COBORDISMS OF
FOLD MAPS OF $2k + 2$-MANIFOLDS INTO $\mathbb{R}^{3k+2}$

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Abstract. We calculate the group of cobordisms of $k$-codimensional maps into Euclidean space with no singularities more complicated than fold for a $2k + 2$-dimensional source manifold in both oriented and unoriented cases.

1. Concept. Given a descending set of singularities $\{\eta\} \cup \tau = \tau'$ with a top singularity $\eta$ and a fixed codimension $k$ of the mappings involved, we can consider the classifying spaces $X_{\tau'}$ and $X_{\tau}$, the homotopy groups of which are isomorphic to the cobordism groups of mappings into Euclidean spaces with all singularities in $\tau'$ and $\tau$, respectively. It is known ([8]) that there is a fibration

$$X_{\tau'} \xrightarrow{X_{\tau}} \Gamma T \xi^\eta,$$

where $\xi^\eta$ is the bundle associated to the universal $G_\eta$-bundle via the representation of $G_\eta$ in the image. Hence we have a long exact sequence

$$\cdots \rightarrow \pi_{N+1}(\Gamma T \xi^\eta) \rightarrow \pi_N(X_{\tau'}) \rightarrow \pi_N(X_{\tau}) \xrightarrow{T} \pi_N(\Gamma T \xi^\eta) \rightarrow \pi_{N-1}(X_{\tau}) \rightarrow \cdots$$

where the mapping $\pi_N(X_{\tau'}) \xrightarrow{T} \pi_N(\Gamma T \xi^\eta)$ assigns to every map $f: M \rightarrow \mathbb{R}^N$ the map that classifies the immersion $f: \eta(f) \rightarrow \mathbb{R}^N$ with the $\xi^\eta$ normal structure added.

This will be applied to $\tau' = \{\Sigma^{1,1,0}\} \cup \{\Sigma^{1,0}, \Sigma^0\}$. We will denote the cobordism groups of $\tau'$-maps ($\tau$-maps) of $n$-dimensional manifolds $\text{Cob}^{1,1}(n, \mathbb{R}^{n+k})$ (respectively, $\text{Cob}^{1,0}(n, \mathbb{R}^{n+k})$); the corresponding classifying space will be called $X_{1,1}$ (respectively, $X_{1,0}$). The classifying space for maps without restrictions on the singularities will be

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called $X_{\infty} = \Omega^\infty M(S)O(k + \infty)$. We have
\[
\cdots \to \pi_{N+1}(X_{1,1}) \xrightarrow{T} \pi_{N+1}^S(T\xi^{1,1}) \to \pi_N(X_{1,0}) \to \pi_N(X_{1,1}) \xrightarrow{T} \pi_N^S(T\xi^{1,1}) \to \cdots
\]
\[\approx\]
$Cob^{1,0}(N - k, \mathbb{R}^N)$
and after calculating the groups and mappings involved we will be able to describe the groups $Cob^{1,0}(2k + 2, \mathbb{R}^{3k+2})$.

2. Calculations

**Lemma 1.** Given a vector bundle $\xi$ of rank $n \geq 1$ over a connected base $B$,
\[
\pi_n(T\xi) = \begin{cases} 
\mathbb{Z} & \text{if } \xi \text{ is orientable,} \\
\mathbb{Z}_2 & \text{if } \xi \text{ is not orientable,}
\end{cases}
\]
and the mapping $[f] \mapsto [f \cap B\xi]$ is an isomorphism. Here $[f \cap B\xi]$ denotes the number of intersection points of $B\xi$ and the image of $f$, taken with sign if $\xi$ is oriented, after a small perturbation to make $f$ transversal to $B\xi$.

*Proof.* Since $T\xi$ is $n - 1$-connected, $\pi_n(T\xi) \approx H_n(T\xi; \mathbb{Z}) \approx H^n(T\xi; \mathbb{Z})$. This group is generated by the Thom class, which is a free generator if $\xi$ is orientable and has order 2 if $\xi$ is not orientable. The mapping $[f] \mapsto [f \cap B\xi]$ is the evaluation of the Thom class on the image of $[f]$ under the Hurewicz homomorphism, hence it is an isomorphism.

**Lemma 2.** Let $\xi$ be an arbitrary vector bundle of rank $n \geq 3$ over a connected base $B$. Then the mapping
\[
C: \pi_{n+1}(T\xi) \ni [f] \mapsto [f \cap B]
\]
is onto and its kernel is either isomorphic to $\mathbb{Z}_2$ or trivial, depending on whether $w_2(\xi)$ vanishes or not.

*Proof.* We can kill $\pi_n(T\xi)$ by constructing a fibration $K(\pi_n(T\xi), n - 1) \to X \to T\xi$ with an $n$-connected $X$ in the usual way (see e.g. [6]), by pulling back the fibration $K(H^n(T\xi), n - 1) \to PK(H^n(T\xi), n) \to K(H^n(T\xi), n)$ with the classifying map of the generator of $H^n(T\xi)$, the Thom class $U$. This way $\pi_{n+1}(T\xi) \approx \pi_{n+1}(X) \approx H_{n+1}(X) \approx H^{n+1}(X)$ can be calculated from the Serre spectral sequence. Indeed, due to dimensional constraints the only potentially non-zero differentials influencing $H^{n+1}(X)$ are transgressions $H^{n-1+j}(K(\pi_n(T\xi), n - 1)) \to H^{n+j}(T\xi) = U \cup H^j(B)$ for $j = 0, 1, 2$. The transgression for $j = 0$ is an isomorphism by design. For $j = 1$, we have $H^n(K(\mathbb{Z}, n - 1)) = 0$ in the oriented case and $H^n(K(Z_2, n - 1)) = \langle Sq^1 \rangle$ if $\xi$ is not oriented; in this latter case, the transgression sends this element to $Sq^1(U) = U \cup w_1(\xi)$ since transgressions commute with Steenrod operations. Finally, for $j = 2$ we have $H^{n+1}(K(\pi_n(T\xi), n - 1)) = \langle Sq^2 \rangle$ in both cases, and the transgression sends this element to $Sq^2 U = U \cup w_2(\xi)$. Therefore,
the $E_{\infty}^{s^*}$ term in dimension $n + 1$ will contain only $E_{\infty}^{n+1,0}$, which can be identified with $U \cup (H^1(B)/\langle w_1(\xi) \rangle)$ and $E_{\infty}^{0,n+1}$, which is 0 when $w_2(\xi) \neq 0$ and $\mathbb{Z}_2$ otherwise. The statement of the lemma follows immediately.

**Corollary 3.** If the bundle $\xi$ is associated to the universal $G$-bundle via the representation $\lambda: G \to \text{Iso}(\mathbb{R}^n)$, $n > 1$, then the mapping $C$ from Lemma 2 is an isomorphism if and only if $\lambda_*(\pi_1(G,e)) = \pi_1(SO(n),e)$, that is, the image of the fundamental group of $G$ under $\lambda$ contains a non-contractible loop in $SO(n)$.

**Proof.** We will check the criterion of Lemma 2. $G$-bundles over $S^2$ correspond in a one-to-one fashion to their gluing maps, which can be identified with the elements of $\pi_1(G)$. For any $[s] \in \pi_2(BG)$ the pullback of the universal $G$-bundle on $S^2$ by $s$ has the gluing map $\partial[s] \in \pi_1(G)$ with $\partial$ being an isomorphism taken from the homotopy long exact sequence of the universal $G$-bundle. Indeed, when we lift $[s]: S^2 \setminus \{\text{point}\} \to BG$ as a homotopy of a trivial mapping of a circle to $EG$, we will get the mapping $\partial[s]$ on the boundary (in the fibre over the excised point), and it gives the difference between the trivialisations of the pullback bundle over the two hemispheres, i.e. the gluing map. Since $\xi$ is associated to the universal bundle via $\lambda$, the gluing map for the pullback of $\xi$ will be the image of the gluing map for the universal bundle under $\lambda$ and hence the degree of $s^*\xi$ can be regarded as $\lambda_*(\partial[s]) \in \pi_1(O(n))$. But as $[s]$ takes all values from $\pi_2(BG)$, $\partial[s]$ takes all values from $\pi_1(G)$, so we will obtain a pulled-back bundle of odd degree if and only if the whole image $\lambda_*(\pi_1(G))$ contains the generator of $\pi_1(O(n)) = \pi_1(SO(n))$, and that completes the proof. We will also need to know what the symmetry group of the singularity $\Sigma^{1,1}$ looks like. $G_{\Sigma^{1,1}}$ in the unoriented case has the homotopy type of the group $\mathbb{Z}_2 \times O(k)$ and the representations $\lambda_1$ (in the source) and $\lambda_2$ (in the image) are of the form

$\lambda_1(\varepsilon, A) = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & \varepsilon A \end{pmatrix}$ and $\lambda_2(\varepsilon, A) = \begin{pmatrix} \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & \varepsilon A & 0 \\ 0 & 0 & 0 & 0 & \varepsilon A \end{pmatrix}$

in an appropriate local coordinate system. Hence the symmetry group in the oriented case, that is, the subgroup of $\mathbb{Z}_2 \times O(k)$ forming the kernel of the orientation mapping of the virtual normal bundle $(\varepsilon, A) \mapsto \det \varepsilon A = \varepsilon^k \det A$, is $\mathbb{Z}_2 \times SO(k)$ for even $k$ and $\{1\} \times SO(k) \cup \{-1\} \times (-1)SO(k)$ for odd $k$. This implies that the connected components of $G_{\Sigma^{1,1}}$ are in all cases separated by the projections $\text{pr}_1(\varepsilon, A) = \varepsilon$ and $\text{pr}_2(\varepsilon, A) = \det \varepsilon A$. When interpreted as projections from $\pi_1(BG_{\Sigma^{1,1}})$, $\text{pr}_1$ is returning the orientability of the kernel bundle over $\Sigma^{1,1}$ (on every loop in $\Sigma^{1,1}(f)$), and $\text{pr}_2$ is returning the orientability of the virtual normal bundle of $f$ over $\Sigma^{1,1}$ (on every loop in $\Sigma^{1,1}(f)$). We will express these projections in terms of the Stiefel-Whitney characteristic classes of the underlying manifold $M$ defined by the Pontryagin-Thom construction from a representative mapping $f$ of $[f] \in \pi_{3k+3}(T\xi)$ (and hence additive notation will be used for convenience). $\text{pr}_2$ is obviously evaluating $\overline{w}_1 \cdot Tp_{\Sigma^{1,1}} = \frac{w_{k+1}^2}{w_{k+2}w_k}w_{k+1}$ on the fundamental class of $M$, $[M]$, since $\overline{w}_1$ gives the orientability of all restrictions of the virtual normal bundle, in
particular, the restriction to the dual of $Tp_{\Sigma^{1,1}}$, represented by $\Sigma^{1,1}(f)$. As to $pr_1$, a direct adaptation of [5] (using [1]) gives us the characteristic number $w_{k+3}w_k + w_{k+2}w_{k+1}$.

2.1. Calculating $\pi_{3k+2}(X_{1,0})$. The long exact sequence (2) gives a short exact sequence

$$0 \to \text{coker } T \to \pi_{3k+2}(X_{1,0}) \to \ker T^{1,0}_{3k+2} \to 0$$

where $\ker T^{1,0}_{3k+2}$ has been calculated in [3], so we need to determine $\text{coker } T$.

First, we claim that Corollary 3 is applicable and the kernel of $C$ is always trivial. Indeed, in all cases the component of unity of the symmetry group $G_{\Sigma^{1,1}}$ is the group $SO(k)$ and the bundle $\xi^{1,1}_i$ is associated to the universal $G_{\Sigma^{1,1}}$-bundle via the image representation. Hence, it is sufficient to check whether the image of a non-contractible loop $\gamma$ in $SO(k)$ under the image representation $\lambda_2$ is non-contractible as well. The representation $\lambda_2$ has the form $(\varepsilon, A) \mapsto \text{diag}(1, 1, A, \varepsilon A, \varepsilon A)$, and it is easy to check that the mapping $[\gamma] \mapsto [\text{diag}(1, 1, \gamma, \gamma, \gamma)]$ is an isomorphism from $\pi_1(SO(k))$ to $\pi_1(SO(3k + 2))$. It follows by applying Corollary 3 that $C$ is indeed an isomorphism.

This fact implies that $\text{coker } T = \text{coker } C \circ T$. Given an element $[f] \in \pi_{3k+3}(X_{1,0})$ let us denote by $\alpha_f$ the corresponding cobordism class of cusp-maps in $\text{Cob}^{b,1}(2k + 3, \mathbb{R}^{3k+3})$. Let $g : \mathbb{M}^{2k+3} \to \mathbb{R}^{3k+3}$ be any representative of $\alpha_f$. We claim that $C \circ T([f])$ depends only on the cobordism class of the source manifold $M$ in $\Omega_{2k+3}$ or $\mathfrak{R}_{2k+3}$ (depending on whether we consider the oriented or the unoriented case). Indeed, if we have an arbitrary cobordism of $M$ and represent it with a generic mapping into $\mathbb{R}^{3k+3} \times [0, 1]$, it will have only isolated $III_{2,2}$-points apart from cusps and folds, so $C \circ T([f])$ is well-defined up to the subgroup generated by the mapping on the boundary of a normal form of a $III_{2,2}$ point. This subgroup is however trivial, because both the kernel bundle and the virtual normal bundle over the cusp-circle are trivial (recall that these two bundles give a complete set of invariants of cobordisms of cusp-maps). The virtual normal bundle is trivial because both the source and the image bundles are trivial as normal bundles of a circle in a $2k + 2$-sphere and a $3k + 2$-sphere, respectively, and the kernel bundle contains the line defined by the vector $(\sin \alpha, -\cos \alpha, 0, 0, 0, 0, 0, 0, \overline{0}, \overline{0})$ over the cone of cusps with the base $(\sin^2 \alpha \cos \alpha, \sin \alpha \cos^2 \alpha, -3 \sin^2 \alpha \cos \alpha, -\sin^3 \alpha, -\cos^3 \alpha, -3 \sin \alpha \cos^2 \alpha, \overline{0}, \overline{0})$ in the canonical form of the $III_{2,2}$ singularity, $(x, y, u, v, w, z, \overline{s}, \overline{t}) \mapsto (xy, x^2 + uy + vy, y^2 + wx + zy, x\overline{s} + y\overline{t}, u, v, w, z, \overline{s}, \overline{t})$ (see [4]).

So, $C \circ T$ can be expressed in terms of Stiefel-Whitney characteristic numbers (and Pontryagin numbers in the oriented case) of the underlying manifold. Hence we have the following cases:

- **Unoriented case:** $\pi_1(G_{\Sigma^{1,1}}) \approx \mathbb{Z}_2 \times \mathbb{Z}_2$, and $C \circ T$ can be identified with the pair

$$w_{k+1}^2w_1 + w_{k+2}w_kw_1, w_{k+2}w_{k+1} + w_{k+3}w_k.$$  

However, the characteristic number

$$(Sq^1 + w_1,w_1)(w_{k+1}^2 + w_{k+2}w_k),$$

which always evaluates to 0 according to [2], is the first element of the given pair when $k$ is odd and the sum of the two elements of the pair when $k$ is even. Therefore it is enough to check whether the second element of the pair always evaluates to 0
or not; it is an easy computation to see that $Y^5$ evaluates to 1 and multiplying by $\mathbb{R}P^2$ does not change this value. So $\pi_{3k+2}(X_{1,0})$ is an extension of $\ker\{T: \pi_{3k+2}(X^{1,1}) \to \pi_{3k+2}(\Gamma T \xi^{1,1})\}$, which is an index 2 subgroup of $\mathfrak{N}_{2k+2}$, by $\mathbb{Z}_2$.

- **Oriented case, $k$ odd:** $\xi^{1,1}_{1,1}$ is orientable, $\pi_2(G_{\Sigma^{1,1}_1}) \approx \mathbb{Z}_2$ and the mapping $C \circ T$ is the characteristic number

$$w_{k+2}w_{k+1} + w_{k+3}w_k.$$ 

Now, $Y^5 \times (\mathbb{R}P^2)^{k-1} \approx \mathfrak{N}_1 Y^5 \times (\mathbb{C}P)^{(k-1)/2}$ evaluates to 1, so $T$ is always onto and $\pi_{3k+2}(X_{1,0}) \approx \ker\{T: \pi_{3k+2}(X^{1,1}) \to \pi_{3k+2}(\Gamma T \xi^{1,1})\}$ is an index 3" subgroup of $\Omega_{2k+2}$ with an appropriate $v$ defined in [7].

- **Oriented case, $k$ even:** $\xi^{1,1}_{1,1}$ changes orientation over all noncontractible loops in $B\xi^{1,1}_{1,1}$, so $T$ is onto and $\pi_{3k+2}(X_{1,0}) \approx \ker\{T: \pi_{3k+2}(X^{1,1}) \to \pi_{3k+2}(\Gamma T \xi^{1,1})\}$ is "the whole" $\Omega_{2k+2} \approx 0$ when $k$ is either 2 and is an index 2 subgroup of $\Omega_{2k+2}$ when $k \geq 4$.

As a reformulation of this result, we have the following theorem:

**Theorem 4.**

a) There is an exact sequence

$$0 \to \mathbb{Z}_2 \to \text{Cob}^{1,0}(2k + 2, \mathbb{R}^{3k+2}) \to G \to 0,$$

where $G$ is an index 2 subgroup of $\mathfrak{N}_{2k+2}$, for all $k > 0$.

b1) If $k$ is odd, then $\text{Cob}^{1,0}_{so}(2k + 2, \mathbb{R}^{3k+2})$ is isomorphic to the kernel of the epimorphic mapping

$$\overline{p(k+1)/2} : \Omega_{2k+2} \to \mathbb{Z}.$$

b2) $\text{Cob}^{1,0}_{so}(6, \mathbb{R}^8) \approx 0$. If $k \geq 4$ is even, then $\text{Cob}^{1,0}_{so}(2k + 2, \mathbb{R}^{3k+2})$ is an index 2 subgroup of $\Omega_{2k+2}$.

**References**


