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ON RAMIFICATION LOCUS OF A POLYNOMIAL MAPPING

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Abstract. Let X be a smooth algebraic hypersurface in $\mathbb{C}^n$. There is a proper polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$, such that the set of ramification values of F contains the hypersurface X.

Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial proper mapping. We say that a point $y \in \mathbb{C}^n$ is a ramification value of F if $\#F^{-1}(y) < \mu(F)$, where $\mu(F) = (\mathbb{C}(x_1, \dots, x_n) : \mathbb{C}(F_1, \dots, F_n))$ denotes the geometric degree of F. The set of all ramification values of F (which always is a hypersurface or the empty set) is called the ramification locus of F. This set will be denoted by $RL(F)$. Some properties of this set were studied in [1] and [2].

In particular in the paper [2] we showed that for a given hypersurface $X \subset \mathbb{C}^n$ there is a proper polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ and a hypersurface X', which is birationally equivalent to X, such that $X' \subset RL(F)$.

However sometimes it is interesting to know whether a hypersurface X itself is a component of the set $RL(F)$ (for some proper mapping F). In particular Professor R. V. Gurjar asked whether the hyperbola $\Gamma = \{(x, y) \in \mathbb{C}^2 : xy = 1\}$ can be a component of the set $RL(F)$ (for a proper mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$). In this note we give the following general answer to this question.

Theorem 1. *Let X be a smooth algebraic hypersurface in $\mathbb{C}^n$. There is a proper polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$, such that $X \subset RL(F)$.*

Proof. Of course, we can assume that $n > 1$. First we construct a mapping $G : \mathbb{C}^n \to \mathbb{C}^n$ such that

a) $G(x_1, \dots, x_n) = (x_1, \dots, x_n)$ on X,
b) $\mathrm{Jac}(G)(x) = 0$ for $x \in X$.

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Let $f=0$ be a reduced equation for X. Since X is smooth, polynomials $\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}$ have no common zeroes on X. Let $f_i=\frac{\partial f}{\partial x_i}$. We look for G among mappings of the form $G=(x_1+\alpha_1 f,\dots,x_n+\alpha_n f)$. We choose α_i in such a way that condition b) be satisfied. It is easy to see that on X the Jacobian of G is equal to

$$\begin{vmatrix} 1+\alpha_1 f_1 & \alpha_1 f_2 & \dots & \alpha_1 f_n \\ \alpha_2 f_1 & 1+\alpha_2 f_2 & \dots & \alpha_2 f_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n f_1 & \alpha_n f_2 & \dots & 1+\alpha_n f_n \end{vmatrix} = 1+\sum_{i=1}^n \alpha_i f_i.$$

Since polynomials $f_1,\dots,f_n$ have no common zeroes on X, by the Hilbert Nullstellensatz there exist polynomials A_i such that

$$1=\sum_{i=1}^n A_i f_i \qquad \text{on } X.$$

Take $\alpha_i=-A_i$. Of course the mapping $G=(x_1+\alpha_1 f,\dots,x_n+\alpha_n f)$ satisfies conditions a) and b). Let $G=(G_1,\dots,G_n)$ and let g denote the restriction of the mapping G to X. The mapping g is the identity (on X), this implies that polynomials x_i-G_i vanish on X. Take

$$\Phi:\mathbb{C}^n \ni x \to (G(x),(x_1-G_1)^2,\dots,(x_n-G_n)^2)\in \mathbb{C}^n\times\mathbb{C}^n.$$

By construction the mapping Φ is finite, $\Phi(X)\subset \mathbb{C}^n\times\{0\}$ and $\dim\Phi(\mathbb{C}^n)=n$. Moreover, we have $d_x(\Phi)=(d_xG,0)$ for $x\in X$. Let $\pi:\Phi(X)\to\mathbb{C}^n\times\{0\}$ be a generic linear projection and let $F=\pi\circ\Phi$. Then the mapping F is finite and $F(x)=G(x)$, $d_xF=d_xG$, for $x\in X$. In particular the mapping F ramifies on X and $F(X)=X$. This means that $X\subset RL(F)$. ∎

Example 1. In the case of the hyperbola $\Gamma=\{(x,y)\in\mathbb{C}^2 : xy=1\}$ our method gives:

$$F(x,y)=(x+(x-x^2y),y+(x-x^2y)^2).$$

In fact it is not difficult to check directly that the mapping F is proper and $\Gamma\subset RL(F)$.

Question. Let X be *an arbitrary hypersurface* in $\mathbb{C}^n$. Is it true that there exists a proper polynomial mapping $F:\mathbb{C}^n\to\mathbb{C}^n$ such that $X\subset RL(F)$?

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References

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