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A HOLOMORPHIC REPRESENTATION FORMULA FOR PARABOLIC HYPERSPHERES

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Abstract. A holomorphic representation formula for special parabolic hyperspheres is given.

Introduction. It was noticed by Blaschke that parabolic spheres in affine 3-space admit parametrisations in terms of holomorphic functions (of one variable). This is related to the fact that the Monge-Ampère equation governing parabolic hyperspheres is completely integrable in dimension two, a fact already known to Monge. The purpose of this note is to derive an explicit formula describing special parabolic hyperspheres in affine $(2n+1)$-space in terms of a holomorphic function of n variables.

1. Special parabolic affine hyperspheres. Let me briefly recall the notion of *special* parabolic affine hypersphere [BC]. We consider $\mathbb{R}^{m+1}$ as affine space with standard connection denoted by $\widetilde{\nabla}$ and parallel volume form vol. A hypersurface is given by an immersion $\varphi : M \to \mathbb{R}^{m+1}$ of an m-dimensional connected manifold. We assume that M admits a transversal vector field ξ and that $m > 1$. This induces on M the volume form $\nu = \mathrm{vol}(\xi, \ldots)$, a torsionfree connection ∇, a quadratic covariant tensor field g, an endomorphism field S (shape tensor) and a one-form θ such that

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(X,Y)\xi, \quad \widetilde{\nabla}_X \xi = SX + \theta(X)\xi. \tag{1.1}$$

Let us call the data (∇, g, S, θ) the *Gauß-Weingarten data* induced by the transversal vector field ξ. We will assume that g is nondegenerate and, hence, is a pseudo-Riemannian metric on M. This condition does not depend on the choice of ξ. According to Blaschke [B], once the orientation of M is fixed, there is a unique choice of transversal vector field ξ

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such that ν coincides with the metric volume form vol^g and $\nabla\nu = 0$. This particular choice of transversal vector field is called the *affine normal* and the corresponding geometric data (g, ∇) are called *Blaschke metric* and *induced connection*. Notice that, for the affine normal, $\theta = 0$ and S is computable from (g, ∇) (Gauß equations). Henceforth we use always the affine normal as transversal vector field.

DEFINITION 1. The hypersurface $\varphi : M \to \mathbb{R}^{m+1}$ is called a *parabolic* (or improper) *hypersphere* if the affine normal is parallel, $\widetilde{\nabla}\xi = 0$. It is called *special* if there exists an almost complex structure J on M which is skew symmetric with respect to the Blaschke metric g and such that the 2-form $\omega := g(J\cdot, \cdot)$ is parallel with respect to the induced connection ∇. Such an almost complex structure J is called *compatible*.

Notice that $\widetilde{\nabla}\xi = 0 \Leftrightarrow S = 0 \Leftrightarrow \nabla$ is flat. It was proven in [BC] that any parabolic two-dimensional sphere with (positive or negative) definite Blaschke metric is special and that a compatible almost complex structure on a special parabolic hypersphere is necessarily integrable. In fact, we proved the following stronger result, Theorem 1 below.

Recall that a *special Kähler manifold* (M, J, g, ∇) is a (pseudo-)Kähler manifold (M, J, g) endowed with a flat torsionfree connection ∇ such that ∇J is symmetric and $\nabla\omega = 0$, where $\omega = g(J\cdot, \cdot)$ is the Kähler form.

THEOREM 1 [BC]. *Let $\varphi : M \to \mathbb{R}^{m+1}$ be a special parabolic hypersphere with Blaschke metric g, induced connection ∇, compatible almost complex structure J and canonical two-form $\omega = g(J\cdot, \cdot)$. Then (M, J, g, ∇) is a special Kähler manifold. Conversely, any simply connected special Kähler manifold (M, J, g, ∇) admits an immersion $\varphi : M \to \mathbb{R}^{m+1}$, which is a special parabolic hypersphere with Blaschke metric g, induced connection ∇ and compatible almost complex structure J. The immersion φ is unique up to a unimodular affine transformation of $\mathbb{R}^{m+1}$.*

2. The holomorphic representation formula. It was proven in [ACD] that any special Kähler manifold is locally defined by a holomorphic function, as follows. Let F be a holomorphic function on a domain (i.e. a connected open set) $U \subset \mathbb{C}^n$ such that the (real) matrix

$$\operatorname{Im} \partial^2 F \quad \text{is invertible,} \tag{2.1}$$

where $\partial^2 F$ denotes the holomorphic Hessian of F. Let us denote by $M_F \subset T^*\mathbb{C}^n$ the image of the holomorphic section $dF : U \to T^*U \subset T^*\mathbb{C}^n$. It is a complex Lagrangian submanifold with respect to the standard complex symplectic structure $\Omega = \sum dz^i \wedge dw_i$, where $(z^1, \ldots, z^n, w_1, \ldots, w_n)$ are canonical coordinates of $T^*\mathbb{C}^n$. We denote its complex structure by J. Using the nondegeneracy condition (2.1), it is shown in [ACD] that the Hermitian form $\gamma := \sqrt{-1}\Omega(\cdot, \bar{\cdot})$ is nondegenerate on M_F and, hence, induces a (pseudo-)Kähler metric $g = \operatorname{Re}\gamma|_{M_F}$. It is also shown that a torsionfree connection ∇ on M_F can be defined by the condition that the real parts $x^i := \operatorname{Re} z^i$ and $y_j := \operatorname{Re} w_j$ are ∇-affine functions on M_F. In fact, it is shown that $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ is a (real) local coordinate system near any point of M_F and that the Kähler form $\omega = g(\cdot, J\cdot)$ is expressed by the formula $\omega = 2\sum dx^i \wedge dy_i$ on M_F.

THEOREM 2 [ACD]. *Let F be a holomorphic function satisfying the nondegeneracy condition (2.1) on a domain $U \subset \mathbb{C}^n$. Then (M_F, J, g, ∇), defined above, is a special Kähler manifold and any special Kähler manifold is locally of this form.*

It is noticed in [BC] that combining Theorem 1 and Theorem 2 we can associate a parabolic hypersphere to any holomorphic function F defined on a simply connected domain $U \subset \mathbb{C}^n$ and satisfying the nondegeneracy condition (2.1). However, the proof of Theorem 1 makes use of the Fundamental Theorem of affine differential geometry [DNV] (the generalisation of Radon's theorem [R] to higher dimensions) and does not involve any explicit parametrisation of the immersion $\varphi : M \to \mathbb{R}^{2n+1}$ realising a simply connected special Kähler manifold (M, J, g, ∇) of real dimension $2n$ as parabolic hypersphere. The aim is now to provide an explicit formula, in terms of the holomorphic function F, for the realisation of (M_F, J, g, ∇) as a parabolic hypersphere $\varphi_F : M_F \cong U \to \mathbb{R}^{2n+1}$.

We will not restrict ourselves to functions F defined on simply connected domains $U \subset \mathbb{C}^n$. More generally, we consider a 'multivalued' function defined on an arbitrary domain $U \subset \mathbb{C}^n$. Or, in other words, a (univalued) function defined on some Riemann domain $\widetilde{U}$ over U. A *Riemann domain* over U is a holomorphic (unramified) covering $\pi : \widetilde{U} \to U$. Any holomorphic function F on $\widetilde{U}$ defines a holomorphic Lagrangian immersion

$$\phi : \widetilde{U} \to T^*U \subset T^*\mathbb{C}^n, \quad \phi(p) := dF \circ (\pi_*|T_p\widetilde{U})^{-1}, \quad p \in \widetilde{U}. \tag{2.2}$$

Let us denote by J the complex structure of $\widetilde{U}$. Pulling back the canonical coordinates of $T^*\mathbb{C}^n$ to $\widetilde{U}$ we obtain holomorphic functions

$$\tilde{z}^i := \phi^* z^i \quad \text{and} \quad \tilde{w}_j := \phi^* w_j$$

on $\widetilde{U}$. The holomorphic functions $\tilde{z}^i$ form a local holomorphic coordinate system near any point of $\widetilde{U}$. We use the compact notation

$$\tilde{z} := (\tilde{z}^1, \ldots, \tilde{z}^n), \quad F_{\tilde{z}} = (F_{\tilde{z}^1}, \ldots, F_{\tilde{z}^n}) = (\frac{\partial F}{\partial \tilde{z}^1}, \ldots, \frac{\partial F}{\partial \tilde{z}^n}), \quad F_{\tilde{z}}\tilde{z} = \sum F_{\tilde{z}^k}\tilde{z}^k \quad \text{etc.}$$

Let $\partial^2 F$ be the Hessian of F with respect to (the flat torsionfree holomorphic connection defined by) the coordinate system $\tilde{z}$. We call F *nondegenerate* if $\operatorname{Im}\partial^2 F$ is invertible. Then, as before, $g := \operatorname{Re}\phi^*\gamma$ is a pseudo-Kähler metric and we can define a flat torsion-free connection ∇ by the condition that the functions $\tilde{x}^i := \operatorname{Re}\tilde{z}^i$ and $\tilde{y}_j := \operatorname{Re}\tilde{w}_j$ are ∇-affine functions on $\widetilde{U}$. We also put $\tilde{u}^i := \operatorname{Im}\tilde{z}^i$ and $\tilde{v}_j := \operatorname{Im}\tilde{w}_j$. Let us abbreviate $M(F) := (\widetilde{U}, J, g, \nabla)$ and define an immersion $\varphi_F : \widetilde{U} \to \mathbb{R}^{2n+1}$ by the formula

$$\begin{aligned} \varphi_F &:= (\operatorname{Re}\tilde{z}, \operatorname{Re}F_{\tilde{z}}, 2\operatorname{Im}F - 2(\operatorname{Re}F_{\tilde{z}})\operatorname{Im}\tilde{z}) \\ &= (\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{y}_1, \ldots, \tilde{y}_n, 2\operatorname{Im}F - 2\sum(\tilde{y}_k)\tilde{u}^k). \end{aligned} \tag{2.3}$$

THEOREM 3. *Let F be a nondegenerate holomorphic function defined on a Riemann domain $\widetilde{U}$. Then $M(F) = (\widetilde{U}, J, g, \nabla)$, defined above, is a special Kähler manifold with Kähler form $\omega = g(\cdot, J\cdot) = 2\sum d\tilde{x}^i \wedge d\tilde{y}_i$. The immersion $\varphi_F : \widetilde{U} \to \mathbb{R}^{2n+1}$ defined by (2.3) is, with respect to the volume form* vol $:= 2^n \det$ *on $\mathbb{R}^{2n+1}$, a special parabolic hypersphere with affine normal $\xi = \partial_{2n+1}$, Blaschke metric g, induced connection ∇ and compatible almost complex structure J. It is unique up to unimodular affine transformations of $\mathbb{R}^{2n+1}$.*

Proof. The first statement is a slight generalisation of the first part of Theorem 2, with essentially the same proof. The uniqueness of φ_F follows, as in the proof of Theorem 1, from the uniqueness statement of the Fundamental Theorem of affine differential geometry. It suffices to prove that φ_F is a parabolic hypersphere with Blaschke metric g, induced connection ∇ and compatible almost complex structure J. Let us compute the Gauß-Weingarten data $(\nabla^v, g^v, S^v, \theta^v)$, see (1.1), induced by the transversal vector field $v = \partial_{2n+1}$ (the 'vertical' vector field). It is immediate that $S^v = 0$ and $\theta^v = 0$. We compute ∇^v and g^v for the coordinate vector fields

$$\partial_{\tilde{x}^i} = \partial_i + \frac{\partial f}{\partial \tilde{x}^i}\partial_{2n+1}, \quad \partial_{\tilde{y}_j} = \partial_{n+j} + \frac{\partial f}{\partial \tilde{y}_j}\partial_{2n+1},$$

where

$$f := 2\mathrm{Im}\, F - 2(\mathrm{Re}\, F_{\tilde{z}})\mathrm{Im}\, \tilde{z} = 2\mathrm{Im}\, F - 2\sum \tilde{y}_k \tilde{u}^k$$

is the last component of φ_F. The covariant derivatives with respect to the connection $\widetilde{\nabla}$ of $\mathbb{R}^{2n+1}$ are given by

$$\widetilde{\nabla}_{\partial_{\tilde{x}^i}} \partial_{\tilde{x}^j} = \frac{\partial^2 f}{\partial_{\tilde{x}^i}\partial_{\tilde{x}^j}} v, \quad \widetilde{\nabla}_{\partial_{\tilde{x}^i}} \partial_{\tilde{y}_j} = \widetilde{\nabla}_{\partial_{\tilde{y}_j}} \partial_{\tilde{x}^i} = \frac{\partial^2 f}{\partial_{\tilde{x}^i}\partial_{\tilde{y}_j}} v, \quad \widetilde{\nabla}_{\partial_{\tilde{y}_i}} \partial_{\tilde{y}_j} = \frac{\partial^2 f}{\partial_{\tilde{y}_i}\partial_{\tilde{y}_j}} v.$$

This shows that the coordinate vector fields $\partial_{\tilde{x}^i}$ and $\partial_{\tilde{y}_j}$ are parallel for the connection ∇^v, so it coincides with ∇. Now $\theta = 0$ implies $\nabla \nu^v = \nabla^v \nu^v = 0$ for the volume form $\nu^v = \mathrm{vol}(v, \ldots)$. Moreover, we see that $g^v = \mathrm{Hess}^\nabla(f) = \nabla^2 f$.

Claim 1. $g^v = g$.

The claim, to be proven below, implies that v is the affine normal and, hence, that $g^v = g$ is the Blaschke metric. Let us see why. The Riemannian volume of the (pseudo-)Kähler manifold $M(F)$ with Kähler form $\omega = g(J\cdot, \cdot) = 2\sum_{i=1}^n d\tilde{x}^i \wedge d\tilde{y}_i$ is given by

$$\mathrm{vol}^g = (-1)^{n(n-1)/2}\frac{\omega^n}{n!} = 2^n d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^n \wedge d\tilde{y}_1 \wedge \ldots \wedge d\tilde{y}_n = 2^n \det(v, \ldots) = \nu^v,$$

if we choose the orientation defined by ν^v. (Notice that $d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^n \wedge d\tilde{y}_1 \wedge \ldots \wedge d\tilde{y}_n = (-1)^{n(n-1)/2} d\tilde{x}^1 \wedge d\tilde{y}_1 \wedge \ldots \wedge d\tilde{x}^n \wedge d\tilde{y}_n$.) This shows that vol^g coincides with the ∇-parallel volume form ν^v. So v is the affine normal and, hence, φ_F is a parabolic hypersphere with Blaschke metric g and induced connection ∇. The fact that $M(F)$ is a special Kähler manifold entails that J is skew symmetric with respect to g and $\nabla\omega = 0$. Therefore, the complex structure J is compatible, in the sense of Definition 1, with the data (g, ∇).

It remains to prove Claim 1. For the calculations we will use the next lemma.

Lemma 1. *The partial derivatives of the functions $\tilde{u}^i$ and $\tilde{v}_j$ on $M(F)$ with respect to the ∇-affine coordinates $(\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{y}_1, \ldots, \tilde{y}_n)$ satisfy the following equations:*

$$\sum_k (\tilde{u}^k_{\tilde{x}^i}(\tilde{v}_k)_{\tilde{y}_j} - \tilde{u}^k_{\tilde{y}_j}(\tilde{v}_k)_{\tilde{x}^i}) = \delta^j_i,$$

$$\sum_k \tilde{u}^k_{\tilde{x}^i}(\tilde{v}_k)_{\tilde{x}^j} = \sum_k \tilde{u}^k_{\tilde{x}^j}(\tilde{v}_k)_{\tilde{x}^i}, \quad \sum_k \tilde{u}^k_{\tilde{y}_i}(\tilde{v}_k)_{\tilde{y}_j} = \sum_k \tilde{u}^k_{\tilde{y}_j}(\tilde{v}_k)_{\tilde{y}_i}$$

$$\tilde{u}^i_{\tilde{x}^j} = -(\tilde{v}_j)_{\tilde{y}_i}, \quad \tilde{u}^i_{\tilde{y}_j} = \tilde{u}^j_{\tilde{y}_i}, \quad (\tilde{v}_i)_{\tilde{x}^j} = (\tilde{v}_j)_{\tilde{x}^i}.$$

Proof. Pulling back the symplectic form Ω of $T^*\mathbb{C}^n$ by means of the Lagrangian immersion $\phi : M(F) \to T^*\mathbb{C}^n$ defined in (2.2), we obtain the equation $\phi^*\Omega = 0$. Decomposing it into real and imaginary parts yields the lemma. ∎

Let us return to the proof of Theorem 3. First we observe that

$$\partial_{\tilde{x}^i} \operatorname{Im} F = \operatorname{Im} \partial_{\tilde{x}^i} F = \sum_j \operatorname{Im}\left(\frac{\partial \tilde{z}^j}{\partial \tilde{x}^i} \partial_{\tilde{z}^j} F\right) = \sum_j \operatorname{Im}\left((\delta_i^j + \sqrt{-1}\tilde{u}^j_{\tilde{x}^i}) F_{\tilde{z}^j}\right) = \tilde{v}_i + \sum_j \tilde{u}^j_{\tilde{x}^i} \tilde{y}_j$$

and

$$\partial_{\tilde{y}_j} \operatorname{Im} F = \operatorname{Im} \partial_{\tilde{y}_j} F = \sum_k \operatorname{Im}\left(\frac{\partial \tilde{z}^k}{\partial \tilde{y}_j} \partial_{\tilde{z}^k} F\right) = \sum_k \operatorname{Im}\left(\sqrt{-1}\tilde{u}^k_{\tilde{y}_j} F_{\tilde{z}_k}\right) = \sum_k \tilde{u}^k_{\tilde{y}_j} \tilde{y}_k.$$

The second derivatives of $\operatorname{Im} F$ are now easily computed with the help of Lemma 1:

$$\partial^2_{\tilde{x}^i \tilde{x}^j} \operatorname{Im} F = (\tilde{v}_i)_{\tilde{x}^j} + \sum_k \tilde{u}^k_{\tilde{x}^i \tilde{x}^j} \tilde{y}_k,\ \partial^2_{\tilde{x}^i \tilde{y}_j} \operatorname{Im} F = \sum_k \tilde{u}^k_{\tilde{y}_j \tilde{x}^i} \tilde{y}_k,\ \partial^2_{\tilde{y}_i \tilde{y}_j} \operatorname{Im} F = \tilde{u}^i_{\tilde{y}_j} + \sum_k \tilde{u}^k_{\tilde{y}_i \tilde{y}_j} \tilde{y}_k.$$

Using this and Lemma 1 one can now evaluate $g^v = \nabla^2 f$:

$$\begin{aligned}
g^v(\partial_{\tilde{x}^i}, \partial_{\tilde{x}^j}) &= \partial^2_{\tilde{x}^i \tilde{x}^j} f = 2((\tilde{v}_i)_{\tilde{x}^j} + \sum_k \tilde{u}^k_{\tilde{x}^i \tilde{x}^j} \tilde{y}_k) - 2 \sum_k \tilde{u}^k_{\tilde{x}^i \tilde{x}^j} \tilde{y}_k = 2(\tilde{v}_i)_{\tilde{x}^j}, \\
g^v(\partial_{\tilde{x}^i}, \partial_{\tilde{y}_j}) &= \partial^2_{\tilde{x}^i \tilde{y}_j} f = 2 \sum_k \tilde{u}^k_{\tilde{y}_j \tilde{x}^i} \tilde{y}_k - 2(\tilde{u}^j_{\tilde{x}^i} + \sum_k \tilde{u}^k_{\tilde{y}_j \tilde{x}^i} \tilde{y}_k) = -2\tilde{u}^j_{\tilde{x}^i}, \qquad (2.4) \\
g^v(\partial_{\tilde{y}_i}, \partial_{\tilde{y}_j}) &= \partial^2_{\tilde{y}_i \tilde{y}_j} f = 2(\tilde{u}^i_{\tilde{y}_j} + \sum_k \tilde{u}^k_{\tilde{y}_i \tilde{y}_j} \tilde{y}_k) - 2(\tilde{u}^j_{\tilde{y}_i} + \tilde{u}^i_{\tilde{y}_j} + \sum_k \tilde{u}^k_{\tilde{y}_i \tilde{y}_j} \tilde{y}_k) = -2\tilde{u}^i_{\tilde{y}_j}.
\end{aligned}$$

Notice that in virtue of (2.4) we have:

$$(\tilde{u}^i)_{\tilde{x}^j} = (\tilde{u}^j)_{\tilde{x}^i}. \qquad (2.5)$$

Let us compare this with g. The simplest way to compute g is using the fact that $g = \omega \circ J$, where $\omega = 2 \sum dx^i \wedge dy_i$ is the Kähler form and we consider g and ω as isomorphisms $TM \to T^*M$ (insertion of a vector in the first argument). It is easier to work with the inverse metric $g^{-1} = J^{-1} \circ \omega^{-1} = -J \circ \omega^{-1} = \omega^{-1} \circ J^*$. Notice that

$$\omega^{-1} = \frac{1}{2} \sum \partial_{\tilde{y}_i} \wedge \partial_{\tilde{x}^i}, \quad J^* d\tilde{x}^i = -d\tilde{u}^i \quad \text{and} \quad J^* d\tilde{y}_j = -d\tilde{v}_j.$$

Let us evaluate g^{-1} with the help of these formulas and Lemma

$$\begin{aligned}
g^{-1}(d\tilde{x}^i, d\tilde{x}^j) &= -\omega^{-1}(d\tilde{u}^i, d\tilde{x}^j) = -\frac{1}{2}\tilde{u}^i_{\tilde{y}_j}, \\
g^{-1}(d\tilde{x}^i, d\tilde{y}_j) &= -\omega^{-1}(d\tilde{u}^i, d\tilde{y}_j) = \frac{1}{2}\tilde{u}^i_{\tilde{x}^j}, \\
g^{-1}(d\tilde{y}_i, d\tilde{y}_j) &= -\omega^{-1}(d\tilde{v}_i, d\tilde{y}_j) = \frac{1}{2}(\tilde{v}_i)_{\tilde{x}^j}.
\end{aligned}$$

Comparing with the formulas for g^v and using Lemma 1 and (2.5) this proves that $g^{-1}g^v = \mathrm{id}$ and, hence, that $g = g^v$. This completes the proof of Claim 1 and Theorem 3. ∎

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