QUADRATIC MAPPINGS AND CONFIGURATION SPACES

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Abstract. We discuss some approaches to the topological study of real quadratic mappings. Two effective methods of computing the Euler characteristics of fibers are presented which enable one to obtain comprehensive results for quadratic mappings with two-dimensional fibers. As an illustration we obtain a complete topological classification of configuration spaces of planar pentagons.

Introduction. The topology of fibers of real polynomial mappings is intensively studied in algebraic geometry and singularity theory [2], [3]. In particular, there exist effective algebraic methods for computing certain topological invariants of real algebraic varieties and mappings, such as the mapping degree and Euler characteristic [6], [11], [4], [17], [12].

Quadratic mappings constitute an important class of polynomial mappings which naturally appear in many problems of geometry and analysis [1], [12], [13]. Recently, some new aspects of quadratic mappings were revealed [12], [13] which assign additional importance to their topological classification and study of their fibers. In particular, this issue is crucial for the topological classification of configuration spaces of polygons and linkages, as well as for the investigation of certain integrable systems. As is well known, families of quadratic forms are also inevitable in calculus of variations and control theory [1].

In many issues an especially important role is played by *proper* quadratic mappings. So it suffices to say that any smooth compact manifold without boundary can be represented as a fiber of a proper quadratic mapping [13] (cf. [18]). For this reason, in the sequel we only deal with *proper quadratic mappings*.

It turns out that in some situations emerging from concrete geometrical problems it is possible to obtain considerable topological information about the fibers of proper

quadratic mappings using the explicit formulae for the Euler characteristic of an intersection of real quadrics which were obtained in [1] and [12]. It should be noted that approaches of [1] and [12] are rather different in nature and can usefully complement each other.

The aim of this paper is to describe those general techniques for the topological study of quadratic mappings and present a number of their applications. We begin with reproducing some results from [12], [13], [1] in the form adapted to our purposes and then apply them to several concrete examples where necessary computations can be performed quite effectively. In particular, we describe the topological structure of fibers for Casimir mappings of the so-called Sklyanin algebras [14] and find the topological types of configuration spaces of certain planar and spatial polygons (cf. [7], [10]).

1. Algebraic formulae for the Euler characteristic. Recall that a polynomial mapping $g : \mathbf{R}^n \to \mathbf{R}^p$ is called *quadratic* if all of its components are polynomials of algebraic degree not exceeding two. As usual, the leader P^* of a polynomial P is defined as the sum of terms of the highest degree in P. If all components of g are homogeneous polynomials of degree two, g is called a *homogeneous quadratic mapping*. Thus the leader of any quadratic mapping g is a homogeneous quadratic mapping g^*.

As usual, a mapping g is called proper if, for any compact subset $X \subset \mathbf{R}^p$, the full pre-image $g^{-1}(K)$ is also compact. Obviously, g is proper if among the leaders of its components there exists a definite (i.e., non-degenerate and without values of opposite signs) quadratic form. Actually, for properness it is sufficient that the pencil of quadratic forms defined by components of g^* contains a definite form but there does not seem to exist a general criterion of that kind [1].

REMARK 1. A homogeneous quadratic mapping into $\mathbf{R}^2$ is proper if and only if the pencil of quadratic forms $\lambda g_1 + \mu g_2$ contains a definite form [1]. In such case one can assume, without loss of generality, that g_1 is positive definite and consider it as a scalar product in $\mathbf{R}^n$. Then the second form can be diagonalized by an orthogonal linear transformation and one can assume that both forms are diagonal in the same basis. In this case it is not difficult to describe the fibers of g in terms of the eigenvalues of g_2. Thus the description of fibers is easily accessible for proper quadratic mappings into $\mathbf{R}^2$ but the problem becomes quite non-trivial for $p \geqslant 3$ [1].

We consider now a proper quadratic mapping $g : \mathbf{R}^n \to \mathbf{R}^p$ and aim at describing the topology of its fibers, i.e. we study the topology of inverse images $g^{-1}(c)$ for $c \in \mathbf{R}^p$. In our situation, for generic $c \in \mathbf{R}^p$, the fiber $g^{-1}(c) \subset \mathbf{R}^n$ is a compact smooth manifold and one of its most important topological invariants is the Euler characteristic $\chi(g^{-1}(c))$ defined as the alternating sum of the ranks of its homology groups with integer coefficients [8].

More generally, the Euler characteristic $\chi(X)$ can be defined for an arbitrary algebraic variety $X \subset \mathbf{R}^n$ [3] so in our case it is defined for all fibers. From general results of singularity theory it is easy to derive that, for a given g as above, the Euler characteristic of its fibers can only assume a finite number of values. More precisely, $\chi(X)$ is constant on each component of the complement to the bifurcation diagram $B(g)$ [2]. Thus as a first step toward description of fibers of g one could try to find all values of the Euler

characteristic of fibers and this is the problem we are going to deal with.

To this end we will use the local topological degree of a certain auxiliary polynomial map which can be naturally associated with algebraic variety $g^{-1}(c)$. Recall that any real algebraic variety X can be represented as a hypersurface $X = \{F = 0\}$, where $F = f_1^2 + \ldots + f_k^2$, whereas f_j are the polynomials determining variety X. Recall also that the local degree of an endomorphism $f : \mathbf{R}^n \to \mathbf{R}^n$, which we will denote by $\deg_p f$, for each $p \in f^{-1}(0)$, is by definition the topological degree of the map

$$\frac{f}{||f||} : S_\varepsilon^{n-1}(p) \to S_1^{n-1}(0),$$

where ε is a small positive number.

Our first aim is to give explicit formulae for the Euler characteristics of level surfaces of a quadratic mapping following the recent monograph [12]. In view of said above it is reasonable to begin with presenting the basic formulae for the Euler characteristic of a real algebraic hypersurface defined as a level surface of a polynomial. Let us introduce some necessary definitions and reproduce some fundamental theorems from [11], [4].

Let $f : U \to \mathbf{R}$ be a smooth function in a neighborhood of $0 \in \mathbf{R}^n$ and let

$$\operatorname{grad} f : \mathbf{R}^n \to \mathbf{R}^n$$

be its gradient map. Denote by B_r the ball of radius r centered at the origin.

The algebraic formulae for the Euler characteristic are based on the following local result which was established in [11].

THEOREM 1. *Let $F : \mathbf{R}^n \to \mathbf{R}^1$ be a polynomial with an isolated critical point at the origin. Then we have*

$$\chi\big(\{F < \lambda\} \cap B_\delta^n\big) = 1 + (-1)^{n+1} \deg_0 \operatorname{grad} F,$$

where $\lambda > 0$ is sufficiently small and B_δ^n is a ball of small radius δ.

Using this formula it becomes possible to find the Euler characteristic of local level surfaces intersected with small spheres. For ε and r sufficiently small, put

$$M_{\varepsilon,r} = \{F = \epsilon\} \cap B_r.$$

Notice that it is a smooth compact manifold without boundary. The preceding theorem implies the following formula which was also obtained in [11].

THEOREM 2. *With the same assumptions and notation as above one has*

$$\chi(M_{\varepsilon,r}) = 1 + (-1)^{n+1} \deg_0 \operatorname{grad} F,$$

where ε and r are sufficiently small positive numbers.

As we will see below, the latter theorem can be used for computing the Euler characteristic of intersections of two homogeneous quadrics. For nonhomogeneous quadratic mappings one can use a more general result which is a particular case of the main theorem of [4] (cf. also [17]).

THEOREM 3 ([4]). *Let* $f_1, \ldots, f_p : \mathbf{R}^n \to \mathbf{R}$ *be real quadratic polynomials. Suppose that* $X = f^{-1}(0)$ *is compact, where* $f = (f_1, \ldots, f_n)$. *Set*

(1) $$h_i(x_0, \ldots, x_n) = x_0^3 f_i\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right), \quad H = \sum_{i=1}^p h_i^2 - \sum_{k=0}^n x_k^8.$$

Then H *has an isolated critical point at the origin and the following equality holds*:

(2) $$\chi(X) = \frac{1}{2}\left((-1)^n - \deg_0 \operatorname{grad} H\right).$$

These formulae enable one to compute the Euler characteristics of fibers of any quadratic mapping g purely algebraically. Indeed, it follows from the results of [11], [6] that the local topological degree of an explicitly given map germ can be computed as the signature of an effectively constructible quadratic form, hence in an algorithmic way. There exists a computer program for calculating the local degree [15] so in any concrete case one can find the Euler characteristic of any fiber by merely using a computer.

One the other hand, it should be noted that this usually requires rather lengthy and involved computations which cannot be performed without using a computer, so one may wonder if there are some other ways of computing the Euler characteristic of fibers which require less complicated manipulations. For homogeneous quadratic mappings this is sometimes possible in the framework of another general approach which is discussed in the next section.

2. Homogeneous quadratic mappings. In [1] A. Agrachev and R. Gamkrelidze consider the spaces of quadratic mappings and investigate topology of their fibers. They give various tools for the calculation of Euler characteristics of the preimages of regular values. In some cases their methods are easier to use than the signature formulae of preceding section. In some low-dimensional cases like the ones considered in the sequel, their technique is especially effective, so we now reproduce relevant constructions and results from [1]. This approach does not require assumption of properness so we first present some results for arbitrary homogeneous quadratic maps.

Denote by $P_N = P(\mathbf{R}^N)$ the set of all (homogeneous) quadratic forms in N variables. The subset of degenerate forms is denoted by Π_N. Denote further by $P(N, k)$ the set of symmetric bilinear maps $p : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^k$. As usual, for any $p \in P(N, k)$, a (homogeneous) quadratic map is defined as $x \mapsto p(x, x)$. A quadratic map $p \in P(N, k)$ is called degenerate, if 0 is a critical value of the map $p|(R^N - \{0\})$. The space of the degenerate quadratic maps is a proper algebraic subvariety of $P(N, k)$ and therefore non-degenerate maps form a dense open subset of $P(N, k)$ which will be denoted by $\widehat{P}(N, k)$.

For dealing with topological invariants it is useful to consider homotopy classes of quadratic maps. Two non-degenerate maps $p_1, p_2 \in \widehat{P}(N, k)$ are called rigid homotopic, if they belong to the same connected component of $\widehat{P}(N, k)$. It is known that if $p \in \widehat{P}(N, k)$, then $p^{-1}(0) \cap S^{N-1}$ is a smooth manifold and if p_1, p_2 are rigid homotopic, then the manifolds $p_1^{-1}(0) \cap S^{N-1}$ and $p_2^{-1}(0) \cap S^{N-1}$ are diffeomorphic.

So for describing fibers of quadratic mappings it is sufficient to do so for one representative of each class of rigid homotopy. It turns out that, for small k, one can construct cer-

tain canonical representatives of rigid homotopy classes. For example, for non-degenerate quadratic self-mappings of the plane $\mathbf{R}^2$ this can be done in an elementary way.

PROPOSITION 1. *Any quadratic map from* $\widehat{P}(2,2)$ *is rigid homotopic to one of the following maps*

$$q_1(x, y; x, y) = (x^2 + y^2, 0);$$
$$q_2(x, y; x, y) = (x^2 - y^2, 2xy);$$
$$q_3(x, y; x, y) = (x^2 - y^2, -2xy).$$

REMARK 2. For each proper $q \in \widehat{P}(2,2)$ one can consider its topological degree $\deg q$. It is evident that $\deg q$ is invariant under rigid homotopy so one can also try to classify such maps by values of their topological degree, i.e. up to the usual homotopy of proper mappings instead of the rigid homotopy.

It is easy to see that in the above case the two classifications coincide. Indeed, from the general estimates for the topological degree in terms of Petrovsky numbers [12] it follows (and in this case can be verified directly) that the topological degree of quadratic self-mapping of the plane can only take three values: $-2, 0, 2$. Thus there are three homotopy classes and the mappings above are their representatives, so the two classifications really coincide, which is specific for two dimensions (cf. [1]).

An analogous comparison can be attempted for $\widehat{P}(N, N)$ with arbitrary N. Recall that the topological degree always vanishes for a homogeneous quadratic mapping in an odd-dimensional space [12]. As a certain substitute one can consider the mod 2 degree of the induced mapping $Q' : \mathbf{RP}^N \to S^N$ (it is obviously well-defined for a homogeneous quadratic mapping) [1]. For $N = 2$, this degree classifies quadratic mappings up to a rigid isotopy (there are exactly two classes) but there is no hope that it can do the same for any odd N. For even N, it is easy to determine all possible values of the topological degree (see [12]) but there does not seem to exist any obvious way to list the classes of rigid homotopy.

Actually, an analogous result holds for pairs of quadratic forms in any dimension. Namely, according to [1] every $q \in \widehat{P}(N, 2)$ considered as map into $\mathbf{C} = \mathbf{R}^2$ is rigid homotopic to a map of the form

(3) $$q(x_1, x_2, \ldots, x_n, z_1, \ldots, z_m) = \sum_{j=1}^{n} x_j^2 e^{\theta_j i} + \sum_{j=1}^{m} z_j^2,$$

where $\mathbf{R}^N = \mathbf{R}^n \oplus \mathbf{C}^m$, $n + 2m = N$, $(x_1, \ldots, x_n) \in \mathbf{R}^n$, $(z_1, \ldots, z_m) \in \mathbf{C}^m$. Moreover, a quadratic map of the form (3) is non-degenerate if and only if $e^{\theta_{j_1} i} + e^{\theta_{j_2} i} \neq 0$, for every pair j_1, j_2, and one can further explicate the form of such canonical maps.

Introduce the following equivalence relation $\sim$ on the set $\{e^{\theta_1 i}, \ldots, e^{\theta_n i}\}$. We call $e^{\theta_{j_1} i}$ and $e^{\theta_{j_2} i}$ equivalent, if one of two arcs connecting the points $e^{\theta_{j_1} i}$ and $e^{\theta_{j_2} i}$ on the circle S^1 does not contain any number from the set $\{-e^{\theta_1 i}, \ldots, -e^{\theta_n i}\}$. This relation divides $\{e^{\theta_1 i}, \ldots, e^{\theta_n i}\}$ into non-intersecting classes. The number of elements of the set $\{e^{\theta_1 i}, \ldots, e^{\theta_n i}\}/ \sim$ is odd, say, $2k - 1$. Identifying equivalent points and marking each equivalence class as a point on S^1 we obtain a configuration of $2k - 1$ points and such configurations enumerate classes of rigid homotopic maps [1]. For example, when $N = 3$

in this way one finds that every quadratic map from $\widehat{P}(3,2)$ is rigid homotopic to one of two maps

$$q_1(x, y, z; x, y, z) = (x^2 + y^2 + z^2, 0);$$
$$q_2(x, y, z; x, y, z) = (x^2 + y^2 - z^2, 2xy).$$

The class of rigid homotopy of a given map can be read off the algebraic properties of the pencil of quadratic forms defined by its components and for each class of rigid homotopy one can find the Euler characteristic of the zero-fiber intersected with the unit sphere by explicit formulae given in [1]. Thus for two quadratic forms the description of fibers is available by this approach. For details of the corresponding algorithm and further examples we refer to [1].

In general case one can use the following topological approach. For a given quadratic map $p : \mathbf{R}^N \to \mathbf{R}^k$, consider the pencil of quadratic forms defined by its components

$$\omega p = \sum_{i=1}^{k} \omega_i p_i, \quad \omega \in (\mathbf{R}^k)^*, \quad |\omega| = 1.$$

This pencil defines a $(k-1)$-sphere S_p in $P(\mathbf{R}^N)$. It turns out that p is non-degenerate if and only if this sphere is not tangent to Π_N at any point. One can show that there exists a deformation $S_p(t)$ of S_p to the sphere of positive definite quadratic forms such that in the process of deformation there only appear isolated points of tangency with Π_N [1]. Moreover, the algebraic number of tangencies with Π_N in such deformation is well-defined [1] and we denote it by T-ind$(S_p(t), \Pi_N)$.

THEOREM 4 ([1]).

$$\text{T-ind}(S_p(t), \Pi_N) = \frac{(-1)^k}{2} \big(\chi(p^{-1}(0) \cap S^{N-1}) + (-1)^N - 1 \big).$$

As examples in [1] show, this theorem can be also used for computing $\chi\big(p^{-1}(0) \cap S^{N-1}\big)$. Furthermore, sometimes one can compute $\chi\big(p^{-1}(0) \cap S^{N-1}\big)$ along the same lines by a more direct method.

Let

$$p : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^k$$

be a real symmetric bilinear map. Let $\omega \in (\mathbf{R}^k)^*$, then $\omega p : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ is a scalar quadratic form. Suppose K is a cone in $\mathbf{R}^k$ and

$$K^o = \big\{ \omega \in (\mathbf{R}^k)^* : \omega y \leqslant 0 \text{ for every } y \in K \big\}$$

is the dual cone. Since the map $p \,|\, S^{N-1}$ is even, it defines a map $\overline{p} : \mathbf{R}\mathbf{P}^{N-1} \to \mathbf{R}^k$. For a fixed p, introduce the function $\text{ind}\,p : \omega \mapsto \text{ind}(\omega p)$ for $\omega \in (\mathbf{R}^k)^*$, where ind denotes the usual index of a scalar-valued quadratic form.

Let $\Omega = S^{k-1} \cap K^o$ and put

$$\Omega_p^n = (\text{ind}\,p)^{-1}([0, n]) \cap \Omega, \quad n \geqslant 0.$$

From the identity $K = K^{oo} = \Omega^o$ it follows that

$$\overline{p}^{-1}(K) = \overline{p}^{-1}(\Omega^o) = \{ x \in \mathbf{R}\mathbf{P}^{N-1} : \omega \overline{p}(x) \leqslant 0 \text{ for every } \omega \in \Omega \}.$$

Suppose that the latter set is a smooth manifold. Then one has another useful formula in the same spirit.

THEOREM 5 ([1]).

$$\chi\big(\overline{p}^{-1}(K)\big) = \frac{1 + (-1)^{N-1}}{2} - \sum_{n=0}^{N-1} (-1)^n \chi(\Omega_p^n).$$

Taking $K = \{0\}$, we see that this theorem gives another way of calculating the Euler characteristic $\chi(\overline{p}^{-1}(0))$. Some visual examples of its application can be found in [1]. It should be noted that methods of [1] are in fact also applicable to nonhomogeneous quadratic mappings using the standard reduction to homogeneous ones by projectivization [1]. For proper mappings like the ones appearing below in relation with configuration spaces of polygons, this reduction is completely equivalent, in most other cases it also works well.

3. Examples. As was already mentioned, proper quadratic mappings and intersections of real quadrics arise in many topics. Two natural and interesting sources of examples are provided by moment mappings of certain Poisson structures with quadratic Casimir mappings [16], [14] and configuration spaces of polygons and linkages [10], [18]. Below we consider some examples of these two types where the results of previous sections appear quite effective.

3.1. *Casimir levels of Sklyanin algebras.* Recall that an interesting class of quadratic Poisson structures on $\mathbf{R}^4$ was introduced and investigated by E. Sklyanin in [16]. As became customary, they are called Sklyanin algebras. Since our main interest lies in topological aspects of the topic, we will work over the field of reals despite the same definition apparently makes sense for arbitrary fields.

It is convenient to use notations and convention from [16]. So let us denote by x_0, x_1, x_2, x_3 coordinates on $\mathbf{R}^4$ and introduce indices i, j, k which can take values 1, 2, 3, with the convention that in formulae (4) below the triple i, j, k denotes an arbitrary cyclic permutation of 1, 2, 3 (so each formula in (4) actually represents three different formulae).

Finally, one introduces three real constants J_{12}, J_{23}, J_{31} satisfying the condition $J_{12} + J_{23} + J_{31} = 0$ and defines a Sklyanin algebra as a quadratic Poisson structure on $\mathbf{R}^4$ with the following brackets of coordinate functions:

(4) $$\{x_i,\, x_j\} = 2J_{ij}\, x_0 x_k, \quad \{x_0, x_i\} = -2x_j x_k.$$

It is well known that such a Poisson structure possesses two functionally independent quadratic Casimir mappings of the form

$$P_1 = x_0^2 + \sum_{i=1}^{3} J_i x_i^2, \quad P_2 = \sum_{i=1}^{3} x_i^2$$

where real constants J_i are chosen so that $J_i - J_j = J_{ij}$ (which is obviously possible since the sum of J_{ij} vanishes).

As is explained in [16], the properties of such a Poisson structure strongly depend on the topological structure of joint level surfaces of its two Casimir mappings called *Casimir levels*. So we come to the problem of topological description of levels of the quadratic mapping $P = (P_1, P_2) : \mathbf{R}^4 \to \mathbf{R}^2$ defined by the Casimir mappings. To this end we can apply the results of previous section and compute the Euler characteristics of Casimir levels

(5) $M_c = \{P_1 = c_1, P_2 = c_2\}, \quad c = (c_1, c_2) \in \mathbf{R^2}.$

As is easy to see, in the case of a Sklyanin algebra the pencil of quadratic forms defined by Casimir mappings always contains a positive definite form so without restriction of generality we can assume that P_1 is positive definite. Notice that in this case the Casimir levels are two-dimensional so the value of Euler characteristic determines the topological type of M_c for generic values of c, i.e., for all c such that M_c is a smooth two-dimensional surface. Note that only cases with $c_1 > 0$ are interesting because otherwise M_c is empty. Obviously, in the given situation one can solve the issue using results of Section 2. This is quite straightforward because P is a homogeneous quadratic mapping.

It is also instructive to see how one can do that quite easily using an appropriate version of results of Section 1. As was shown in [12], from Theorem 2 a similar formula follows for the global level surface of a homogeneous polynomial, which in our case has the following form.

THEOREM 6. *For $c_1 > 0$ and sufficiently small generic c_2, one has*

(6) $\chi(M_c) = 2(1 - \deg_0 \operatorname{grad} f),$

where $f := c_2 P_1 - c_1 P_2$.

Obviously by adding a constant we can achieve that all $J_i > 0$, which is equivalent to the fact that P_1 is positive definite. Using Theorem 6 we obtain for $M_c = \{P_1 = c_1, P_2 = c_2\}$:

$$f = c_2 x_0^2 + (c_2 J_1 - c_1)x_1^2 + (c_2 J_2 - c_1)x_2^2 + (c_2 J_3 - c_1)x_3^2,$$
$$\chi(M_c) = 2(1 - \deg_0 \operatorname{grad} f) = 2\left(1 - \operatorname{sgn}\left(J_1 - \frac{c_1}{c_2}\right)\left(J_2 - \frac{c_1}{c_2}\right)\left(J_3 - \frac{c_1}{c_2}\right)\right).$$

Without loss of generality we can suppose that $J_3 \leqslant J_2 \leqslant J_1$ and distinguish four cases:

  a) $c_1 \in (c_2 J_1, \infty)$,
  b) $c_1 \in (c_2 J_2, c_2 J_1)$,
  c) $c_1 \in (c_2 J_3, c_2 J_2)$,
  d) $c_1 \in (0, c_2 J_3)$.

It follows that $\chi(M_c) = 4$ in cases a) and c) and then M_c is homeomorphic to a disjoint union of two spheres S^2. In case b) $\chi(M_c) = 0$ and M_c is in fact homeomorphic to the torus T^2. In the last case d) we get an empty set $M_c = \emptyset$. Thus we obtain a complete description of the topological structure of regular (smooth) Casimir levels. These results of course agree with the description given in [16]. It is also possible to find all possible values of the Euler characteristic for singular levels using Theorem 3 but this already

requires using a computer and does not seem appropriate to be described in further detail.

3.2. *Configuration spaces of planar polygons.* Configuration spaces of polygons in Euclidean spaces naturally appear in mechanics and engineering (cf. [7]) and suggest some interesting geometric problems. In particular, the topology of such configuration spaces has recently gained considerable interest [18], [10]. For polygons with a small number of sides the description of configuration spaces becomes a geometric problem which can be effectively handled by using Morse theory [10], [7]. At the same time some topological invariants of such configuration spaces (for example, the Euler characteristic) can be effectively computed by using the results of previous sections. For simplicity we basically deal with the so-called *configuration spaces of planar polygons.*

Let P_n denote a polygon with n angles and vertices $v_1, \ldots, v_n$ in the Euclidean plane $\mathbf{R}^2$ and let l_i denote the length of the side $v_i v_{i+1}$, $i = 1, \ldots, n$, $v_{n+1} = v_1$. Informally speaking we want to consider the set of all possible positions of P_n in Euclidean space $\mathbf{R}^k$ which preserve lengths of the sides (so the sides are considered as rigid sticks). In order to achieve certain normalization (in particular, to avoid shifts of P_n as a whole), one does not distinguish two realizations of a given P_n which can be obtained from each other by an orientation preserving homothety of the plane [18].

With this convention we can assume without loss of generality that one of the sides, say the first one, has length one (by applying an appropriate homothety). Furthermore, by applying a shift and rotation one can always achieve that the first side coincides with the first unit vector of the standard basis whose endpoint $(1, 0)$ will be denoted by e (in the sequel we always assume that $l_1 = 1$). Then it is sufficient to consider only realizations of P_n with the first side "frozen" in this position.

As usual, the distance between points $a, b \in \mathbf{R}^k$ is denoted by $d(a, b)$. For convenience of notation we introduce a "virtual" vertex $v_{n+1} = v_1$.

DEFINITION 1. The *planar configuration space* $C(P_n, 2)$ of the polygon P_n is defined as the set of all maps
$$F : V \to \mathbf{R}^2$$
of the set of vertices V into $\mathbf{R}^2$ such that

   a) for each $i = 2, \ldots, n$, $d(F(v_i), F(v_{i+1})) = l_i$;

   b) $F(v_1) = 0 \, (= (0, 0))$, $F(v_2) = e \, (= (1, 0))$.

We always consider these spaces endowed with natural topologies and wonder about their possible homeomorphy types over the set of all n-gons. Notice that with our assumptions, the set of all n-gons is parametrized by a non-negative vector of lengths $l = (l_2, \ldots, l_n) \in \mathbf{R}_+^{n-1}$. We write $P_n(l)$ for the corresponding polygon. It is well known (see, e.g., [10]) that, for l in an open dense subset of the parameter space, $C(P_n(l), 2)$ is a smooth manifold. In such cases we speak of a smooth configuration space.

PROPOSITION 2. *For almost all $l \in \mathbf{R}_+^{n-1}$, $C(M_n(l), 2)$ is a smooth compact orientable manifold of dimension $n - 3$.*

An easy but useful observation is that the configuration space of a planar n-gon $P(l)$ can be represented as a fiber of quadratic mapping in the following manner. Denote by (x_j, y_j) the coordinates in $\mathbf{R}^2$ of vertex v_j, $j = 1, \ldots, n$. Then we have

(7) $$(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2 = l_j^2, \quad j = 2, \ldots, n$$

(as above we assume that $v_{n+1} = v_1$). From this we obtain that to any n-gon $P_n(l)$ there corresponds a quadratic map

$$Q : \mathbf{R}^{2(n-2)} \to \mathbf{R}^{n-1}$$

and the configuration space $C(P_n, k)$ can be represented as a preimage of the point $(l_2, \ldots, l_n)$ under this map. Here $(l_2, \ldots, l_n) \in \mathbf{R}_+^{n-1}$ denotes the fixed side lengths of a particular n-gon. Now it is clear that the above proposition follows from the Sard lemma.

Let us show now how one can describe all topological types of configuration spaces of planar quadrangles, i.e. for $n = 4$. In this case the configuration space is that of a pentagon which, in terms of (7), is determined by the quadratic equations

$$(x_1 - 1)^2 + x_2^2 = l_2^2,$$
$$(x_3 - x_1)^2 + (x_4 - x_2)^2 = l_3^2,$$
$$x_3^2 + x_4^2 = l_4^2.$$

The configuration space $C(P_4(l), 2)$ is naturally isomorphic to the full preimage of the origin under mapping $\Phi_l : \mathbf{R}^4 \to \mathbf{R}^3$ defined by the left hand sides of the above equations. Notice that this mapping is proper but it is not homogeneous so to compute the Euler characteristics of fibers we need Theorem 3 in full generality.

Following the standard paradigm of singularity theory we now need to determine the bifurcation diagram of the mapping defined by the left hand sides of the above equations. This amounts to finding those collections of side lengths l for which the configuration space of corresponding quadrangle is singular. This can be done using the following special case of a general criterion which is well known in the theory of configuration spaces [10].

PROPOSITION 3. *The quadrangle $M_4(l)$ has a singular configuration space if and only if there exist numbers $c_i \in \{-1, 1\}$, $i = 2, 3, 4$, such that*

(8) $$\sum c_i l_i = 1.$$

This means that a configuration space is singular exactly in those cases when the corresponding quadrangle can degenerate to a line segment, i.e. when the side lengths are such that all vertices can be put on the same line (the first coordinate axis in our case). Thus the bifurcation diagram $B(4, 2)$ coincides with the union of eight hyperplanes H_c in $\mathbf{R}^3$ defined by the above linear equations of the form (8). Moreover, the topological type of the configuration space behaves exactly in such way as predicted by singularity theory.

PROPOSITION 4. *The topological type of the configuration space does not change on each component of the complement to $B(4, 2)$.*

We can now determine all possible values of the Euler characteristic of configuration spaces. In view of said above in order to do that, it is sufficient to choose one point

in each component of the complement to the bifurcation diagram and in each of its faces on hyperplanes H_c (a face is defined as a component of the complement in H_c of $(B(4,2) - H_c) \cap H_c$) and compute the Euler characteristics of their preimages using Theorem 3. For smooth configuration spaces we do not get of course anything interesting since for dimension reasons they are homeomorphic to disjoint unions of several circles. From the general theory it is known that the number of such circles can be either one or two and one can distinguish these two cases by checking some linear inequalities for side lengths [10]. So a smooth configuration space can be homeomorphic to a circle (for quadrangles with a "long" third side, like $(1, 1, 2, 1)$) or homeomorphic to a disjoint union of two circles (for quadrangles with a "short" third side, like $(1, 1, 0.5, 1)$).

However for singular configuration spaces the answer is not quite trivial. Namely, by choosing points in all faces of $B(4,2)$ we find out that the Euler characteristic $\chi(C(P_4(l), 2))$ can equal -1, -2, and -3. For example, the first value is realized for lengths $(1, 1, 1.5, 0.5)$, the second for $(1, 2, 1, 2)$, and the third for $(1, 1, 1, 1)$ (square). It is known that singular configuration spaces have singularities of quadratic cone type [10] so in this case we just need to know how many branches each singular point has. This can be done using a general method for computing the number of branches of one-dimensional analytic sets (see, e.g., [12]) and we arrive to the final result.

PROPOSITION 5. *All possible topological types of the non-singular configuration spaces $C(M_4, 2)$ are contained in the following list: circle, disjoint union of two circles, two circles glued at one point (bouquet of two circles), two circles glued at two points, two circles glued at three points.*

It is not difficult to verify that configuration spaces of the three last types are realized exactly for those three collections of side lengths which were given before the theorem.

REMARK 3. This result is now new but the known proofs used Morse theory or some other nontrivial geometric considerations (cf. [7], [10]) while with our approach it emerges in a purely algorithmic way.

The same approach appears quite effective for planar pentagons because the non-singular fibers in this case are compact closed orientable surfaces and their topological types are completely determined by values of the Euler characteristic. It is easy to check by our method that, for smooth fibers the Euler characteristic can equal -6, -4, -2, 0, 2 and we arrive to another well-known result (cf. [10]).

PROPOSITION 6. *All possible topological types of the non-singular configuration spaces $C(P_5, 2)$ are contained in the following list: $S^0 \times S^1 \times S^1$ (disjoint union of two tori $\mathbf{T}^2$); compact closed orientable surface M_g^2 of genus g not exceeding four.*

By the same method it is also possible to find the topological types of singular fibers corresponding to the points of the bifurcation diagram. In particular, it turns out that the Euler characteristic can be any integer in the segment $[-6, 2]$. The final result requires a rather boring description of occurring homeomorphy types and is omitted (it can be also extracted from results of [10]).

3.3. *Configuration spaces of spatial quadrangles.* One can also consider realizations of a given polygon in higher-dimensional spaces. This leads to a more general concept of configuration space. The standard definition is as follows. Realizations of a given P_n in $\mathbf{R}^k$ are defined as the set $R(n, k)$ of all systems of n points such that distances between them taken in cyclic order coincide with the side lengths of P_n. This set is factored over the natural action of the group $\mathrm{Iso}^+(\mathbf{R}^k)$ of orientation preserving isometries of $\mathbf{R}^k$.

DEFINITION 2. *The k-th configuration space $C(P_n, k)$ is defined as $R(n, k)/\mathrm{Iso}^+(\mathbf{R}^k)$.*

For $k = 3$, these spaces are called configuration spaces of *spatial polygons*. Notice that these spaces are not a priori related to quadratic mappings but this can be arranged quite easily. Let us confine to spatial quadrangles, i.e. take $n = 3$.

Notice that one can always place the first side along the first coordinate axis by an isometry and further rotate the whole configuration so that the fourth side lies in the plane orthogonal to the second coordinate axis. Then it is sufficient to consider only such realizations of P_n in $\mathbf{R}^3$. Obviously, there are five "free parameters"—three coordinates of the second vertex and two coordinates of the third vertex (its ordinate always vanishes by our choice)—and three quadratic relations of the same type as above so we end up with a quadratic mapping from $\mathbf{R}^5$ to $\mathbf{R}^3$.

Its fibers are two-dimensional, so in order to determine the configuration space $C(P_4(l), 3)$, it is sufficient to compute the Euler characteristic of the fiber over the origin which can be done using Theorem 3. For example, for a square we readily obtain the value of Euler characteristic equal to two and conclude that its spatial configuration space is homeomorphic to the sphere S^2. Applying the same reasoning with the bifurcation diagram as above, we eventually conclude that this value is the same for all spatial quadrangles.

PROPOSITION 7. *The configuration space of any spatial quadrangle is homeomorphic to two-dimensional sphere S^2.*

In the same way one can compute the Euler characteristics of configuration spaces of spatial regular polygons with any number of sides.

4. Concluding remarks. Our arguments can be used to calculate the Euler characteristics of Casimir levels for the so-called generalized Sklyanin algebras considered in [14]. There also arises a quadratic map $P = (P_1, P_2) : \mathbf{R}^4 \to \mathbf{R}^2$ and for the generic $\mathbf{c} = (c_1, c_2)$ the Euler characteristic of the algebraic variety $M_{\mathbf{c}} = P^{-1}(\mathbf{c})$ determines the topological type of $M_{\mathbf{c}}$. One can again find all possible values of the Euler characteristic of $M_{\mathbf{c}}$ and their topological types using the methods of preceding sections. In particular, for smooth Casimir fibers it can be shown (see [13]) that $\chi(M_{\mathbf{c}})$ may only equal 4 or 0 as above.

As to configuration spaces, by these methods one can compute the Euler characteristic in many interesting cases, for example, for spatial regular polygons. One can also identify some of two-dimensional configuration spaces of planar linkages (see [18] for the corresponding definitions and basic results concerned with planar linkages).

Finally, the moduli spaces of complex structures on a compact Riemann surface with n distinct marked points can also be investigated using the formulae for the Euler characteristic. Indeed, according to [9] the construction of such moduli spaces given by Deligne-Mostow [5] can be interpreted as the configuration space of a stable polygon in Euclidean 3-space. Thus the methods described above permit one to compute the Euler characteristics of the latter moduli spaces as well.

These and other applications of the methods described above will be discussed in further publications of the author. The author is grateful to Professor G. Khimshiashvili for emphasizing his attention on signature formulae and the possibility of their use for investigation of configuration spaces in the spirit of [13]. The author is also grateful to Professor D. Siersma for pointing at the paper [7] and several useful discussions. The last but not the least, thanks are addressed to the unknown referee for several valuable comments and helpful suggestions.

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