

# GAUDIN'S MODEL AND THE GENERATING FUNCTION OF THE WROŃSKI MAP

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**Abstract.** We consider the Gaudin model associated to a point  $z \in \mathbb{C}^n$  with pairwise distinct coordinates and to the subspace of singular vectors of a given weight in the tensor product of irreducible finite-dimensional  $sl_2$ -representations, [G]. The Bethe equations of this model provide the critical point system of a remarkable rational symmetric function. Any critical orbit determines a common eigenvector of the Gaudin hamiltonians called a Bethe vector.

In [ReV], it was shown that for generic  $z$  the Bethe vectors span the space of singular vectors, i.e. that the number of critical orbits is bounded from below by the dimension of this space. The upper bound by the same number is one of the main results of [SV].

In the present paper we get this upper bound in another, “less technical”, way. The crucial observation is that the symmetric function defining the Bethe equations can be interpreted as the generating function of the map sending a pair of complex polynomials into their Wroński determinant: the critical orbits determine the preimage of a given polynomial under this map. Within the framework of the Schubert calculus, the number of critical orbits can be estimated by the intersection number of special Schubert classes. Relations to the  $sl_2$  representation theory ([F]) imply that this number is the dimension of the space of singular vectors.

We prove also that the spectrum of the Gaudin hamiltonians is simple for generic  $z$ .

**1. Introduction.** The Gaudin model of statistical mechanics is a completely integrable quantum spin chain associated to the Lie algebra  $sl_2 = sl_2(\mathbb{C})$ , [G]. Denote by  $L_{\lambda_j}$  the irreducible  $sl_2$ -module with highest weight  $\lambda_j \in \mathbb{C}$ . The space of states of the model is the tensor product  $L = L_{\lambda_1} \otimes \dots \otimes L_{\lambda_n}$ . Associate with any  $L_{\lambda_j}$  a complex number  $z_j$ , and assume  $z_1, \dots, z_n$  to be pairwise distinct,  $z = (z_1, \dots, z_n)$ . The Gaudin hamiltonians  $\mathcal{H}_1(z), \dots, \mathcal{H}_n(z)$  are mutually commuting linear operators on  $L$  which are defined as follows:

$$\mathcal{H}_j(z) = \sum_{i \neq j} \frac{C_{ij}}{z_j - z_i}, \quad j = 1, \dots, n,$$

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where  $C_{ij}$  is the operator which acts as the Casimir element on the  $i$ -th and  $j$ -th factors of  $L$  and as the identity on the others.

One of the main problems in the Gaudin model is to find common eigenvectors and eigenvalues of the Gaudin hamiltonians, and the algebraic Bethe Ansatz is one of the most effective methods for solving this problem. The idea of this method is to find some function with values in the space of states, and to determine a certain special value of its argument in such a way that the value of this function is an eigenvector. The equations which determine these special values of the argument are called the *Bethe equations*, and the common eigenvector corresponding to a solution of the Bethe equations, is called the *Bethe vector*, [FaT].

It is enough to diagonalize the Gaudin hamiltonians in the subspace of singular vectors of a given weight. The Bethe equations associated to the space  $\text{Sing}_k(L)$  of singular vectors of the weight  $\lambda_1 + \dots + \lambda_n - 2k$ , where  $k$  is a positive integer, have the form

$$\sum_{l=1}^n \frac{\lambda_l}{t_i - z_l} = \sum_{j \neq i} \frac{2}{t_i - t_j}, \quad i = 1, \dots, k.$$

This system is symmetric with respect to permutations of unknowns  $t_1, \dots, t_k$ , and any solution  $t^0 = (t_1^0, \dots, t_k^0)$  to this system defines an eigenvector  $v(t^0) \in \text{Sing}_k(L)$  of the Gaudin hamiltonians  $\{\mathcal{H}_j(z)\}$ , [G].

A conjecture related to the Bethe Ansatz says that the Bethe vectors give a basis of the space of states. For the  $sl_2$  Gaudin model, the conjecture was proved in [ReV] in the case of generic  $z$  and generic (non-resonant) weights  $\lambda_1, \dots, \lambda_n$ . In the case of *integral dominant* weights, which is a non-generic one, results of Sec. 9 of [ReV] imply only that for generic  $z$  the number of Bethe vectors is *at least* the dimension of the space of singular vectors, see also Theorem 8 in [SV]. The proof of the Bethe Ansatz conjecture for the  $sl_2$  Gaudin model was completed in our work with A. Varchenko [SV], where the bound from above for the number of the Bethe vectors by the dimension of the space of singular vectors was obtained.

In [SV], we studied the function

$$\Phi(t) = \prod_{i=1}^k \prod_{l=1}^n (t_i - z_l)^{-\lambda_l} \prod_{1 \leq i < j \leq k} (t_i - t_j)^2.$$

If all  $\lambda_l$  are positive integers, then this function is rational, and the Bethe equations are exactly the equations on the critical points of this function with non-zero critical values,

$$\frac{1}{\Phi(t)} \cdot \frac{\partial \Phi}{\partial t_i}(t) = 0, \quad i = 1, \dots, k.$$

The critical points split into orbits of the permutation group. One of the main results of [SV] was as follows.

CLAIM. *The number of critical orbits with non-zero critical values of the function  $\Phi(t)$  is at most the dimension of  $\text{Sing}_k$ .*

Our proof was difficult, see Theorems 9–11 of [SV]. As we pointed out in Sec. 1.4 of [SV], the critical orbits are labeled by certain two-dimensional planes in the linear space

of complex polynomials. This observation suggests to apply the Schubert calculus in order to prove the Claim. In the present paper, this approach is realized via the *Wroński map*.

The Wroński determinant of two polynomials  $f(x), g(x)$  in one variable is a non-zero polynomial iff  $f(x)$  and  $g(x)$  are linearly independent, i.e. span a two-dimensional plane in the linear space of polynomials. Another basis of the same plane has the same Wrońskian, up to a non-zero constant factor. This defines a map from the Grassmannian of two-dimensional planes of complex polynomials to the space of monic polynomials called the Wroński map.

The function  $\Phi(t)$  turns out to be the generating function of the Wroński map: the critical orbits label elements in the preimage under this map of the polynomial

$$W(x) = (x - z_1)^{\lambda_1} \cdots (x - z_n)^{\lambda_n} .$$

In fact, this is a reformulation of a classical result of Heine and Stieltjes, see Ch. 6.8 of [Sz].

To calculate the cardinality of a preimage of the Wroński map is a problem of enumerative algebraic geometry, and an upper bound can be easily obtained in terms of the intersection number of special Schubert classes. A well-known relation between representation theory and the Schubert calculus (see [F]) implies that the obtained upper bound coincides with the dimension of  $\text{Sing}_k(L)$ .

In Sec. 2 relevant facts related to Bethe Ansatz in the Gaudin model are collected. In Sec. 3 we show that the function defining the Bethe equations is the generating function of the Wroński map, and in Sec. 4 we obtain an upper bound for the number of critical orbits of the generating function by the intersection number of special Schubert classes.

Another conjecture related to the Gaudin model says that for generic  $z$  the Gaudin hamiltonians have a simple spectrum. In [ReV] it was proved that Bethe vectors differ by their eigenvalues for generic  $\lambda_1, \dots, \lambda_n$  and for real  $z$  of the form  $z_j = s^j$ , where  $s \gg 1$ . If it is known that the Bethe vectors form a basis, then the simplicity of the spectrum follows.

In Sec. 5 we deduce the simplicity of the spectrum of the Gaudin hamiltonians for generic  $z$  and for integral dominant  $sl_2$  weights (which are non-generic) from the relation to linear differential equations.

The arguments which are presented here work and give the similar results for the  $sl_p$  Gaudin model associated with the tensor product of symmetric powers of the standard  $sl_p$ -representation, see Sec. 5 of [S1]. The link between the Bethe vectors and the Wroński map seems to be useful for study rational maps; see [S2] where the case of rational functions is treated.

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**2. Gaudin’s model.** We collect here some known facts on the Bethe Ansatz in Gaudin’s model, following [G], [FaT], [Fr], [ReV], [SV].

**2.1. The Gaudin hamiltonians.** Assume that  $m_1, \dots, m_n$  are nonnegative integers,  $M = (m_1, \dots, m_n)$ . Denote by  $L_{m_j}$  the irreducible  $sl_2$ -module with highest weight  $m_j$ .

The *space of states* of the Gaudin model is the tensor product

$$(1) \quad L^{\otimes M} = L_{m_1} \otimes \dots \otimes L_{m_n}.$$

Associate with any  $L_{m_j}$  a complex number  $z_j^0$ , and assume  $z_1^0, \dots, z_n^0$  to be pairwise distinct,  $z^0 = (z_1^0, \dots, z_n^0)$ . Let  $e, f, h$  be the standard generators of  $sl_2$ ,

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$$

and

$$C = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h \in sl_2 \otimes sl_2$$

the Casimir element. For  $1 \leq i < j \leq n$ , denote by  $C_{ij} : L^{\otimes M} \rightarrow L^{\otimes M}$  the operator which acts as  $C$  on the  $i$ -th and  $j$ -th factors of  $L^{\otimes M}$  and as the identity on all others. The *hamiltonians* of the Gaudin model are defined as follows:

$$(2) \quad \mathcal{H}_i(z^0) = \sum_{j \neq i} \frac{C_{ij}}{z_i^0 - z_j^0}, \quad i = 1, \dots, n.$$

**2.2. Subspaces of singular vectors.** Let  $|M| = m_1 + \dots + m_n$ . For a nonnegative integer  $k$  such that  $|M| - 2k \geq 0$ , define  $\text{Sing}_k$ , the subspace of singular vectors of weight  $|M| - 2k$  in  $L^{\otimes M}$ ,

$$(3) \quad \text{Sing}_k = \text{Sing}_k(L^{\otimes M}) = \{w \in L^{\otimes M} \mid ew = 0, hw = (|M| - 2k)w\}.$$

THEOREM 1 ([SV], Theorem 5). *We have*

$$\dim \text{Sing}_k(L^{\otimes M}) = \sum_{q=0}^n (-1)^q \sum_{1 \leq i_1 < \dots < i_q \leq n} \binom{k + n - 2 - m_{i_1} - \dots - m_{i_q} - q}{n - 2},$$

here we set  $\binom{a}{b} = 0$  for  $a < b$ .

The subspace  $\text{Sing}_k$  is an invariant subspace of the Gaudin hamiltonians for any  $0 \leq k \leq |M|/2$ , and the singular vectors generate the whole of  $L^{\otimes M}$ . Therefore it is enough to diagonalize Gaudin hamiltonians in a given  $\text{Sing}_k$ .

**2.3. Bethe equations associated to  $\text{Sing}_k$  and  $z^0$ .** The Bethe equations for the Gaudin model associated to  $\text{Sing}_k$  and  $z^0$  have the form

$$(4) \quad \sum_{l=1}^n \frac{m_l}{t_i - z_l^0} = \sum_{j \neq i} \frac{2}{t_i - t_j}, \quad i = 1, \dots, k.$$

Any solution  $t^0 = (t_1^0, \dots, t_k^0)$  to this system defines an eigenvector  $v(t^0) \in \text{Sing}_k$  of the Gaudin hamiltonians,  $\mathcal{H}_j(z^0)v(t^0) = \mu_j v(t^0)$ , with eigenvalues

$$(5) \quad \mu_j = \mu_j(t^0) = \sum_{l \neq j} \frac{m_l m_j}{2(z_j^0 - z_l^0)} - \sum_{i=1}^k \frac{m_j}{z_j^0 - t_i^0}, \quad j = 1, \dots, n.$$

**2.4. The master function of the Gaudin model.** Consider the following function in variables  $t = (t_1, \dots, t_k)$  and  $z = (z_1, \dots, z_n)$ ,

$$(6) \quad \Psi(t, z) = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{m_i m_j / 2} \prod_{i=1}^k \prod_{l=1}^n (t_i - z_l)^{-m_i} \prod_{1 \leq i < j \leq k} (t_i - t_j)^2,$$

defined on

$$(7) \quad \mathcal{C} = \mathcal{C}(t; z) = \{t \in \mathbb{C}^k, z \in \mathbb{C}^n \mid t_i \neq z_j, t_i \neq t_l, z_j \neq z_q, 1 \leq i \neq l \leq k, 1 \leq j \neq q \leq n\}.$$

Consider the (multivalued) function

$$(8) \quad \begin{aligned} S(t, z) &= \ln \Psi(t, z) \\ &= \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{2} \ln(z_i - z_j) - \sum_{i=1}^k \sum_{l=1}^n m_l \ln(t_i - z_l) + \sum_{1 \leq i < j \leq k} 2 \ln(t_i - t_j). \end{aligned}$$

The both functions  $S(t, z)$  and  $\Psi(t, z)$  have clearly the same critical set in  $\mathcal{C}(t; z)$ . Define

$$Z = \{z \in \mathbb{C}^n \mid z_j \neq z_q, 1 \leq j \neq q \leq n\}.$$

Let  $\pi : \mathcal{C} \rightarrow Z$  be the natural projection,  $\pi(t, z) = z$ . For fixed  $z^0 \in Z$ , the equations (4) form a critical point system of the function  $S(t, z^0)$  considered as a function in  $t$  on  $\pi^{-1}(z^0)$ ,

$$\frac{\partial S}{\partial t_i}(t, z^0) = 0, \quad i = 1, \dots, k,$$

and the equalities (5) can be re-written in the form

$$\mu_j = \frac{\partial S}{\partial z_j}(t^0, z^0), \quad j = 1, \dots, n,$$

where  $t^0 = t^0(z^0)$  is a solution to the Bethe equations. In terms of the function  $\Psi(t, z)$ , the Bethe equations (4) are

$$\frac{1}{\Psi(t, z^0)} \cdot \frac{\partial \Psi}{\partial t_i}(t, z^0) = 0, \quad i = 1, \dots, k,$$

and

$$\mu_j = \frac{1}{\Psi(t^0, z^0)} \cdot \frac{\partial \Psi}{\partial z_j}(t^0, z^0), \quad j = 1, \dots, n.$$

**2.5. The Shapovalov form.** Let  $m$  be a nonnegative integer. For the  $sl_2$ -module  $L_m$  with highest weight  $m$ , fix the highest weight singular vector

$$v_m \in L_m, \quad hv_m = mv_m, \quad ev_m = 0.$$

Denote by  $B_m$  the unique bilinear symmetric form on  $L_m$  such that

$$B_m(v_m, v_m) = 1, \quad B_m(hx, y) = B_m(x, hy), \quad B_m(ex, y) = B_m(x, fy)$$

for all  $x, y \in L_m$ . The vectors  $v_m, fv_m, \dots, f^m v_m$  are orthogonal with respect to  $B_m$  and form a basis of  $L_m$ .

The bilinear symmetric form on  $L^{\otimes M}$  given by

$$(9) \quad B = B_{m_1} \otimes \dots \otimes B_{m_n}$$

is called the *Shapovalov form*.

Let  $j_1, \dots, j_n$  be integers such that  $0 \leq j_i \leq m_i$  for any  $1 \leq i \leq n$ . Write  $J = (j_1, \dots, j_n)$  and  $|J| = j_1 + \dots + j_n$ . Let

$$(10) \quad f^J v_M = f^{j_1} v_{m_1} \otimes \dots \otimes f^{j_n} v_{m_n}.$$

The vectors  $\{f^J v_M\}$  are orthogonal with respect to the Shapovalov form  $B$  and provide a basis of the space  $L^{\otimes M}$ . We have

$$h(f^J v_M) = (|M| - 2|J|)f^J v_M, \quad e(f^J v_M) = 0,$$

i.e. the vector  $f^J v_M$  is a *singular vector of weight*  $|M| - 2|J|$ . The space  $\text{Sing}_k$  is generated by the vectors  $f^J v_M$  with  $|J| = k$ .

**2.6. Bethe vectors.** For  $J = (j_1, \dots, j_n)$  with integer coordinates such that  $0 \leq j_i \leq m_i$  and  $|J| = k$ , and for  $(t, z)$  in  $\mathcal{C}$  given by (7) set

$$A_J(t, z) = \frac{1}{j_1! \cdots j_n!} \text{Sym}_t \left[ \prod_{l=1}^n \prod_{i=1}^{j_l} \frac{1}{t_{j_1 + \dots + j_{l-1} + i} - z_l} \right],$$

where

$$\text{Sym}_t F(t) = \sum_{\sigma \in S^k} F(t_{\sigma(1)}, \dots, t_{\sigma(k)})$$

is the sum over all permutations of  $t_1, \dots, t_k$ .

THEOREM 2 ([ReV]).

(i) If  $t^{(i)}$  is a nondegenerate critical point of the function  $S(t, z^0)$ , given by (8), then the vector

$$(11) \quad v(t^{(i)}, z^0) = \sum_{J: |J|=k} A_J(t^{(i)}, z^0) f^J v_M$$

is an eigenvector of the operators  $\mathcal{H}_1(z^0), \dots, \mathcal{H}_n(z^0)$ .

(ii) For generic  $z^0$ , the eigenvectors  $v(t^{(i)}, z^0)$  corresponding to all critical point of the function  $S(t, z^0)$  generate  $\text{Sing}_k$ .

The words “generic  $z^0$ ” mean that  $z^0$  does not belong to a suitable proper algebraic set of  $\mathbb{C}^n$ .

The set of critical points of the function  $S(t, z^0)$  is invariant with respect to the permutations of  $t_1, \dots, t_k$ , and critical points belonging to the same orbit clearly define the same vector. Theorem 2 gives a lower bound for the number of critical orbits of the function  $S(t, z^0)$  by the dimension of  $\text{Sing}_k$ . In [SV] the upper bound by the same number was obtained.

THEOREM 3 ([SV]). For fixed generic  $z^0$ , all critical points of the function  $S(t, z^0)$  are nondegenerate, and the number of orbits of critical points is at most the dimension of  $\text{Sing}_k$ .

COROLLARY 1 ([SV]). For generic  $z^0$ , the Bethe vectors of the  $sl_2$  Gaudin model form a basis in the space of singular vectors of a given weight.

The statement that all critical points of the function  $S(t, z^0)$  are nondegenerate is easy, see Theorem 6 in [SV]. The difficult part of [SV] was to estimate from above the number of critical orbits (see Theorems 9–11 in [SV]), i.e. to prove the following statement.

PROPOSITION. The number of orbits of critical points of the function  $S(t, z^0)$  is at most the dimension of  $\text{Sing}_k$ .

This statement is clearly equivalent to the Claim of the Introduction. As we will see below, within the framework of the Schubert calculus, the Proposition is a direct and immediate corollary of easy Theorem 7 of Sec. 4.2.

**3. The generating function of the Wroński map**

**3.1. The Wroński map.** Let  $\text{Poly}_d$  be the vector space of complex polynomials of degree at most  $d$  in one variable. Denote by  $G_2(\text{Poly}_d)$  the Grassmannian of two-dimensional planes in  $\text{Poly}_d$ . The complex dimension of  $G_2(\text{Poly}_d)$  is  $2d - 2$ .

For any  $V \in G_2(\text{Poly}_d)$  define the *degree* of  $V$  as the maximal degree of its polynomials and the *order* of  $V$  as the minimal degree of its non-zero polynomials. Let  $V \in G_2(\text{Poly}_d)$  be a plane of order  $a$  and of degree  $b$ . Clearly  $0 \leq a < b \leq d$ . Choose in  $V$  two monic polynomials,  $F(x)$  and  $G(x)$ , of degrees  $a$  and  $b$  respectively. They form a basis of  $V$ . The *Wrońskian* of  $V$  is defined as the monic polynomial

$$W_V(x) = \frac{F'(x)G(x) - F(x)G'(x)}{a - b}.$$

The following lemma is evident.

LEMMA 1.

- (i) *The degree of  $W_V(x)$  is  $a + b - 1 \leq 2d - 2$ .*
- (ii) *The polynomial  $W_V(x)$  does not depend on the choice of a monic basis.*
- (iii) *All polynomials of degree  $a$  in  $V$  are proportional.*

Thus the mapping sending  $V \in G_2(\text{Poly}_d)$  to  $W_V(x)$  is a well-defined map from the Grassmannian  $G_2(\text{Poly}_d)$  to  $\mathbb{C}\mathbb{P}^{2d-2}$ . We call it the *Wroński map*. This is an algebraic map between smooth complex algebraic varieties of the same dimension, and hence the preimage of any polynomial consists of a finite number of planes. On Wroński maps see, e.g., [EGa].

**3.2. Planes with a given Wrońskian**

LEMMA 2. *Any element of  $G_2(\text{Poly}_d)$  with a given Wrońskian is uniquely determined by any of its polynomial.*

*Proof.* Let  $W(x)$  be the Wrońskian of a plane  $V \in G_2(\text{Poly}_d)$  and  $f(x) \in V$ . Take any polynomial  $g(x) \in V$  linearly independent with  $f(x)$ . The plane  $V$  is the solution space of the following second order linear differential equation with respect to unknown function  $u(x)$ ,

$$\begin{vmatrix} u(x) & f(x) & g(x) \\ u'(x) & f'(x) & g'(x) \\ u''(x) & f''(x) & g''(x) \end{vmatrix} = 0.$$

The Wrońskian of the polynomials  $f(x)$  and  $g(x)$  is proportional to  $W(x)$ , therefore this equation can be re-written in the form

$$W(x)u''(x) - W'(x)u'(x) + h(x)u(x) = 0,$$

where

$$h(x) = \frac{-W(x)f''(x) + W'(x)f'(x)}{f(x)},$$

as  $f(x)$  is clearly a solution to this equation. ■

We call a plane  $V \in G_2(\text{Poly}_d)$  *generic* if for any  $x_0 \in \mathbb{C}$  there is a polynomial  $P(x) \in V$  such that  $P(x_0) \neq 0$ . In a generic plane, the polynomials of any basis do not have common roots, and almost all polynomials of the bigger degree do not have multiple roots.

The following lemma is evident.

LEMMA 3. *Let  $V \in G_2(\text{Poly}_d)$  be a generic plane.*

(i) *If  $P(x) = (x - x_1) \cdots (x - x_l) \in V$  is a polynomial without multiple roots, then  $W_V(x_i) \neq 0$  for all  $1 \leq i \leq l$ .*

(ii) *If  $x_0$  is a root of multiplicity  $\mu > 1$  of a polynomial  $Q(x) \in V$ , then  $x_0$  is a root of  $W_V(x)$  of multiplicity  $\mu - 1$ .*

A generic plane  $V$  is *nondegenerate* if the polynomials of the smaller degree in  $V$  do not have multiple roots.

LEMMA 4. *Let  $V \in G_2(\text{Poly}_d)$  be a nondegenerate plane. If the Wrońskian  $W_V(x)$  has the form*

$$W_V(x) = x^m \tilde{W}(x), \quad \tilde{W}(0) \neq 0,$$

*then there exists a polynomial  $F_0(x) \in V$  of the form*

$$F_0(x) = x^{m+1} \tilde{F}(x), \quad \tilde{F}(0) \neq 0.$$

*Proof.* Let  $G(x) \in V$  be a polynomial of the smaller degree. We have  $G(0) \neq 0$ , due to Lemma 3. Let  $F(x) \in V$  be a polynomial of the bigger degree. The polynomial

$$F_0(x) = F(x) - \frac{F(0)}{G(0)} G(x) \in V$$

satisfies  $F_0(0) = 0$  and therefore has the form

$$F_0(x) = x^l \tilde{F}(x), \quad \tilde{F}(0) \neq 0,$$

for some integer  $l \geq 1$ . The polynomials  $G$  and  $F_0$  form a basis of  $V$ , therefore the polynomial  $F'_0(x)G(x) - F_0(x)G'(x)$  is proportional to  $W_V(x)$ . The smallest degree term in this polynomial is  $l \cdot a_l \cdot G(0)x^{l-1}$ , where  $a_l$  is the coefficient of  $x^l$  in  $F_0(x)$ . Therefore  $l = m + 1$ . ■

**3.3. Nondegenerate planes in  $\text{Poly}_d$  with a given Wrońskian.** Recall that the *resultant*  $\text{Res}(P, Q)$  of polynomials  $P(x)$  and  $Q(x)$  is an irreducible integral polynomial in the coefficients of  $P(x)$  and  $Q(x)$  which vanishes whenever  $P(x)$  and  $Q(x)$  have a common root, and the *discriminant*  $\Delta(P)$  of  $P(x)$  is an irreducible integer polynomial in the coefficients of  $P(x)$  which vanishes whenever  $P(x)$  has a multiple root.

Our aim is to describe the nondegenerate planes of a given order in the preimage of the polynomial

$$(12) \quad W(x) = (x - z_1)^{m_1} \cdots (x - z_n)^{m_n}.$$

under the Wroński map.

If plane  $V$  of order  $k$  and of degree  $d$  has the Wrońskian  $W(x)$ , then  $d = |M|+1-k > k$ . Let  $F(x)$  be an unknown polynomial of degree  $1 \leq k \leq |M|/2$ . Consider the function

$$\Phi = \Phi(F; W) = \frac{\Delta(F)}{\text{Res}(W, F)}.$$

This function appeared originally in an unpublished note of V. Zakalyukin on the Wroński map, [Z].

If the polynomial  $F(x)$  belongs to a plane in  $\text{Poly}_{|M|+1-k}$  with the Wrońskian  $W(x)$ , then  $W(x) = F'(x)Q(x) - F(x)Q'(x)$  for some polynomial  $Q(x)$ . Differentiating gives  $W'(x) = F''(x)Q(x) - F(x)Q''(x)$ , and we have

$$\frac{W'}{W} = \frac{F''Q - FQ''}{F'Q - FQ'}.$$

If the plane spanned by  $F(x)$  and  $G(x)$  is non-degenerate, then polynomials  $F$  and  $Q$  do not have common roots, polynomial  $F$  does not have multiple roots, and polynomials  $W$  and  $F$  do not have common roots, by Lemmas 3 and 4. Therefore at each root  $t_i$  of  $F$  we get

$$(13) \quad \frac{W'(t_i)}{W(t_i)} = \frac{F''(t_i)}{F'(t_i)}, \quad i = 1, \dots, k.$$

**3.4. Theorem of Heine-Stieltjes.** In order to re-write down the function  $\Phi$  as a function in unknown roots of  $F(x)$ , recall that if  $F(x) = (x - t_1) \cdots (x - t_k)$  and if  $W(x)$  is as in (12), then

$$\Delta(F) = \prod_{1 \leq i < j \leq k} (t_i - t_j)^2, \quad \text{Res}(W, F) = \prod_{i=1}^k \prod_{j=1}^n (t_i - z_j)^{m_j}.$$

We have

$$(14) \quad \Phi = \Phi(t; z, M) = \prod_{i=1}^k \prod_{l=1}^n (t_i - z_l)^{-m_l} \prod_{1 \leq i < j \leq k} (t_i - t_j)^2.$$

The critical points of this function were studied by Heine and Stieltjes in connection with second order linear differential equations having polynomial coefficients and a polynomial solution of a prescribed degree. The result of Heine and Stieltjes can be formulated as follows.

**THEOREM 4** (Heine-Stieltjes, cf. [Sz], Ch. 6.8). *Let  $t^0$  be a critical point with non-zero critical value of the function  $\Phi(t)$  given by (14). Then  $F(x) = (x - t_1^0) \cdots (x - t_k^0)$  is a polynomial of the smaller degree in a nondegenerate plane with the Wrońskian  $W(x)$  given by (12).*

*Conversely, if  $F(x) = (x - t_1^0) \cdots (x - t_k^0)$  is a polynomial of the smaller degree in a non-degenerate plane  $V$  with the Wrońskian  $W_V(x) = W(x)$ , then  $t^0 = (t_1^0, \dots, t_k^0)$  is a critical point with non-zero critical value of the function  $\Phi(t)$ .*

The function  $\Phi(t)$  is symmetric with respect to permutations of  $t_1, \dots, t_k$ , critical points belonging to one orbit define the same polynomial  $F(x)$  and hence, according to Lemma 2, the same plane.

**3.5. Bethe vectors and nondegenerate planes.** The function  $\Phi(t)$  given by (14) and the function  $\Psi(t, z)$  given by (6) differ in a factor depending only on  $z$  and  $M$ ,

$$\Psi(t, z) = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{m_i m_j / 2} \cdot \Phi(t; z, M).$$

An easy calculation shows that (13) is exactly the system defining the critical points with non-zero critical values of the function  $\Phi(t)$ . The theorem of Heine-Stieltjes implies the following result.

**COROLLARY 2.** *There is a one-to-one correspondence between the critical orbits with non-zero critical value of the function (14) and the nondegenerate planes of order  $k$  and of degree  $|M| + 1 - k$  having Wrońskian (12).*

**COROLLARY 3.** *For fixed  $z \in Z$ ,  $M = (m_1, \dots, m_n)$  and  $k$ , there is a one-to-one correspondence between the Bethe vectors of the  $sl_2$  Gaudin model associated to  $z$  and  $Sing_k$  given by (3) and the nondegenerate planes of order  $k$  with the Wrońskian (12).*

**COROLLARY 4.** *The Bethe equations associated to  $z$  and (3) coincide with the critical point system of the generating function associated to order  $k$  and the Wrońskian (12).*

The function  $\Phi$  is called the *generating function* of the Wroński map: for any given monic polynomial  $W(x)$  and any given order  $k$ ,  $1 \leq k \leq \deg W/2$ , the critical orbits of the function  $\Phi$  determine the non-degenerate planes of order  $k$  in the preimage of  $W(x)$ . We shall show below that for generic  $z$  the critical orbits determine *all* planes of order  $k$  in the preimage, see Corollary 5 in Sec. 4.2.

**4. The preimage of a given Wrońskian.** The number of nondegenerate planes with a given Wrońskian can be estimated from above by the intersection number of Schubert classes. We recall first some basic facts related to the Schubert calculus, following [GrH] and [F].

**4.1. Schubert calculus.** Let  $G_2(d + 1) = G_2(\mathbb{C}^{d+1})$  be the Grassmannian variety of two-dimensional subspaces  $V \subset \mathbb{C}^{d+1}$ . A chosen basis  $e_1, \dots, e_{d+1}$  of  $\mathbb{C}^{d+1}$  defines the flag of linear subspaces

$$E_\bullet : E_1 \subset E_2 \subset \dots \subset E_d \subset E_{d+1} = \mathbb{C}^{d+1},$$

where  $E_i = \text{Span}\{e_1, \dots, e_i\}$ ,  $\dim E_i = i$ . For any integers  $a_1$  and  $a_2$  such that  $0 \leq a_2 \leq a_1 \leq d - 1$ , the *Schubert cell*  $\Omega_{a_1, a_2}^\circ(E_\bullet) \subset G_2(d + 1)$  is formed by the elements  $V \in G_2(d + 1)$  such that

$$\dim(V \cap E_i) = \begin{cases} 0, & 1 \leq i < d - a_1 \\ 1, & d - a_1 \leq i \leq d - a_2 \\ 2, & d - a_2 < i \leq d + 1. \end{cases}$$

The *Schubert variety*  $\Omega_{a_1, a_2}(E_\bullet)$  is the closure of the Schubert cell,  $\Omega_{a_1, a_2} = \Omega_{a_1, a_2}(E_\bullet) = \{V \in G_2(d + 1) \mid \dim(V \cap E_{d-a_1}) \geq 1, \dim(V \cap E_{d+1-a_2}) \geq 2\}$ . The variety  $\Omega_{a_1, a_2} = \Omega_{a_1, a_2}(E_\bullet)$  is an irreducible closed subvariety of  $G_2(d + 1)$  of the complex codimension  $a_1 + a_2$ .

The homology classes  $[\Omega_{a_1, a_2}]$  of Schubert varieties  $\Omega_{a_1, a_2}$  are independent of the choice of flag, and form a basis for the integral homology of  $G_2(d + 1)$ . Denote by  $\sigma_{a_1, a_2}$

the cohomology class in  $H^{2(a_1+a_2)}(G_2(d+1))$  whose cap product with the fundamental class of  $G_2(d+1)$  is the homology class  $[\Omega_{a_1,a_2}]$ . The classes  $\sigma_{a_1,a_2}$  are called *Schubert classes*. They give a basis over  $\mathbb{Z}$  for the cohomology ring of the Grassmannian. The product or *intersection* of any two Schubert classes  $\sigma_{a_1,a_2}$  and  $\sigma_{b_1,b_2}$  has the form

$$\sigma_{a_1,a_2} \cdot \sigma_{b_1,b_2} = \sum_{c_1+c_2=a_1+a_2+b_1+b_2} C(a_1, a_2; b_1, b_2; c_1, c_2) \sigma_{c_1,c_2},$$

where  $C(a_1, a_2; b_1, b_2; c_1, c_2)$  are nonnegative integers called the *Littlewood-Richardson coefficients*.

If the sum of the codimensions of classes equals  $\dim G_2(d+1) = 2d - 2$ , then their intersection is an integer (identifying the generator of the top cohomology group  $\sigma_{d-1,d-1} \in H^{4d-4}(G_2(d+1))$  with  $1 \in \mathbb{Z}$ ) called the *intersection number*.

When  $(a_1, a_2) = (q, 0)$ ,  $0 \leq q \leq d - 1$ , the Schubert varieties  $\Omega_{q,0}$  are called *special* and the corresponding cohomology classes  $\sigma_q = \sigma_{q,0}$  are called *special Schubert classes*.

As it is well-known, the Littlewood-Richardson coefficients appear also in the decomposition of the tensor product of two irreducible finite-dimensional  $sl_2$ -modules into the direct sum of irreducible  $sl_2$ -modules. This leads to the following claim connecting the Schubert calculus and the representation theory of the Lie algebra  $sl_2(\mathbb{C})$ .

Denote by  $L_q$  the irreducible  $sl_2$ -module with highest weight  $q$ .

**THEOREM 5 ([F]).** *Let  $q_1, \dots, q_{n+1}$  be integers such that  $0 \leq q_i \leq d - 1$  for all  $1 \leq i \leq n + 1$  and  $q_1 + \dots + q_{n+1} = 2d - 2$ . The intersection number of the special Schubert classes,  $\sigma_{q_1} \dots \sigma_{q_{n+1}}$ , coincides with the multiplicity of the trivial  $sl_2$ -module  $L_0$  in the tensor product  $L_{q_1} \otimes \dots \otimes L_{q_{n+1}}$ .*

Theorems 1 and 5 imply the following explicit formula.

**THEOREM 6.** *Let  $q_1, \dots, q_{n+1}$  be integers such that  $0 \leq q_i \leq d - 1$  for all  $1 \leq i \leq n + 1$  and  $q_1 + \dots + q_{n+1} = 2d - 2$ . Then*

$$\sum_{l=1}^n (-1)^{n-l} \sum_{1 \leq i_1 < \dots < i_l \leq n} \binom{q_{i_1} + \dots + q_{i_l} + l - d - 1}{n - 2}$$

is the intersection number  $\sigma_{q_1} \dots \sigma_{q_{n+1}}$ .

We did not find this formula in the literature on the Schubert calculus.

**4.2. Planes with a given Wroński and Schubert classes.** Applying the Schubert calculus to the Wroński map, we arrive at the following result.

**THEOREM 7.** *Let  $m_1, \dots, m_n$  be positive integers,  $|M| = m_1 + \dots + m_n$ . For generic  $z \in Z$  and for any integer  $k$  such that  $1 \leq k < |M| + 1 - k$ , the preimage of the polynomial (12) under the Wroński map consists of at most*

$$\sigma_{m_1} \cdot \dots \cdot \sigma_{m_n} \cdot \sigma_{|M|-2k}$$

nondegenerate planes of order  $k$  and of degree  $< |M| + 1 - k$ .

*Proof.* Any plane of order  $k$  with the Wroński of degree  $|M|$  lies in  $G_2(\text{Poly}_{|M|+1-k})$ , as Lemma 1 shows. Hence it is enough to consider the Wroński map on  $G_2(\text{Poly}_{|M|+1-k})$ .

For any  $z_j$ , define the flag  $\mathcal{F}_{z_j}$  in  $\text{Poly}_{|M|+1-k}$ ,

$$\mathcal{F}_0(z_j) \subset \mathcal{F}_1(z_j) \subset \dots \subset \mathcal{F}_{|M|+1-k}(z_j) = \text{Poly}_{|M|+1-k},$$

where  $\mathcal{F}_i(z_j)$  consists of the polynomials  $P(x) \in \text{Poly}_{|M|+1-k}$  of the form

$$P(x) = a_i(x - z_j)^{|M|+1-k-i} + \dots + a_0(x - z_j)^{|M|+1-k}.$$

We have  $\dim \mathcal{F}_i(z_j) = i + 1$ . Lemma 4 implies that the nondegenerate planes with a Wrońskian having at  $z_j$  a root of multiplicity  $m_j$  lie in the special Schubert cell

$$\Omega_{m_j,0}^\circ(\mathcal{F}_{z_j}) \subset G_2(\text{Poly}_{|M|+1-k}).$$

The maximal possible degree of the Wrońskian  $W_V(x)$  for  $V \in \text{Poly}_{|M|+1-k}$  is clearly  $2|M| - 2k$ . Denote by  $m_\infty = |M| - 2k$  the difference between  $2|M| - 2k$  and  $|M|$ . If  $m_\infty$  is positive, it is the *multiplicity of  $W(x)$  at infinity*.

The nondegenerate planes with a Wrońskian having given multiplicity  $m_\infty$  at infinity lie in the special Schubert cell  $\Omega_{m_\infty,0}^\circ(\mathcal{F}_\infty)$ , where  $\mathcal{F}_\infty$  is the flag

$$\text{Poly}_0 \subset \text{Poly}_1 \subset \dots \subset \text{Poly}_{|M|+1-k}.$$

We conclude that the nondegenerate planes of order  $k$  which have the Wrońskian (12) lie in the intersection of special Schubert cells

$$(15) \quad \Omega_{m_1,0}^\circ(\mathcal{F}_{z_1}) \cap \Omega_{m_2,0}^\circ(\mathcal{F}_{z_2}) \cap \dots \cap \Omega_{m_n,0}^\circ(\mathcal{F}_{z_n}) \cap \Omega_{m_\infty,0}^\circ(\mathcal{F}_\infty).$$

The dimension of  $G_2(\text{Poly}_{|M|+1-k})$  is exactly  $m_1 + \dots + m_n + m_\infty$ , therefore this intersection consists of a finite number of planes, and the intersection number of the special Schubert classes

$$\sigma_{m_1} \cdot \dots \cdot \sigma_{m_n} \cdot \sigma_{|M|-2k}$$

provides an upper bound. ■

Theorems 5 and 7, together with Corollary 2, imply the Proposition of Sec. 2.6 and hence, the Claim of the Introduction.

**COROLLARY 5.** *For generic  $z \in Z$ , all planes in the preimage under the Wroński map of the polynomial  $W(x)$  given by (12) are nondegenerate.*

**COROLLARY 6.** *For generic  $z \in Z$ , the intersection of Schubert varieties*

$$\Omega_{m_1,0}(\mathcal{F}_{z_1}) \cap \Omega_{m_2,0}(\mathcal{F}_{z_2}) \cap \dots \cap \Omega_{m_n,0}(\mathcal{F}_{z_n}) \cap \Omega_{m_\infty,0}(\mathcal{F}_\infty)$$

*is transversal, coincides with (15) and consists of nondegenerate elements.*

### 5. The simplicity of the spectrum of the Gaudin hamiltonians

**5.1. Fuchsian differential equations with only polynomial solutions and the Wroński map.** Consider a second order Fuchsian differential equation with regular singular points at  $z_1, \dots, z_n$ ,  $n \geq 2$ , and at infinity. If the exponents at  $z_j$  are 0 and  $m_j + 1$ ,  $1 \leq j \leq n$ , then this equation has the form

$$(16) \quad \begin{aligned} &F(x)u''(x) + G(x)u'(x) + H(x)u(x) = 0, \\ &F(x) = \prod_{j=1}^n (x - z_j), \quad \frac{G(x)}{F(x)} = \sum_{j=1}^n \frac{-m_j}{x - z_j}, \end{aligned}$$

where  $H(x)$  is a polynomial of degree not greater than  $n - 2$ . On Fuchsian equations see Ch. 6 of [R].

**THEOREM 8** (cf. Sec. 1.4 of [SV]). *Any nondegenerate plane  $V \in G_2(\text{Poly}_d)$  with the Wrońskian (12) is the solution space of such an equation.*

*Proof.* Let  $f(x)$  and  $g(x)$  be non-zero monic polynomials in  $V$  such that  $\deg f < \deg g$ . As we have seen in course of the proof of Lemma 2, the plane  $V$  is the solution space of the equation

$$W(x)u''(x) - W'(x)u'(x) + h(x)u(x) = 0,$$

where

$$h(x) = \frac{-W(x)f''(x) + W'(x)f'(x)}{f(x)}$$

is a polynomial proportional to the Wrońskian of  $f'(x)$  and  $g'(x)$ . One can easily check that

$$\frac{W'(x)}{W(x)} = \sum_{j=1}^n \frac{m_j}{x - z_j}.$$

Moreover, if  $z_j$  is a root of  $W(x)$  of multiplicity  $m_j > 1$ , then all coefficients of the equation have  $(x - z_j)^{m_j-1}$  as a common factor, and the equation can be reduced to the required form (16). ■

In [Fr], a correspondence between eigenvalues of the Gaudin hamiltonians and the  $sl_2$ -opers which determine a trivial monodromy representation

$$\pi_1(P^1 \setminus \{z_1, \dots, z_n, \infty\}) \rightarrow \text{PGL}_2$$

was established. Notice that any nondegenerate plane with a given Wrońskian defines such an oper, and any oper with trivial monodromy defines a plane in  $G_2(\text{Poly}_d)$  for some  $d$ .

**5.2. Bethe vectors differ by their eigenvalues.** Let  $t^{(1)}$  and  $t^{(2)}$  be two solutions to the Bethe equations (4) associated to fixed  $z^0 \in Z$  and  $\text{Sing}_k$ . Denote by  $v(t^{(i)}, z^0)$  the corresponding Bethe vectors given by (11) and  $\mu^{(i)} = (\mu_1(t^{(i)}), \dots, \mu_n(t^{(i)}))$  their eigenvalues given by (5),  $i = 1, 2$ .

**THEOREM 9.** *If  $\mu^{(1)} = \mu^{(2)}$ , then  $v(t^{(1)}, z^0) = v(t^{(2)}, z^0)$ .*

*Proof.* Let  $f_i(x) = (x - t_1^{(i)}) \cdots (x - t_k^{(i)})$  be the corresponding polynomials,  $i = 1, 2$ . Any of these polynomials defines a differential equation of the form (16). A classical fact of the theory of Fuchsian equations is that if an equation of the form (16) with positive integer  $m_1, \dots, m_n$  has a polynomial solution without multiple roots, then all solutions to this equation are polynomials, see Ch. 6 of [R] or Sec. 3.1 of [SV]. We have

$$H_i(x) = -\frac{F(x)f_i''(x) + G(x)f_i'(x)}{f_i(x)}, \quad i = 1, 2.$$

Polynomials  $H_1(x)$  and  $H_2(x)$  have degree at most  $n - 2$ . We will show that these polynomials in fact coincide. Indeed, the substitution of  $z_j$  into  $H_i$  gives

$$H_i(z_j) = m_j \cdot \frac{f_i'(z_j)}{f_i(z_j)} \prod_{l \neq j}^n (z_j - z_l).$$

An easy calculation shows that

$$\frac{f'_i(z_j)}{f_i(z_j)} = \sum_{l=1}^k \frac{1}{z_j - t_l^{(i)}}, \quad i = 1, 2,$$

and the condition  $\mu^{(1)} = \mu^{(2)}$  together with (5) imply

$$\frac{f'_1(z_j)}{f_1(z_j)} = \frac{f'_2(z_j)}{f_2(z_j)}, \quad j = 1, \dots, n.$$

We conclude that  $H_1(x)$  and  $H_2(x)$  coincide at  $n$  points, therefore  $H_1(x) = H_2(x)$ . Thus polynomials  $f_1(x)$  and  $f_2(x)$  are monic polynomials of the minimal degree in the same solution plane. Hence  $f_1(x) = f_2(x)$ , the solutions  $t^{(1)}$  and  $t^{(2)}$  of the Bethe equations lie in the same orbit, and  $v(t^{(1)}, z^0) = v(t^{(2)}, z^0)$ . ■

**COROLLARY 7.** *For generic  $z^0 \in Z$ , the  $sl_2$  Gaudin hamiltonians (2) have a simple spectrum.*

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