be of minimal corank, i.e. of corank 1 for $n < p$ or of corank $n - p + 1$ for $n \geq p$):

- 0-stable invariants and M-deformations
- All $\mathcal{A}_e$-codimension-1 mono-germs have an M-deformation
- All mono-germs in the $\mathcal{K}$-class of A_k, $k \leq n/(p - n + 1)$ have an M-deformation
- Counting arguments for detecting positive $\mathcal{A}$-modality

- New results on the existence of M-deformations for $\mathcal{A}$-simple germs: the dimensions $(n,p)=(n,2n-1)$
- $\mathcal{A}$-simple, singular germs of minimal corank without an M-deformation

2. 0-stable invariants and M-deformations. Let $f:\mathbb{K}^n,0\to\mathbb{K}^p,0$ be an $\mathcal{A}$-simple, singular map-germ of minimal corank, i.e. of corank 1 (for $p\geq n$) or $n-p+1$ (for $n>p$). In fact, for $n>p$, any $\mathcal{A}$-simple germ f of corank $n-p+1$ is $\mathcal{A}$-equivalent to the "suspension" of some $\mathcal{A}$-simple equidimensional corank-1 germ $\tilde{f}$, and the discriminants of f and $\tilde{f}$ are diffeomorphic. We will therefore consider corank-1 germs in dimensions (n,p), $p\geq n$, of the (pre-normal) form

$$f:\mathbb{K}^n,0\to\mathbb{K}^p,0,\quad (x,y)\mapsto(x,g_n(x,y),\ldots,g_p(x,y)).$$

The $\mathcal{K}$-class of such an f is A_k, $k\geq 0$, if

$$m_f(0):=\dim_{\mathbb{K}}\mathcal{O}_n/f^*\mathcal{M}_p=k+1.$$

The $\mathcal{K}$-type of an s-germ $f=(f_1,\ldots,f_s):\mathbb{K}^n,S\to\mathbb{K}^p,q$, where

$$S=\{(x,y_1),\ldots,(x,y_s)\}\mapsto q=f_1(x,y_1)=\ldots=f_s(x,y_s),$$

whose ith component germ f_i is of type A_{k_i}, will be denoted by $A_{k(s,m)}=A_{(k_1,\ldots,k_s)}$, where

$$\sum_{i=1}^{s}k_i=m=b_1k_1+\sum_{i=2}^{r}b_ik_{l_i}.$$

Here $k(s,m)$ denotes a "partition" of m with s summands $k_i\geq 0$, and for $r=1$ the sum on the RHS is zero (the RHS says that k_1 appears b_1 times in the partition, and so on). For equidimensional germs we only consider singular component germs of type $A_{\geq 1}$, but for $p>n$ we also have to consider immersive A_0 components. Notice that for corank-1 maps f we can embed the space of s-fold points in the source in $\mathbb{K}^{n+s-1}$ (with coordinates $(x,y_1,\ldots,y_s)$). On $A_{k(s,m)}\subset\mathbb{K}^{n+s-1}$ there acts a subgroup $S_{k(s,m)}$, of order $c_{k(s,m)}=\prod_{i=1}^{r}(b_i!)$, of the permutation group S_s that preserves the partition $k(s,m)$ of m.

The closures of the sets $A_{k(s,m)}\subset\mathbb{K}^{n+s-1}$ in the source of f are the 0-sets of maps $G_{k(s,m)}$, which are defined as follows. For $D:=p-n+1$ and $\ell\geq m+s$ let $G_{k(s,m)}:=\varphi\circ j^{\ell}_s f$ be the composition

$$\mathbb{K}^{n+s-1},0\xrightarrow{j^{\ell}_s f}J^{\ell}_s\xrightarrow{\varphi}\mathbb{K}^{(m+s-1)D},0,$$

where $\varphi^{-1}(0)$ defines the closure of the set of $A_{k(s,m)}$ points in the space J^{ℓ}_s of ℓ-jets of s-germs (including the diagonal). This set is a smooth submanifold. Let $f^s:\bar{A}_{k(s,m)}=G^{-1}_{k(s,m)}(0)\to\mathbb{K}^p$ be the restriction of the map $\mathbb{K}^{n+s-1}\to\mathbb{K}^{n-1+sD}$, given by

$$(x,y_1,\ldots,y_s)\mapsto(X,Y_{n,1},\ldots,Y_{n,s},\ldots,Y_{p,1},\ldots,Y_{p,s}),\quad X=x,\ Y_{i,j}:=g_i(x,y_j),$$

to the main diagonal $Y_{i,1}=\ldots=Y_{i,s}$, $i=n,\ldots,p$, in $\mathbb{K}^{n-1+sD}$, which is isomorphic to $\mathbb{K}^p$. The map f^s (not to be confused with a deformation f^t of f with parameter t) has degree $c_{k(s,m)}$ and its image is the $\bar{A}_{k(s,m)}$ stratum in the target of f. (Also notice that $f^s=f\circ\pi$, where π is the restriction of the projection $(x,y_1,\ldots,y_s)\mapsto(x,y_1)$ to

$\bar{A}_{k(s,m)} \subset \mathbb{K}^{n+s-1}$, which has degree $c_{k(s,m)}/b_1$, and the restriction of f to the image of π has degree b_1.)

By a result of Mather, f is stable if and only if, for all $k(s,m)$, $j_s^\ell f$ is transverse to the closure of $A_{k(s,m)}$ (or, equivalently, the maps $G_{k(s,m)}$ are submersions). The relation between the sets $A_{k(s,m)}$ and the $\mathcal{A}$-finiteness of f has been studied in [11] for $p > n$. A map-germ f is $\mathcal{A}$-finite if and only if the maps $G_{k(s,m)}$ are $\mathcal{K}$-finite (i.e. they are submersive outside 0, equivalently the set-germs $G_{k(s,m)}^{-1}(0)$ are complete intersections with isolated singularity at 0.) In fact, for equidimensional germs f the $\mathcal{K}$-finiteness of the $G_{k(s,m)}$ for partitions $k(s,m)$ with summands $k_i \in \{1,2\}$ is already sufficient for the $\mathcal{A}$-finiteness of f (see [16]), and for $p > n$ one can restrict to "partitions" $k(s,m)$ consisting of sequences of 0s (see [11]).

The sets $A_{k(s,m)} \subset \mathbb{K}^{n+s-1}$ are of dimension 0 for $n+s-1 = (m+s-1)D$. For $D = 1$ (i.e. $p = n$) this condition holds for all partitions $k(s,n)$ of n, for $D > 1$ ($p > n$) it holds for all "partitions" $k(s,m)$ (possibly with 0 summands) with $s = iD + 1 - n$, $m = i(1-D)+n$, with $i \in [n/D, n/(D-1)]$ a positive integer. For each $k(s,m)$ satisfying the above condition we define the following "0-stable invariant" of f:

$$r_{k(s,m)}(f) := c_{k(s,m)}^{-1} m_{G_{k(s,m)}}(0).$$

Let $\Delta(f)$ denote the image (for $n < p$) or the discriminant (for $n \geq p$) of f. For $\mathbb{K} = \mathbb{C}$ any stable deformation f^t of f has precisely $r_{k(s,m)}(f)$ points of type $A_{k(s,m)}$ in $\Delta(f^t)$. For $\mathbb{K} = \mathbb{R}$ the number $r_{k(s,m)}^{\mathbb{R}}(f^t)$ of real $A_{k(s,m)}$ points in $\Delta(f^t)$ depends on the choice of deformation f^t, obviously $r_{k(s,m)}^{\mathbb{R}}(f^t) \leq r_{k(s,m)}(f)$. We say that a real deformation f^t is an *M-deformation* of f if $r_{k(s,m)}^{\mathbb{R}}(f^t) = r_{k(s,m)}(f)$ for all $k(s,m)$ satisfying the condition $n+s-1 = (m+s-1)D$. Notice that for $D > n+1$ (i.e. $p > 2n$) there are no 0-stable invariants, so that any deformation is trivially an M-deformation.

In the study of M-deformations we have to relate real deformations f^t of f to the corresponding deformations $G_{k(s,m)}^t$ of the maps $G_{k(s,m)}$. The maps $G_{k(s,m)}$ are defined by iteration, see below (some general properties of the $G_{k(s,m)}$ for corank-1 maps in dimensions (n,p), for $p \geq n$, and their relation to alternative ways – see e.g. [6, 11, 13] – of defining the sets $\bar{A}_{k(s,m)}$ are described in [16]). Replacing the coordinates $(x, y_1, \ldots, y_s) \in \mathbb{K}^{n+s-1}$ (in the space of s-fold points in the source) by $(x, y_1, \epsilon_2, \ldots, \epsilon_s) := (x, y_1, y_2 - y_1, \ldots, y_s - y_{s-1})$, and setting for $r = n, n+1, \ldots, p$

$$g_{r,1}^{(i)} := \partial^i g_r / \partial y^i, \quad i \geq 1,$$

we define by iteration for $j = 1, \ldots, s-1$

$$g_{r,j+1}^{(0)} := \sum_{\alpha \geq k_j+1} g_{r,j}^{(\alpha)} \epsilon_{j+1}^{\alpha-k_j-1}/\alpha!, \qquad g_{r,j+1}^{(i)} := \partial^i g_{r,j+1}^{(0)} / \partial \epsilon_{j+1}^i, \quad i \geq 1.$$

The component functions $G_1, \ldots, G_{(m+s-1)D}$ of $G_{k(s,m)}$ are then given for $r = n, n+1, \ldots, p$ by

$$g_{r,1}^{(1)}, \ldots, g_{r,1}^{(k_1)}; \;\; g_{r,j}^{(0)}, \ldots, g_{r,j}^{(k_j)} \;\; (j = 2, \ldots, s)$$

(here $\{g_{r,1}^{(1)}, \ldots, g_{r,1}^{(0)}\}$ denotes the empty set).

For $\mathcal{A}_e$-codimension-1 germs, and for low multiplicity germs (i.e. of multiplicity at most $(n/D)+1$), we only have to consider $s = 1, 2$, but we require explicit expressions for the component functions of $G_{k(s,m)}$. The case $s=1$ is trivial, hence consider $s=2$ and take coordinates $(x,y,\epsilon):=(x,y_1,y_2-y_1)\in\mathbb{K}^{n+1}$. The maps $G_{k(2,m)}$ associated with a corank-1 map $f=(x,g_n,\ldots,g_p)$ have $(m+1)(p-n+1)=n+1$ component functions, namely $m+1$ component functions $G_{i,1},\ldots,G_{i,m+1}$ for each g_i.

Each term $\varphi(x)y^r$ in g_i generates a term $\varphi(x)h_a(y,\epsilon)$ in $G_{i,a}$, and for $k(2,m)=(k-1-l,l)$ with $l=0,\ldots,[(k-1)/2]$ the h_a are given by:

$$h_a = r^{\underline{a}}y^{r-a},\quad a=1,2,\ldots,k-1-l,$$

$$h_{k-l+b} = \sum_{j\geq k-l+b}\frac{r^{\underline{j}}(j-k+l)^{\underline{b}}}{j!}y^{r-j}\epsilon^{j-k+l-b},\quad b=0,1,\ldots,l$$

with $n^{\underline{0}}=1$ and $n^{\underline{s}}=n(n-1)\ldots(n-s+1)$, as usual.

Notice that for $r=k+1$ and $k+2$

$$h_k = l!\big((k+1)y+(l+1)\epsilon\big)$$

and

$$h_k = l!\Big(\frac{1}{2}(k+1)(k+2)y^2+(k+2)(l+1)y\epsilon+\frac{1}{2}(l+1)(l+2)\epsilon^2\Big),$$

respectively. The coordinate change $(y,\epsilon)\mapsto(y,\epsilon-\frac{k+1}{l+1}y)$ reduces these equations to

$$(l+1)!\epsilon\quad\text{and}\quad\frac{l!(k+1)(k-l)}{2(l+1)}y^2+\epsilon L(y,\epsilon),$$

where ϵL can be removed using the first equation. Notice that the coefficients of ϵ and y^2 are positive for all relevant k,l. Also notice that for $r=k$ we get $h_k=l!>0$.

EXAMPLE 2.1. Let us apply this to the following 1-parameter deformation

$$f^t=(x,y^{k+1}+x_1y+\ldots+x_{k-1}y^{k-1},\\ y^{k+2}+x_ky+\ldots+x_{2k-2}y^{k-1}+t\cdot y^k,g_{n+2},\ldots,g_p),$$

where $k=(n+1)/(p-n+1)$ and $g_i=x_{i_1}y+\ldots+x_{i_k}y^k$ such that all the x_i appearing in f^t are distinct. Then f^t induces maps $G^t_{(k-1-l,l)}\sim_{\mathcal{K}}(x,\epsilon,\frac{l!(k+1)(k-l)}{2(l+1)}y^2+l!t)$. Hence, for all $l=0,\ldots,[(k-1)/2]$, we get for $t<0$ two real $A_{(k-1-l,l)}$ points in the source (for $k-1$ even this gives one real $A_{((k-1)/2,(k-1)/2)}$ point in the target). We will see in the next section that f_0 is a pre-normal form for the $\mathcal{A}$-orbit of lowest codimension in A_k (which has $\mathcal{A}_e$-codimension at least 1). Notice that any deformation of f^0 that preserves its $\mathcal{K}$-type A_k induces $\mathcal{K}$-trivial deformations of the maps $G^0_{(k-1-l,l)}$. It follows that the germs in A_k of minimal $\mathcal{A}$-codimension have an M-deformation.

3. All $\mathcal{A}_e$-codimension-1 mono-germs have an M-deformation. In this section we will show that all singular $\mathcal{A}_e$-codimension-1 germs $\mathbb{R}^n,0\to\mathbb{R}^p,0$ of minimal corank have an M-deformation. This result is best possible, because for dimensions $(4,5)$ there is a corank-1 map-germ of $\mathcal{A}_e$-codimension 2 that fails to have an M-deformation [18]. For $n\geq p$ the claim follows from the results in [17] (because all the rank $p-1$ germs of positive $\mathcal{A}$-modality have $\mathcal{A}_e$-codimension greater than one).

Hence consider corank-1 germs in dimensions (n, p), $p > n$. Set $D := p - n + 1 \geq 2$ (for $p > n$) and consider a germ f in A_k of $\mathcal{A}_e$-codimension at most 1, then we have the conditions:

$$(C_s) \qquad m + s \leq k + 1, \quad kD \leq n + 1, \quad (m + s - 1)D = n + s - 1.$$

The first and third condition are from [16] ($r_{k(s,m)}(f) = 0$ for f of multiplicity $k + 1 < m + s$, and for $(m + s - 1)D = n + s - 1$ the $A_{k(s,m)}$ points in the source of an $\mathcal{A}$-finite f are isolated). And the second condition says that the codimension of the $\mathcal{K}$-orbit of A_k is at most $n + 1$ – notice that $\mathcal{K}$-orbits of codimension greater than $n + 1$ cannot contain germs f of $\mathcal{A}_e$-codimension less than two.

For $s = 1$ we deduce from (C_1) that $m = k$, $mD = n$ and the dimension range $(n, p) = (rm, (m + 1)r - 1)$, for all $m \geq 1, r \geq 2$. Notice that $m \leq k$ (first condition), and putting $k = m+c$ for some non-negative integer c we get $kD = mD+cD = n+cD \geq n+2c$. This implies $c = 0$ (for otherwise $kD > n + 1$, contradicting the second condition).

For $s = 2$ we obtain in the same way from (C_2) that $k = m + 1$, $(m + 1)D = n + 1$ and $(n, p) = (r(m + 1) - 1, r(m + 2) - 2)$, for all $m \geq 0, r \geq 2$.

For $s \geq 3$ the conditions in (C_s) cannot hold simultaneously – for $k \geq m+2$ we obtain $kD \geq (m + 2)D \geq n + 2 > n + 1$, contradicting the second condition.

Hence we only have to consider $s = 1$ and 2 further, and notice that the dimension ranges (n, p) are disjoint for $s = 1$ and $s = 2$. Thus for the (n, p) listed in case $s = 1$ only the invariant $r_{(m)}(f)$ can be positive for f of $\mathcal{A}_e$-codimension 1, for the (n, p) listed in case $s = 2$ only the invariants $r_{(m-l,l)}(f)$, $l = 0, \ldots, [m/2]$, can be positive for f of $\mathcal{A}_e$-codimension 1 and for all other (n, p), $p > n$, the $\mathcal{A}_e$-codimension-1 germs have no positive 0-stable invariants (in fact, one also easily checks that for $p > 2n$ there are no germs f of $\mathcal{A}_e$-codimension 1).

For $s = 1$ (and the associated $m, k, (n, p)$ specified above) we can take the pre-normal form

$$f = (x, y^{k+1} + P_1(x)y + \ldots + P_{k-1}(x)y^{k-1}, g_{n+1}, \ldots, g_p),$$

with $g_i = P_{i_1}(x)y + \ldots + P_{i_k}(x)y^k + y^{k+2}R_i$, $R_i \in C_n$. Then $G_{(k)} \sim_{\mathcal{K}} (y, P(x))$ with $P = (P_1, \ldots, P_{n-1}) : \mathbb{R}^{n-1}, 0 \to \mathbb{R}^{n-1}, 0$ the map $x \mapsto P(x)$ with component functions the above P_i. A deformation P^t of P lifts to a deformation f^t of f, hence we can argue as in [17] and conclude that the $\mathcal{K}$-simplicity of $G_{(k)}$ (and hence of P) is equivalent to the $\mathcal{A}$-simplicity of f, and that the $\mathcal{K}$-simplicity of $G_{(k)}$ implies that we can split off A_k points from f one by one. (A word on terminology: we have a map $f^t \mapsto G^t_{(k)}$ and in the above situation we may say that $G^t_{(k)} \sim_{\mathcal{K}} (y, P^t)$ "lifts" over this map.) Hence an $\mathcal{A}$-simple f (and $\mathcal{A}_e$-codimension-1 corank-1 germs are $\mathcal{A}$-simple) has an M-deformation. But we can be more precise: if f (as in the above pre-normal form) has $\mathcal{A}_e$-codimension 1 then the associated P (and hence $G_{(k)}$) must be of type A_1, and hence $r_{(k)}(f) = 2$. To see this, note that A_k has (for $k = m$ and $p - n + 1 = r$) codimension $rm = n$. The open $\mathcal{A}$-orbit in A_k has $\mathcal{A}$-codimension n and is stable, and the associated P (and $G_{(k)}$) is a diffeomorphism. The P associated with an $\mathcal{A}$-orbit in A_k of $\mathcal{A}_e$-codimension 1 must be of type A_1. Notice: the f with such a P must be unstable, and any f with a P of type A_2 or worse must be of $\mathcal{A}_e$-codimension greater than 1, because we can deform such a

P to A_1. Such a deformation of P lifts to a deformation of the associated f and hence decreases the $r_{(k)}$ number of f – this means that the $\mathcal{A}_e$-codimension of f decreases and is still positive, hence it must be greater than 1.

For $s = 2$ we refer to the pre-normal form in the example 2.1 in Section 2.

Hence we can conclude

PROPOSITION 3.1. *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, $p > n$, be a map-germ of corank 1 and $\mathcal{A}_e$-codimension 1. Then f has an M-deformation and furthermore the invariants of f satisfy the following conditions: for*

$(n,p) = (rm, r(m+1)-1)$, $m \geq 1$, $r \geq 2$: $m_f(0) = m+1$ and $r_{(m)}(f) = 2$, and for

$(n,p) = (r(m+1)-1, r(m+2)-2)$, $m \geq 0$, $r \geq 2$: $m_f(0) = m+2$, $r_{(m-l,l)}(f) = 2$ for $0 \leq l < m/2$ and $r_{(m/2,m/2)}(f) = 1$ for even m.

For dimensions (n,p) outside the ranges listed above there are no 0-stable invariants $r_{k(s,m)}(f)$ or these are equal to zero for f as above (or there are no $\mathcal{A}_e$-codimension 1 germs, as for $p > 2n$).

REMARK 3.2. (i) For general (n,p) the above result cannot be improved: for $(n,p) = (4,5)$ there is a corank-1 germ of $\mathcal{A}_e$-codimension 2 without an M-deformation [18].

(ii) It follows from the results in [17] that all singular map-germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, $n \geq p$, of minimal corank $n-p+1$ and $\mathcal{A}_e$-codimension 1 have an M-deformation. Furthermore, the 0-stable invariants of such f are all zero for $m_f(0) \leq p$, otherwise $r_{(p)}(f) = 2$ (for f with $m_f(0) = p+1$) and $r_{(p)}(f) = 2$, $r_{(p-l,l)}(f) = 2$ $(0 < l < p)$ and $r_{(p/2,p/2)}(f) = 1$, provided p is even, (for f with $m_f(0) = p+2$) are the only non zero 0-stable invariants.

4. Germs of low multiplicity and M-deformations. The argument for $s = 1$ (where $k(p-n+1) = n$) before the statement of Proposition 3.1, which implies that for $k = \frac{n}{p-n+1}$ the deformations of $G_{(k)}$ lift to deformations of the corresponding maps f in the $\mathcal{K}$-orbit A_k, yields the following

PROPOSITION 4.1. *All $\mathcal{A}$-simple corank-1 germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, $p \geq n$, in A_k, with $k \leq \frac{n}{p-n+1}$ have an M-deformation.*

Notice that for $k < \frac{n}{p-n+1}$ or n not a multiple of $p-n+1$ there are no 0-stable invariants. Otherwise, for $k = \frac{n}{p-n+1}$, $r_{(k)}(f)$ is the only 0-stable invariant of f. In the latter case if f is $\mathcal{A}$-simple then $G_{(k)}$ is $\mathcal{K}$-simple (using the liftability of deformations of $G_{(k)}$ as in [17]). And the $\mathcal{K}$-simple real equidimensional germs $G_{(k)}$ can be deformed to give $m_{G_{(k)}}(0) = r_{(k)}(f)$ real points in some fibre near 0 (by noticing that these real points can be split off 0 one by one [17]).

EXAMPLE 4.2. The proposition implies the existence of M-deformations in the following three examples (the first two recover results from [17] and [18], and the third is a new result). (i) For $p = n$ all $\mathcal{A}$-simple germs f in $A_{(n)}$ have an M-deformation with $r_{(n)}(f)$ real A_n-points. (ii) For $p = n+1$ and n even all $\mathcal{A}$-simple germs f in $A_{(n/2)}$ have an M-deformation with $r_{(n/2)}(f)$ real $A_{n/2}$-points. (iii) And for $p = 2n-1$ all $\mathcal{A}$-simple germs f in $A_{(1)}$ have an M-deformation with $r_{(1)}(f)$ real A_1-points (i.e. cross-caps).

For map-germs f in A_k, with $k > \frac{n}{p-n+1}$, of higher multiplicity the invariants $r_{k(s,m)}(f)$ with $s > 1$ can be positive. For example, equidimensional map-germs have 0-stable invariants $r_{k(s,n)}(f)$, for all partitions $k(s,n)$ of n with s summands. But $r_{k(s,n)}(f) = 0$ for germs f of multiplicity $m_f(0) < n+s$, and for $n \geq 3$ any $\mathcal{A}$-simple germ f has multiplicity at most $n+2$. Hence the only non-zero invariants $r_{k(s,n)}(f)$ are those with $s = 1, 2$.

For germs in dimensions (n,p) with $p > n$ the following 0-stable invariants $r_{k(s,m)}(f)$ can appear. Set $D := p - n + 1 \geq 2$ (for $p > n$), the conditions

$$(m + s - 1)D = n + s - 1, \quad s \geq 1, \quad m \geq 0,$$

imply that for any positive integer i with $\frac{n}{D} \leq i \leq \frac{n}{D-1}$ there are 0-stable invariants $r_{k(s,m)}(f)$ with

$$s = iD + 1 - n \quad \text{and} \quad m = i(1 - D) + n.$$

Furthermore, $r_{k(s,m)}(f) = 0$ for all germs f of multiplicity $m_f(0) < m + s = i + 1$. Notice that if $i = n/D$ is an integer then $s = 1$ and $m = n/D$ – this corresponds to the low multiplicity case $k = n/D$ in Proposition 4.1 above. In general, as i increases from $i_0 := \lceil n/D \rceil$ to $i_m := \lfloor n/(D-1) \rfloor$ the corresponding s (respectively m) increase in steps of D (respectively decrease in steps of $D - 1$). If $i = n/(D-1)$ is an integer then $s = p/(D-1)$ and $m = 0$, and the corresponding invariant $r_{k(s,m)}(f)$ (corresponding to $p/(D-1)$-fold points) is zero for all f of multiplicity less than $m + s = p/(D-1)$. The higher i (with $\frac{n}{D} < i \leq \frac{n}{D-1}$) often require multiplicities $m_f(0) \geq i + 1$ that $\mathcal{A}$-simple germs f cannot have, this can be shown with the techniques described in the next section.

5. Detecting positive $\mathcal{A}$-modality. In order to show that the $\mathcal{A}$-simple germs in dimensions (n,p) have an M-deformation we need criteria for detecting positive $\mathcal{A}$-modality. For $n \geq p$ this is done by partially classifying $\mathcal{A}$-orbits of a certain multiplicity (see [17]), using the techniques described in Section 5.1 below. In [18] a new counting technique for detecting positive $\mathcal{A}$-modality of corank-1 germs in dimensions $(n, n+1)$ is used, which is more efficient and avoids partial classifications. In Section 5.2 we describe this counting technique for corank-1 germs in dimensions (n,p), $p \geq n$, and illustrate it by some examples. Example 5.1 describes the technique, in its most basic form, for certain germs in dimensions $(n, n+1)$. Example 5.2 shows how the arguments in [17] in the equidimensional case $n = p$ (based on partial classifications) can be replaced by much shorter counting arguments. Example 5.3 illustrates the counting technique for a germ with integer weights ≥ 2 (in practise, we so far – in ruling out germs of positive modality – only had to consider germs with one weight equal to 1, but the counting technique is not limited to this case). Finally, for large $p - n$, modality often appears already at filtration 0 – Example 5.4 shows how modality at filtration 0 can be detected with the counting technique for two germs in dimensions $(n, 2n)$ (namely for the "bordering germs" (B.1) and (B.2) in the classification of $\mathcal{A}$-simple orbits in dimensions $(n, 2n)$, $n \geq 2$, in [10]). In fact, the counting argument is applied in dimensions $(2, 4)$ and the positive $\mathcal{A}$-modality for the corresponding germs in dimensions $(n, 2n)$, $n \geq 3$, can then be deduced using a simple trick.

5.1. *Necessary and sufficient conditions for positive $\mathcal{A}$-modality.* Here the strategy is the following. First, we rule out certain $\mathcal{K}$-orbits that cannot contain any $\mathcal{A}$-simple orbit. This uses the necessary and sufficient condition for an $\mathcal{A}$-orbit to be open in its $\mathcal{K}$-orbit in [19]. Second, for the remaining $\mathcal{K}$-orbits one carries out a partial classification (ruling out orbits of positive $\mathcal{A}$-modality using Mather's lemma).

5.2. *Sufficient conditions: the counting technique.* Here we use certain sufficient conditions for positive $\mathcal{A}$-modality that are based on counting arguments in $\mathcal{A}$-tangent spaces filtered by weighted degrees.

For map-germs $f : \mathbb{K}^n, 0 \to \mathbb{K}^p, 0, p \geq n$, of corank 1 we use source coordinates $(x, y) = (x_1, \ldots, x_{n-1}, y)$ such that $f(x, y) = (x, g_n(x, y), \ldots, g_p(x, y))$, and target coordinates $(X_1, \ldots, X_p)$. In describing elements of $T\mathcal{A} \cdot f$ we sometimes use the shorter notation e_i for the target and source vector fields $\partial/\partial X_i$ and $\partial/\partial x_i$ (where $x_n = y$).

Let $f : \mathbb{K}^n, 0 \to \mathbb{K}^p, 0$ be weighted-homogeneous with weights $w = (w_1, \ldots, w_n)$ and weighted degrees $\delta = (\delta_1, \ldots, \delta_p)$, and let $(\theta_n)_s$, $(\theta_p)_s$ and $(\theta_f)_s$ denote the filtration (more precisely, weighted degree) s parts of the modules of source-, target-vector fields and vector fields along f, respectively. (Recall that the monomial vector fields $u \cdot e_i \in \theta_n$, $v \cdot e_i \in \theta_p$ and $m \cdot e_i \in \theta_f$, with exponent vectors α_u, α_v and α_m, have filtration s if $\langle \alpha_u, w\rangle - w_i$, $\langle \alpha_v, \delta\rangle - \delta_i$ and $\langle \alpha_m, w\rangle - \delta_i$ are equal to s.) For integers $s \geq 0$ consider the linear maps

$$\gamma_s(f) : (\theta_n)_s \oplus (\theta_p)_s \to (\theta_f)_s, \quad (a, b) \mapsto tf(a) - wf(b),$$

of $\mathbb{K}$-vector spaces. Notice that (E_w, E_δ), where $E_w := \sum_i w_i x_i \cdot e_i$ (where $x_n = y$) and $E_\delta := \sum_j \delta_j X_j \cdot e_j$ are the Euler vector fields in source and target, is in the kernel of $\gamma_0(f)$, and we call $e(f) := \gamma_0(f)(E_w, E_\delta)$ the Euler relation. From the C_p-module generated by $e(f)$ we get further relations in the higher filtration-s parts of $T\mathcal{A} \cdot f$ (notice: $e(f) = 0$ implies $\gamma_s(f)(f^* X^\alpha \cdot E_w, X^\alpha E_\delta) = 0$ for target monomials X^α with $\langle \alpha, \delta\rangle = s$).

Now let H_n^r and H_{n-1}^r denote the vector spaces (possibly 0-dimensional) generated by monomials in x, y and in x, respectively, of weighted degree r, then we can write

$$(\theta_f)_s = H_n^{s+\delta_1} \oplus \ldots \oplus H_n^{s+\delta_p}, \quad H_n^r = \bigoplus_{i=0}^{[r/w_n]} y^i H_{n-1}^{r-iw_n}.$$

And (since $w_i = \delta_i$, $i < n$)

$$(\theta_n)_s = H_n^{s+\delta_1} \oplus \ldots \oplus H_n^{s+\delta_{n-1}} \oplus H_n^{s+w_n}.$$

Let K_p^r denote the vector space generated by target monomials X^α of weighted degree r and set $B^r := \{X^\alpha \in K_p^r : \alpha_n + \ldots + \alpha_p > 0\}$, then we have the decomposition

$$(\theta_p)_s = K_p^{s+\delta_1} \oplus \ldots \oplus K_p^{s+\delta_p}, \quad K_p^r \cong H_{n-1}^r \oplus B^r$$

(remark: substitute $x_1, \ldots, x_{n-1}$ in H_{n-1}^r by $X_1, \ldots, X_{n-1}$). And we can decompose

$$B^r \cong \bigoplus_{0 < \sum_{i=n}^p \alpha_i \delta_i \leq r} \prod_{i=n}^p X_i^{\alpha_i} H_{n-1}^{r - \sum_{i=n}^p \alpha_i \delta_i}.$$

Finally, note that we have a vector space $e(f) K_p^s$ of relations at filtration s from the Euler relation.

The counting arguments involve comparing the source and target dimensions of the maps $\gamma_s(f)$ (and taking into account the relations coming from $e(f)K_p^s$). More precisely, we have that

$$c_s := \dim(\theta_f)_s - \dim(\theta_n)_s \oplus (\theta_p)_s + \dim\ e(f)K_p^s$$

is a lower bound for the dimension of the cokernel of the linear map $\gamma_s(f)$, and hence for the dimension of a complete transversal of f at filtration s (see [2] for the definition of a complete transversal). Using the above decompositions for the vector spaces appearing on the RHS of the formula for c_s we can cancel most of the direct summands (isomorphic to $H^i := H^i_{n-1}$ for some i) and count only the dimensions of the few remaining terms. Notice that we can cancel all the H^i summands between the first $n-1$ components of $(\theta_f)_s$ and of $(\theta_n)_s$ – hence it is enough to count the H^i in the $\partial/\partial X_j$-components, $j \geq n$, of $(\theta_f)_s$ and in the $\partial/\partial y$-component of $(\theta_n)_s$ (and, of course, in $(\theta_p)_s$ and $e(f)K_p^s$).

The most basic form of the counting argument is then as follows. Suppose the kernel of $\gamma_0(f)$ is 1-dimensional and therefore generated by (E_w, E_δ), and that $c_s \geq 2$, for some $s > 0$, then $\dim(\theta_f)_s/\mathrm{im}\gamma_s(f) \geq 2$ (so that the filtration-s complete transversal of f is at least 2-dimensional). Then $g := f + c_1m_1 + c_2m_2 + \ldots$ (where the monomial vectors m_i generate the filtration-s complete transversal) is at least uni-modal, because the filtration-0 part of any generator $tg(a) - wg(b)$, $(a, b) \in \theta_n \oplus \theta_p$, relating the m_i must lie in the kernel of $\gamma_0(f)$, which is 1-dimensional.

In order to apply this counting argument we typically find a germ f as above – that is, f weighted homogeneous and with essentially unique weights (i.e. unique up to common multiplicative factor) such that $\gamma_0(f)$ has 1-dimensional kernel generated by (E_w, E_δ) – which is "best possible" within its $\mathcal{K}$-orbit in the sense that $\gamma_0(f)$ is surjective (so that the $F^0\mathcal{A}$-orbit of f is open in its $F^0\mathcal{K}$-orbit, where F^s denotes the filtration on $C_n^{\times p}$ induced by the weights and weighted degrees of f).

Example 5.1. The following example from [18] uses the counting argument in its most basic form (described above). The map-germ $f : \mathbb{K}^{2l+1}, 0 \to \mathbb{K}^{2l+2}, 0$, where $l \geq 2$, defined by

$$f(x, y) = (x, y^{l+2} + x_1y + \ldots + x_ly^l, x_{l+1}y + \ldots + x_{2l-1}y^{l-1} + x_{2l}y^{l+1} + y^{l+3})$$

is weighted homogeneous for the weights $w = (l+1, l, \ldots, 2, l+2, l+1, \ldots, 4, 2, 1)$ with weighted degree $\delta = (l+1, l, \ldots, 2, l+2, l+1, \ldots, 4, 2, l+2, l+3)$. The map $\gamma_0(f)$ is surjective with 1-dimensional kernel generated by (E_w, E_δ), and cancelling H^i direct summands we find that $c_1 = \dim H^4 - 3\dim H^0 = 5 - 3$ (for $l > 2$) and $c_1 = \dim H^4 - 2\dim H^0 = 4 - 2$ (for $l = 2$). Hence we have at least one modulus at filtration 1.

Example 5.2. Detecting orbits of positive $\mathcal{A}$-modality in the case $n = p$. In the argument in [17] that all equidimensional corank-1 germs have an M-deformation there are three lemmas, whose proofs occupy about 6 pages: Lemma 4.5 that shows that there are no $\mathcal{A}$-simple orbits of multiplicity $\geq n+3$ (for any $n \geq 3$) and Lemmas 4.9 and 4.10 which yield pre-normal forms for the $\mathcal{A}$-simple germs of multiplicity $n+2$. Using the counting argument we can give short alternative proofs of these lemmas (see below).

First, consider corank-1 germs of multiplicity $n+3$, $n \geq 3$. The map-germ $f = (x, y^{n+3} + x_1y + \ldots + x_{n-1}y^{n-1})$ is weighted homogeneous for the weights $n+2$,

$n+1,\ldots,4,1$ with weighted degrees $n+2,n+1,\ldots,4,n+3$. And $\gamma_0(f)$ is surjective with 1-dimensional kernel generated by (E_w,E_δ). Then (always after cancelling H^i summands and omitting 0-dimensional H^is) $c_1=\dim H^4-\dim H^0=1-1$ and in fact $\gamma_1(f)$ is an isomorphism. Next $c_2=\dim H^4+\dim H^5-\dim H^0=1+1-1$ (for any $n\geq 3$), while $c_3=\dim H^5+\dim H^6-\dim H^0=1+1-1$ (for $n\geq 4$) and $c_3=\dim H^5=1$ (for $n=3$). It follows that (for any $n\geq 3$) we either have $\dim(\theta_f)_2/\operatorname{im}\gamma_2(f)>1$ (then there is a modulus at filtration 2, as in the basic argument above) or else $\dim(\theta_f)_2/\operatorname{im}\gamma_2(f)=1$ so that $\gamma_2(f)$ has 0 kernel and the complete filtration-2 transversal is generated by a single monomial vector m. In the latter case the complete filtration-3 transversals of f and $f+\lambda m$, $\lambda\in\mathbb{K}$, coincide (because $\ker\gamma_1(f)=0$) and $\dim(\theta_f)_2/\operatorname{im}\gamma_2(f)\geq 1$ then implies that there is a modulus at filtration 3. Notice that this argument implies Lemma 4.5 in [17], because the $\mathcal{A}$-orbits in A_{n+2} of lowest codimension have filtration-0 part equivalent to f and contain all $\mathcal{A}$-orbits in A_k, $k>n+2$, in their closures.

Next, consider germs of multiplicity $n+2$. In Lemma 4.9 of [17] map-germs f_j have been defined whose filtration-0 parts are given by

$$g_j := (x, y^{n+2}+x_1y+\ldots+x_{j-1}y^{j-1}+x_jy^{j+1}+\ldots+x_{n-1}y^n),$$

where $j=1,\ldots,n$, and it was shown that all f_j with $j\leq n-2$ are non-simple. It is easy to see that any f_j with $j<n-2$ can be deformed to f_{n-2}, hence it is enough to apply the counting technique to g_{n-2} (alternatively, we could apply the counting technique to g_j for all $j\leq n-2$). With the obvious weights and weighted degrees for $f:=g_{n-2}$ we have that $\gamma_0(f)$ is surjective with 1-dimensional kernel generated by (E_w,E_δ). Furthermore we find that $c_1=\dim H^5-\dim H^0=2-1$ and $c_2=\dim H^2+\dim H^6-\dim H^0=1+3-1$, hence (by the same reasoning as above) we see that any germ f_{n-2} with filtration-0 part equal to $f=g_{n-2}$ is non-simple.

Finally, Lemma 4.10 in [17] gives explicit normal forms for any $\mathcal{A}$-simple germ of type f_n and f_{n-1} – but the invariants (and the existence of an M-deformation) of germs of type f_n and (for odd n) of type f_{n-1} are determined by their filtration-0 parts g_n and g_{n-1}. For showing the existence of an M-deformation the explicit normal forms are therefore not required in these cases. So it remains to consider f_{n-1} for even n – in Lemma 4.10 this is done by showing that the series $\tilde{f}_k=g_{n-1}+(0,x_{n-1}^ky^{n-1})$ gives the complete classification of $\mathcal{A}$-orbits with filtration-0 part g_{n-1}. Notice that only one invariant depends on k, namely $r_{(n/2,n/2)}(\tilde{f}_k)=k$, the others are already determined by the initial part g_{n-1}. One can replace the proof of Lemma 4.10 by a shorter argument analogous to the proof of part (iv) of Proposition 4.6 in [18]. Briefly, this argument uses the maps $G_{(n/2,n/2)}$ associated with g_{n-1} and $\tilde{f}_k$ (for the former $G_{(n/2,n/2)}\sim_{\mathcal{K}}(x,\epsilon_2,0)$ and for the latter $G_{(n/2,n/2)}\sim_{\mathcal{K}}(x,\epsilon_2,y^{2k})$) together with the possible outcomes of the counting argument (without actually carrying it out). The conclusion then is that either we get the series $\tilde{f}_k$ (whose members have M-deformations, and by the results of [17] this is actually the case) or else the $\mathcal{A}$-orbits over the filtration-0 orbit of g_{n-1} are all non-simple.

Example 5.3. Next, we consider an example of a corank-1 germ whose integer weights are all greater than 1 (allowing rational weights we could always assume that $w(y)=1$, of course). Such examples are difficult to find in the existing classifications of germs of $\mathcal{A}$-modality zero or one, so we construct an example in dimensions $(2, 2)$ (of modality at least two). The map-germ $f = (x, x^3y + y^5)$ is weighted homogeneous for the weights $w = (4, 3)$ and weighted degrees $\delta = (4, 15)$. Then $\gamma_0(f)$ is surjective with 1-dimensional kernel generated by (E_w, E_δ) and $c_1 = 0$, in fact $\gamma_1(f)$ is an isomorphism. Finally, $c_2 = \dim H^8 = 1$ and $c_3 = \dim H^{12} = 1$, so we get at least one modulus at filtration 2 or 3. In fact, calculating the "best possible" $F^3\mathcal{A}$-orbit over f we obtain $(x, x^3y+y^5+x^2y^3+ay^6)$, where a is a modulus. This confirms the "counting result".

Example 5.4. In the first three examples, positive $\mathcal{A}$-modality appears in positive filtration – this is typical for germs in dimensions (n, p) with small $p - n$. For large $p - n$ we frequently have positive $\mathcal{A}$-modality already in filtration zero. In the classification of $\mathcal{A}$-simple germs in dimensions $(n, 2n)$, $n \geq 2$, there are non-simple bordering germs (B.1) to (B.8) whose orbits contain together all non-simple germs in their closures, and these bordering germs often have moduli in filtration 0 (see [10]). The simplest examples are the map-germs (B.1) and (B.2) from [10] for $n = 2$, given by

$$f_a = (x, y^2, x^2y + y^5, xy^5 + ay^7) \quad \text{and} \quad (x, y^2, x^3y + xy^3, y^5 + ax^4y).$$

The first is quasi-homogeneous for the weights $w = (2, 1)$ with weighted degrees $\delta = (2, 2, 5, 7)$. We find that $c_0 = \dim H^6 + 2\dim H^4 - 2\dim H^0 = 1$, hence we have at least one modulus at filtration zero. The second is homogeneous with degrees $\delta = (1, 2, 4, 5)$. We find that $c_0 = \dim H^4 + \dim H^3 + \dim H^2 - \dim H^1 - \dim H^0 = 1$, hence we have at least one modulus at filtration zero. The above map-germs in dimensions $(2, 4)$ "correspond" to map-germs in dimensions $(n, 2n)$ for any $n \geq 3$, and these have positive $\mathcal{A}$-modality as well. For example, f_a corresponds for each $n \geq 3$ to exactly one map-germ

$$\tilde{f}_a = (f_a(x_1, y), x_2, \dots, x_{n-1}, x_2y, \dots, x_{n-1}y)$$

(in the sense that the normal spaces $N\mathcal{A}_e \cdot f_a$ and $N\mathcal{A}_e \cdot \tilde{f}_a$ are isomorphic, see [10]). That the parameter a is not only a modulus for f_a but also for $\tilde{f}_a$ can be checked by simply giving the new extra variables a sufficiently high weight. Assigning x_i, $i \geq 2$, the weight 8, the new extra component functions have weighted degrees 8 and 9. And the filtration-0 vector fields in source and target that involve the new source and target variables contribute nothing to the θ_{f_a} subspace of $\theta_{\tilde{f}_a}$.

6. M-deformations of $\mathcal{A}$-simple germs from $\mathbb{R}^n$ to $\mathbb{R}^{2n-1}$, $n \geq 3$. In Example 4.2 (iii), Section 4 we have already seen that all $\mathcal{A}$-simple multiplicity 2 germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n-1}, 0$ of corank 1, have an M-deformation. In this section we deal with germs of multiplicity greater than 2.

Lemma 6.1. *Let $f : \mathbb{K}^n, 0 \to \mathbb{K}^{2n-1}, 0$, $n \geq 3$, of corank 1.*

(i) If $m_f(0) = 3$, the $\mathcal{A}$-orbit of f has the pre-normal form

$$\tilde{f} = (x, x_1y + q_1(x, y), \dots, x_{n-2}y + q_{n-2}(x, y), ax_{n-1}y + q_{n-1}(x, y), q_n(x, y)), \quad q_i \in \mathcal{M}_n^3,$$

or it lies in the closure of some $\mathcal{A}$-orbit of the type

$f' = (x, x_1y + q_1(x,y), \ldots, x_{n-3}y + q_{n-3}(x,y), q_{n-2}(x,y), q_{n-1}(x,y), q_n(x,y)),\ \ q_i \in \mathcal{M}_n^3$ *and the latter are non-simple.*

(ii) If $m_f(0) \geq 4$, *then* f *is non-simple.*

Proof. The normal forms in (i) follow easily using the complete transversal method. Applying Mather's Lemma one can see that f' is non-simple.

To prove (ii) we now use the weighted version of the complete transversal: given weights $w = (2,\ldots,2,1)$, and weighted degree $\delta = (2,\ldots,2,3,\ldots,3,4)$, we find that the "best possible" $F^0\mathcal{A}$-orbit of multiplicity 4 is $f_0 = (x, x_1y, \ldots, x_{n-1}y, x_1y^2 + y^4)$.

When $n = 2$ it follows from Mond's classification ([13]) that $(x_1, x_1y, x_1y^2 + y^4)$ is adjacent to $(x_1, x_1y, x_1y^2 + y^4 + ay^6)$, which is non-simple ($a$ is a modulus). The general case follows similarly since f_0 is adjacent to $f_a = (x, x_1y, \ldots, x_{n-1}y, x_1y^2 + y^4 + ay^6)$, which is also non-simple. ■

Remark 6.2. Here is a quick alternative proof of (ii). Consider the weighted homogeneous germ $f = (x_1, x_1y + ay^3, x_1y^2 + by^4)$ which for weights $w = (2,1)$ and weighted degrees $\delta = (2,3,4)$ lies in the is "best possible" $F^0\mathcal{A}$-orbit. By the counting argument $c_0 = \dim H^2 = 1$, hence we have at least one modulus at filtration 0 (in fact, in Sec. 4.2.2 of [13] it is shown by heavy calculations that the unimodal germ $(x_1, x_1y + y^3, x_1y^2 + cy^4)$ yields the lowest codimension orbit over the 3-jet (x_1, x_1y+y^3, x_1y^2)). Now we argue as in Example 5.4 that the moduli of f are also moduli of $F := (f, x_2, x_2y, \ldots, x_{n-1}, x_{n-1}y)$ (just give the x_i, $i > 1$, weight > 4), and finally notice that the orbit of $(x, x_1y, \ldots, x_{n-1}y, x_1y^2+y^4)$ lies in the closure of that of F. Also note that for $a = 0$ the $\tilde{f}$ in part (i) of the lemma lies in the closure of the orbit of F, hence $\tilde{f}$ is non-simple for $a = 0$.

Proposition 6.3. *All $\mathcal{A}$-simple germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^{2n-1}, 0$ of corank 1 have an M-deformation.*

Proof. It follows from the classification in [13] that all $\mathcal{A}$-simple corank-1 germs in dimensions $(2,3)$ have an M-deformation (see the concluding remarks in [17]). Then, we can consider $n \geq 3$. For pairs $(n, 2n-1)$, $n \geq 3$, it follows from section 4 that $r_{(1)}(f)$ is the only 0-stable invariant.

As we saw in Lemma 6.1, if f is $\mathcal{A}$-simple and $m_f(0) > 2$, then $m_f(0) = 3$ and its prenormal form is $\tilde{f}$. The map-germ $G_{(1)}$ is defined by $G_{(1)}(x,y) = (x_1 + \frac{\partial q_1}{\partial y}(x,y), \ldots, ax_{n-1} + \frac{\partial q_{n-1}}{\partial y}(x,y), \frac{\partial q_n}{\partial y}(x,y))$.

Suppose $a \neq 0$. Then $G_{(1)}$ is an A_k singularity and so we can apply Lemma 4.8 of [17] to find a germ G' of lower $\mathcal{K}$-codimension, to which $G_{(1)}$ is $\mathcal{K}$-adjacent to, such that $m_{G_{(1)}}(0) - m_{G'}(0) \leq 1$. Write $G'(x,y) = (G'_1(x,y), \ldots, G'_n(x,y))$. Of course G' is also an A_l singularity and we can suppose that $\operatorname{rank}(G'_1, \ldots, G'_{n-1}) = n-1$. Write $q_n(x,y) = q_n(0,y) + x_1\tilde{q}_1(x,y) + \cdots + x_{n-1}\tilde{q}_{n-1}(x,y)$, for some $\tilde{q}_i$, $i = 1, \ldots, n-1$, and define $q'_n(0,y) = \int G'_n(0,y)dy$ and $g(x,y) = (x, x_1y + q_1(x,y), \ldots, ax_{n-1}y + q_{n-1}(x,y), q'_n(0,y) + x_1\tilde{q}_1(x,y) + \cdots + x_{n-1}\tilde{q}_{n-1}(x,y))$. The germ g is $\mathcal{A}$-finite and adjacent to f. Also $r_{(1)}(f) - r_{(1)}(g) \leq 1$. Therefore we can split off A_1-points (i.e. cross-caps) from f one by one. ■

Notice that $\tilde{f}$ also has an M-deformation for $a = 0$ (by the previous remark such a $\tilde{f}$ is non-simple). In this case $G_{(1)}$ has corank 2 and is therefore $\mathcal{K}$-equivalent to a germ

$F(x,y) = (x_1,\ldots,x_{n-2},F_{n-1}(x,y),F_n(x,y))$. Let (H,h) be the $\mathcal{K}$-equivalence taking $G_{(1)}$ to F. Applying Lemma 4.8 of [17] to F we obtain a germ F' with multiplicity at least one less than the multiplicity of F. Define $G' = (H,h)^{-1}.F'$, then

$$G'(x,y) = (x_1+\lambda_1(x,y),\ldots,x_{n-2}+\lambda_{n-2}(x,y),G'_{n-1}(x,y),G'_n(x,y)).$$

We have

$$\int (x_i+\lambda_i(x,y))dy = x_iy+q'_i(x,y), \quad i=1,\ldots,n-2,$$

$$\int G'_i(x,y)dy = q'_i(x,y), \quad i=n-1,n.$$

Define

$$g'(x,y) = (x,x_1y+q'_1(x,y),\ldots,x_{n-2}y+q'_{n-2}(x,y),q'_{n-1}(x,y),q'_n(x,y)).$$

Again g' is $\mathcal{A}$-finite, adjacent to f and $r_{(1)}(g') < r_{(1)}(f)$.

7. $\mathcal{A}$-simple singular germs of minimal corank without an M-deformation. In [18] we show that all corank-1, $\mathcal{A}$-simple germs of multiplicity $l+1$ from $\mathbb{R}^{2l}$ into $\mathbb{R}^{2l+1}$ have an M-deformation. We also show the existence of an $\mathcal{A}$-simple map-germ from $\mathbb{R}^4$ into $\mathbb{R}^5$ of multiplicity 4 that does not have M-deformation, namely

$$g = (x_1,x_2,x_3,y^4+x_1y,y^6+y^7+x_2y+x_3y^2).$$

In what follows we generalize this result. That is, we construct unfoldings of the germ g, which are $\mathcal{A}$-simple germs from $\mathbb{R}^{4+3k}$ into $\mathbb{R}^{5+4k}$ of multiplicity 4 that do not have M-deformations.

PROPOSITION 7.1. *The map-germ $f:\mathbb{R}^{4+3k}\to\mathbb{R}^{5+4k}$ $(k\ge 1)$ given by*

$$f = (x_1,\ldots,x_{3(k+1)},y^4+x_1y+x_6y^2,y^6+x_2y+x_3y^2,x_4y+x_5y^2+x_6y^3,\ldots,$$
$$x_{3k+1}y+x_{3k+2}y^2+x_{3k+3}y^3),$$

is 6-determined, is $\mathcal{A}$-simple and has $\mathcal{A}_e$-codimension two. The 0-stable invariant of f is $r_{(1,0,0)}(f)=3$. The bifurcation set of f in the parameter (u,v)-plane (the unfolding is given in the proof below) is the union of the non-positive part of the u-axis and the cuspidal curve $8u^3+27v^2=0$ and divides the parameter plane into 3 connected regions. The numbers of real $A_{(1,0,0)}$-points in these regions are 2, 2 and 0. Hence there is no M-deformation for f.

Proof. The argument follows the proof of Prop. 4.8 in [18], and is in fact slightly simpler (the remark following this proof describes the differences between the $k=0$ and the $k\ge 1$ cases). Using the unipotent group $\mathcal{A}_1$ one checks that f is 7-determined. But since the 7-weighted transversal is empty we can conclude that actually f is 6-determined. The $\mathcal{A}_e$-versal unfolding of f is given by

$$F = (u,v,x,y^4+x_1y+x_6y^2+uy^2,y^6+x_2y+x_3y^2+vy^3,x_4y+x_5y^2+x_6y^3,\ldots,$$
$$x_{3k+1}y+x_{3k+2}y^2+x_{3k+3}y^3).$$

The bifurcation set $B(f)$ is the union of sets $B_{k(s,m)}$ consisting of (u,v) for which $G_{k(s,m)}$ is non-submersive for some $(x,y,\epsilon_2,\ldots,\epsilon_s)$ (corresponding to the s-tuple of source points

(x,y), $(x,y+\epsilon_2),\ldots,(x,y+\epsilon_2+\ldots+\epsilon_s))$. The map $G_{k(s,m)}$ defines the closure of the $A_{k(s,m)}$ set in the source, and we only have to consider "partitions" $k(s,m)$ with $m+s\le m_f(0)=4$ and $k(s,m)\neq(0)$ (see [16]). Arguing as in the proof of Prop. 4.8 in [18], we see that it is sufficient to determine the three sets $B_{(0,0)}$, $B_{(0,0,0)}$ and $B_{(0,0,0,0)}$ (essentially we obtain all relevant $B_{k(s,m)}$ by cutting these three sets with suitable parts of the diagonal). One can check easily that $G_{(0,0)}$ and $G_{(0,0,0)}$ are submersions for all (u,v) and that $B_{(0,0,0,0)}=\{8u^3+27v^2=0\}\cup\{v=0,u\le 0\}$ (see [18]). Now one calculates the numbers of real $A_{(1,0,0)}$-points of $F_{(u,v)}$ at a point (u,v) in each of the three regions in the complement of $\mathbb{R}^2\setminus B(f)$. ∎

Remark 7.2. Define $f:\mathbb{R}^{4+3k}\to\mathbb{R}^{5+4k}$ for $k\ge 1$ as above and for $k=0$ by $f:=(x_1,x_2,x_3,y^4+x_1y,y^6+x_2y+x_3y^2)$ (notice that then $g=f+y^7\cdot e_5$ for $k=0$). There are two differences between the $k=0$ and the $k\ge 1$ cases: (1) for $k=0$ there is an additional 0-stable invariant, namely $r_{(2)}(f)$. And (2) for $k=0$ the map-germ f fails to be 6-determined, but it is topologically 6-determined and furthermore its negative versal unfolding, given by

$$F=(u,v,x,y^4+x_1y+uy^2,y^6+x_2y+x_3y^2+vy^3),$$

is topologically versal (this is shown in the proof of Prop. 4.8 in [18] using a result of Damon [4]). Hence the bifurcation set in the base of a versal unfolding of g is homeomorphic to that in the base of a negative versal unfolding of f. And the latter has for $k=0$ the v-axis as an extra component – $F_{(u,v)}$ has two A_2-points for $v<0$ and none for $v>0$ (recall that for $k\ge 1$ there are no isolated stable A_2 points, so it makes geometric sense that the v-axis is not a component of the bifurcation set for $k\ge 1$).

Acknowledgements. This work was supported by a joint CAPES-DAAD project between our institutes.

References

[1] N. A'Campo, *Le groupe de monodromie du déploiement des singularités isolées de courbes planes*. I, Math. Ann. 213 (1975), 1–32.

[2] J. W. Bruce, N. P. Kirk and A. A. du Plessis, *Complete transversals and the classification of singularities*, Nonlinearity 10 (1997), 253–275.

[3] T. Cooper, D. Mond and R. Wik Atique, *Vanishing topology of codimension 1 multi-germs over $\mathbb{R}$ and $\mathbb{C}$*, Compositio Math. 131 (2002), 121–160.

[4] J. Damon, *Topological triviality and versality of subgroups of $\mathcal{A}$ and $\mathcal{K}$: II. Sufficient conditions and applications*, Nonlinearity 5 (1992), 373–412.

[5] J. Damon and D. Mond, *$\mathcal{A}$-codimension and the vanishing topology of discriminants*, Invent. Math. 106 (1991), 217–242.

[6] V. V. Goryunov, *Semi-simplicial resolutions and homology of images and discriminants of mappings*, Proc. London Math. Soc. 70 (1995), 363–385.

[7] S. M. Gusein-Zade, *Dynkin diagrams of singularities of functions of two variables*, Funct. Anal. Appl. 8 (1974), 295–300.

[8] K. Houston, *A note on good real perturbations of singularities*, Math. Proc. Cambridge Phil. Soc. 132 (2002), 301–310.

[9] K. Houston, *On the topology of augmentations and concatenations of singularities*, Manuscripta Math. 117 (2005), 383–405.

[10] C. Klotz, O. Pop and J.H. Rieger, *Real double-points of deformations of $\mathcal{A}$-simple map-germs from $\mathbb{R}^n$ to $\mathbb{R}^{2n}$*, Math. Proc. Cambridge Phil. Soc. 142 (2007), 341–363.

[11] W. L. Marar and D. Mond, *Multiple point schemes for corank-1 maps*, J. London Math. Soc. 39 (1989), 553–567.

[12] W. L. Marar and D. Mond, *Real map-germs with good perturbations*, Topology 35 (1996), 157–165.

[13] D. Mond, *On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$*, Proc. London Math. Soc. (3) 50 (1985), 333–369.

[14] D. Mond, *Vanishing cycles for analytic maps*, in: Singularity Theory and Applications (Warwick, 1989), Part I, Lecture Notes in Math. 1462, Springer, New York, 1991, 221–234.

[15] D. Mond, *How good are real pictures?*, in Algebraic Geometry and Singularities (La Rábida, 1991), Progr. Math. 134, Birkhäuser, Basel, 1996, 259–276.

[16] J. H. Rieger, *Invariants of equidimensional corank-1 maps*, in: Geometry and Topology of Caustics — Caustics '02, Banach Center Publ. 62, 2004, 239–248.

[17] J. H. Rieger and M. A. S. Ruas, *M-deformations of $\mathcal{A}$-simple Σ^{n-p+1}-germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, $n \geq p$*, Math. Proc. Cambridge Phil. Soc. 139 (2005), 333–349.

[18] J. H. Rieger, M. A. S. Ruas and R. Wik Atique, *M-deformations of $\mathcal{A}$-simple germs from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$*, Math. Proc. Cambridge Phil. Soc. 144 (2008), 181–195.

[19] M. A. S. Ruas, *C^l-Determinação finita e aplicações*, Thesis, ICMC, USP, São Carlos, 1983.