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COBORDISMS OF FOLD MAPS OF $2k+2$-MANIFOLDS INTO $\mathbb{R}^{3k+2}$

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Abstract. We calculate the group of cobordisms of k-codimensional maps into Euclidean space with no singularities more complicated than fold for a $2k+2$-dimensional source manifold in both oriented and unoriented cases.

1. Concept. Given a descending set of singularities $\{\eta\} \cup \tau = \tau'$ with a top singularity η and a fixed codimension k of the mappings involved, we can consider the classifying spaces $X_{\tau'}$ and X_τ, the homotopy groups of which are isomorphic to the cobordism groups of mappings into Euclidean spaces with all singularities in τ' and τ, respectively. It is known ([8]) that there is a fibration

$$X_{\tau'} \xrightarrow{X_\tau} \Gamma T\xi^\eta,$$

where ξ^η is the bundle associated to the universal G_η-bundle via the representation of G_η in the image. Hence we have a long exact sequence

$$\begin{array}{ccccccccc} \cdots \to & \pi_{N+1}(\Gamma T\xi^\eta) & \to \pi_N(X_\tau) \to & \pi_N(X_{\tau'}) & \xrightarrow{T} & \pi_N(\Gamma T\xi^\eta) & \to \pi_{N-1}(X_\tau) \to \cdots \\ & \approx\downarrow & & & & \approx\downarrow & \\ & \pi_N^S(T\xi^\eta) & & & & \pi_N^S(T\xi^\eta) & \end{array} \tag{1}$$

where the mapping $\pi_N(X_{\tau'}) \xrightarrow{T} \pi_N(\Gamma T\xi^\eta)$ assigns to every map $f\colon M \to \mathbb{R}^N$ the map that classifies the immersion $f\colon \eta(f) \to \mathbb{R}^N$ with the ξ^η normal structure added.

This will be applied to $\tau' = \{\Sigma^{1,1,0}\} \cup \{\Sigma^{1,0}, \Sigma^0\}$. We will denote the cobordism groups of τ'-maps (τ-maps) of n-dimensional manifolds $Cob^{1,1}(n, \mathbb{R}^{n+k})$ (respectively, $Cob^{1,0}(n, \mathbb{R}^{n+k})$); the corresponding classifying space will be called $X_{1,1}$ (respectively, $X_{1,0}$). The classifying space for maps without restrictions on the singularities will be

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called $X_\infty = \Omega^\infty M(S)O(k+\infty)$. We have

$$\begin{array}{c} \dots \to \pi_{N+1}(X_{1,1}) \xrightarrow{T} \pi^S_{N+1}(T\xi^{1,1}) \to \pi_N(X_{1,0}) \to \pi_N(X_{1,1}) \xrightarrow{T} \pi^S_N(T\xi^{1,1}) \to \dots \\ \approx \\ Cob^{1,0}(N-k, \mathbb{R}^N) \end{array} \tag{2}$$

and after calculating the groups and mappings involved we will be able to describe the groups $Cob^{1,0}(2k+2, \mathbb{R}^{3k+2})$.

2. Calculations

LEMMA 1. *Given a vector bundle ξ of rank $n \geq 1$ over a connected base B,*

$$\pi_n(T\xi) = \begin{cases} \mathbb{Z} & \text{if } \xi \text{ is orientable,} \\ \mathbb{Z}_2 & \text{if } \xi \text{ is not orientable,} \end{cases} \tag{3}$$

and the mapping $[f] \to [f \cap B\xi]$ is an isomorphism. Here $[f \cap B\xi]$ denotes the number of intersection points of $B\xi$ and the image of f, taken with sign if ξ is oriented, after a small perturbation to make f transversal to $B\xi$.

Proof. Since $T\xi$ is $n-1$-connected, $\pi_n(T\xi) \approx H_n(T\xi;\mathbb{Z}) \approx H^n(T\xi;\mathbb{Z})$. This group is generated by the Thom class, which is a free generator if ξ is orientable and has order 2 if ξ is not orientable. The mapping $[f] \to [f \cap B\xi]$ is the evaluation of the Thom class on the image of $[f]$ under the Hurewicz homomorphism, hence it is an isomorphism.

LEMMA 2. *Let ξ be an arbitrary vector bundle of rank $n \geq 3$ over a connected base B. Then the mapping*

$$C\colon \pi_{n+1}(T\xi) \ni [f] \mapsto [f \cap B] \in \begin{cases} \{[\gamma] \in \mathfrak{N}_1(B)\colon \gamma^*\xi \text{ is orientable}\} \approx \ker w_1(\xi) \leq H_1(B;\mathbb{Z}_2) \\ \qquad\qquad \text{if } \xi \text{ is not orientable,} \\ \Omega_1(B) \approx H_1(B;\mathbb{Z}) \quad \text{if } \xi \text{ is orientable.} \end{cases} \tag{4}$$

is onto and its kernel is either isomorphic to $\mathbb{Z}_2$ or trivial, depending on whether $w_2(\xi)$ vanishes or not.

Proof. We can kill $\pi_n(T\xi)$ by constructing a fibration $K(\pi_n(T\xi), n-1) \to X \to T\xi$ with an n-connected X in the usual way (see e.g. [6]), by pulling back the fibration $K(H^n(T\xi), n-1) \to PK(H^n(T\xi), n) \to K(H^n(T\xi), n)$ with the classifying map of the generator of $H^n(T\xi)$, the Thom class U. This way $\pi_{n+1}(T\xi) \approx \pi_{n+1}(X) \approx H_{n+1}(X) \approx H^{n+1}(X)$ can be calculated from the Serre spectral sequence. Indeed, due to dimensional constraints the only potentially non-zero differentials influencing $H^{n+1}(X)$ are transgressions $H^{n-1+j}(K(\pi_n(T\xi), n-1)) \to H^{n+j}(T\xi) = U \cup H^j(B)$ for $j = 0,1,2$. The transgression for $j=0$ is an isomorphism by design. For $j=1$, we have $H^n(K(\mathbb{Z}, n-1)) = 0$ in the oriented case and $H^n(K(\mathbb{Z}_2, n-1)) = \langle Sq^1 \rangle$ if ξ is not oriented; in this latter case, the transgression sends this element to $Sq^1(U) = U \cup w_1(\xi)$ since transgressions commute with Steenrod operations. Finally, for $j=2$ we have $H^{n+1}(K(\pi_n(T\xi), n-1) = \langle Sq^2 \rangle$ in both cases, and the transgression sends this element to $Sq^2 U = U \cup w_2(\xi)$. Therefore, the E_∞^{**} term in dimension $n+1$ will contain only $E_\infty^{n+1,0}$, which can be identified with $U \cup (H^1(B)/\langle w_1(\xi)\rangle)$ and $E_\infty^{0,n+1}$, which is 0 when $w_2(\xi) \neq 0$ and $\mathbb{Z}_2$ otherwise. The statement of the lemma follows immediately.

Corollary 3. *If the bundle ξ is associated to the universal G-bundle via the representation $\lambda\colon G \to Iso(\mathbb{R}^n)$, $n > 1$, then the mapping C from Lemma 2 is an isomorphism if and only if $\lambda_*(\pi_1(G,e)) = \pi_1(SO(n),e)$, that is, the image of the fundamental group of G under λ contains a non-contractible loop in $SO(n)$.*

Proof. We will check the criterion of Lemma 2. G-bundles over $\mathbb{S}^2$ correspond in a one-to-one fashion to their gluing maps, which can be identified with the elements of $\pi_1(G)$. For any $[s] \in \pi_2(BG)$ the pullback of the universal G-bundle on $\mathbb{S}^2$ by s has the gluing map $\partial[s] \in \pi_1(G)$ with ∂ being an isomorphism taken from the homotopy long exact sequence of the universal G-bundle. Indeed, when we lift $[s]\colon \mathbb{S}^2 \setminus \{point\} \to BG$ as a homotopy of a trivial mapping of a circle to EG, we will get the mapping $\partial[s]$ on the boundary (in the fibre over the excised point), and it gives the difference between the trivialisations of the pullback bundle over the two hemispheres, i.e. the gluing map. Since ξ is associated to the universal bundle via λ, the gluing map for the pullback of ξ will be the image of the gluing map for the universal bundle under λ and hence the degree of $s^*\xi$ can be regarded as $\lambda_*(\partial[s]) \in \pi_1(O(n))$. But as $[s]$ takes all values from $\pi_2(BG)$, $\partial[s]$ takes all values from $\pi_1(G)$, so we will obtain a pulled-back bundle of odd degree if and only if the whole image $\lambda_*(\pi_1(G))$ contains the generator of $\pi_1(O(n)) = \pi_1(SO(n))$, and that completes the proof. We will also need to know what the symmetry group of the singularity $\Sigma^{1,1}$ looks like. $G_{\Sigma^{1,1}}$ in the unoriented case has the homotopy type of the group $\mathbb{Z}_2 \times O(k)$ and the representations λ_1 (in the source) and λ_2 (in the image) are of the form

$$\lambda_1(\varepsilon, A) = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & \varepsilon A \end{pmatrix} \quad \text{and} \quad \lambda_2(\varepsilon, A) = \begin{pmatrix} \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & \varepsilon A & 0 \\ 0 & 0 & 0 & 0 & \varepsilon A \end{pmatrix}$$

in an appropriate local coordinate system. Hence the symmetry group in the oriented case, that is, the subgroup of $\mathbb{Z}_2 \times O(k)$ forming the kernel of the orientation mapping of the virtual normal bundle $(\varepsilon, A) \mapsto \det \varepsilon A = \varepsilon^k \det A$, is $\mathbb{Z}_2 \times SO(k)$ for even k and $\{1\} \times SO(k) \cup \{-1\} \times (-1)SO(k)$ for odd k. This implies that the connected components of $G_{\Sigma^{1,1}}$ are in all cases separated by the projections $pr_1(\varepsilon, A) = \varepsilon$ and $pr_2(\varepsilon, A) = \det \varepsilon A$. When interpreted as projections from $\pi_1(BG_{\Sigma^{1,1}})$, pr_1 is returning the orientability of the kernel bundle over $\Sigma^{1,1}$ (on every loop in $\Sigma^{1,1}(f)$), and pr_2 is returning the orientability of the virtual normal bundle of f over $\Sigma^{1,1}$ (on every loop in $\Sigma^{1,1}(f)$). We will express these projections in terms of the Stiefel-Whitney characteristic classes of the underlying manifold M defined by the Pontryagin-Thom construction from a representative mapping f of $[f] \in \pi_{3k+3}(T\xi)$ (and hence additive notation will be used for convenience). pr_2 is obviously evaluating $\overline{w_1} \cdot Tp_{\Sigma^{1,1}} = \overline{w_{k+1}^2 w_1} + \overline{w_{k+2} w_k w_1}$ on the fundamental class of M, $[M]$, since $\overline{w_1}$ gives the orientability of all restrictions of the virtual normal bundle, in particular, the restriction to the dual of $Tp_{\Sigma^{1,1}}$, represented by $\Sigma^{1,1}(f)$. As to pr_1, a direct adaptation of [5] (using [1]) gives us the characteristic number $\overline{w_{k+3}w_k} + \overline{w_{k+2}w_{k+1}}$.

2.1. *Calculating* $\pi_{3k+2}(X_{1,0})$. The long exact sequence (2) gives a short exact sequence

$$0 \to \operatorname{coker} T \to \pi_{3k+2}(X_{1,0}) \to \ker T^{1,0}_{3k+2} \to 0$$

where $\ker T^{1,0}_{3k+2}$ has been calculated in [3], so we need to determine $\operatorname{coker} T$.

First, we claim that Corollary 3 is applicable and the kernel of C is always trivial. Indeed, in all cases the component of unity of the symmetry group $G_{\Sigma^{1,1}}$ is the group $SO(k)$ and the bundle $\xi^{1,1}$ is associated to the universal $G_{\Sigma^{1,1}}$-bundle via the image representation. Hence, it is sufficient to check whether the image of a non-contractible loop γ in $SO(k)$ under the image representation λ_2 is non-contractible as well. The representation λ_2 has the form $(\varepsilon, A) \mapsto diag(1, 1, A, \varepsilon A, \varepsilon A)$, and it is easy to check that the mapping $[\gamma] \mapsto [diag(1, 1, \gamma, \gamma, \gamma)]$ is an isomorphism from $\pi_1(SO(k))$ to $\pi_1(SO(3k+2))$. It follows by applying Corollary 3 that C is indeed an isomorphism.

This fact implies that $\operatorname{coker} T$=$\operatorname{coker} C \circ T$. Given an element $[f] \in \pi_{3k+3}(X_{1,1})$ let us denote by α_f the corresponding cobordism class of cusp-maps in $Cob^{1,1}(2k+3, \mathbb{R}^{3k+3})$. Let $g : M^{2k+3} \to \mathbb{R}^{3k+3}$ be any representative of α_f. We claim that $C \circ T([f])$ depends only on the cobordism class of the source manifold M in Ω_{2k+3} or $\mathfrak{N}_{2k+3}$ (depending on whether we consider the oriented or the unoriented case). Indeed, if we have an arbitrary cobordism of M and represent it with a generic mapping into $\mathbb{R}^{3k+3} \times [0, 1]$, it will have only isolated $III_{2,2}$-points apart from cusps and folds, so $C \circ T([f])$ is well-defined up to the subgroup generated by the mapping on the boundary of a normal form of a $III_{2,2}$ point. This subgroup is however trivial, because both the kernel bundle and the virtual normal bundle over the cusp-circle are trivial (recall that these two bundles give a complete set of invariants of cobordisms of cusp-maps). The virtual normal bundle is trivial because both the source and the image bundles are trivial as normal bundles of a circle in a $2k+2$-sphere and a $3k+2$-sphere, respectively, and the kernel bundle contains the line defined by the vector $(\sin\alpha, -\cos\alpha, 0, 0, 0, 0, 0, 0, \overline{0}, \overline{0})$ over the cone of cusps with the base $(\sin^2\alpha\cos\alpha, \sin\alpha\cos^2\alpha, -3\sin^2\alpha\cos\alpha, -\sin^3\alpha, -\cos^3\alpha, -3\sin\alpha\cos^2\alpha, \overline{0}, \overline{0})$ in the canonical form of the $III_{2,2}$ singularity, $(x, y, u, v, w, z, \overline{s}, \overline{t}) \mapsto (xy, x^2 + ux + vy, y^2 + wx + zy, x\overline{s} + y\overline{t}, u, v, w, z, \overline{s}, \overline{t})$ (see [4]).

So, $C \circ T$ can be expressed in terms of Stiefel-Whitney characteristic numbers (and Pontryagin numbers in the oriented case) of the underlying manifold. Hence we have the following cases:

- **Unoriented case:** $\pi_1(G_{\Sigma^{1,1}}) \approx \mathbb{Z}_2 \times \mathbb{Z}_2$, and $C \circ T$ can be identified with the pair

  $$(\overline{w_{k+1}^2 w_1 + w_{k+2}w_k w_1}, \overline{w_{k+2}w_{k+1} + w_{k+3}w_k}).$$

  However, the characteristic number

  $$(Sq^1 + w_1 \cdot)(\overline{w_{k+1}^2 + w_{k+2}w_k}),$$

  which always evaluates to 0 according to [2], is the first element of the given pair when k is odd and the sum of the two elements of the pair when k is even. Therefore it is enough to check whether the second element of the pair always evaluates to 0 or not; it is an easy computation to see that Y^5 evaluates to 1 and multiplying by $\mathbb{R}P^2$ does not change this value.

So $\pi_{3k+2}(X_{1,0})$ is an extension of $\ker\{T\colon \pi_{3k+2}(X^{1,1}) \to \pi_{3k+2}(\Gamma T\xi^{1,1})\}$, which is an index 2 subgroup of $\mathfrak{N}_{2k+2}$, by $\mathbb{Z}_2$.

- **Oriented case,** k **odd:** $\xi^{1,1}$ is orientable, $\pi_1(G_{\Sigma^{1,1}}) \approx \mathbb{Z}_2$ and the mapping $C \circ T$ is the characteristic number
$$\overline{w_{k+2}w_{k+1} + w_{k+3}w_k}.$$
Now, $Y^5 \times (\mathbb{R}P^2)^{k-1} \approx_{\mathfrak{N}} Y^5 \times (\mathbb{C}P)^{(k-1)/2}$ evaluates to 1, so T is always onto and $\pi_{3k+2}(X_{1,0}) \approx \ker\{T\colon \pi_{3k+2}(X^{1,1}) \to \pi_{3k+2}(\Gamma T\xi^{1,1})\}$ is an index 3^v subgroup of Ω_{2k+2} with an appropriate v defined in [7].
- **Oriented case,** k **even:** $\xi^{1,1}$ changes orientation over all noncontractible loops in $B\xi^{1,1}$, so T is onto and $\pi_{3k+2}(X_{1,0}) \approx \ker\{T\colon \pi_{3k+2}(X^{1,1}) \to \pi_{3k+2}(\Gamma T\xi^{1,1})\}$ is "the whole" $\Omega_{2k+2} \approx 0$ when k is either 2 and is an index 2 subgroup of Ω_{2k+2} when $k \geq 4$.

As a reformulation of this result, we have the following theorem:

Theorem 4.

a) *There is an exact sequence*
$$0 \to \mathbb{Z}_2 \to Cob^{1,0}(2k+2, \mathbb{R}^{3k+2}) \to G \to 0,$$
where G *is an index 2 subgroup of* $\mathfrak{N}_{2k+2}$*, for all* $k > 0$.

b1) *If* k *is odd, then* $Cob^{1,0}_{so}(2k+2, \mathbb{R}^{3k+2})$ *is isomorphic to the kernel of the epimorphic mapping*
$$\overline{p_{(k+1)/2}}[\cdot]\colon \Omega_{2k+2} \to \mathbb{Z}.$$

b2) $Cob^{1,0}_{so}(6, \mathbb{R}^8) \approx 0$. *If* $k \geq 4$ *is even, then* $Cob^{1,0}_{so}(2k+2, \mathbb{R}^{3k+2})$ *is an index 2 subgroup of* Ω_{2k+2}.

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