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ON THE GEOMETRY OF CONVEX REFLECTORS

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In this paper we consider a special class of convex hypersurfaces in Euclidean space which arise as weak solutions to some inverse problems of recovering reflectors from scattering data. For this class of hypersurfaces we study the notion of the focal function which, while sharing the important convexity property with the classical support function, has the advantage of being exactly the "right tool" for such inverse problems. We also discuss briefly the close analogy between one such inverse problem and the classical Minkowski problem.

1. Introduction. The notion of support function plays a crucial role in many problems of convexity theory. For a compact convex subset L in Euclidean space R^{n+1}, $n \geq 2$[1], the support function is given by

$$h(u) = max_{x \in L} \langle x, u \rangle,$$

where u is a point on a unit $n$sphere S^n centered at the origin and $\langle , \rangle$ is the usual inner product in R^{n+1}. When the support function is differentiable the closed convex hypersurface $F = \partial L$ can be represented as an envelope of the family of its tangent hyperplanes given by the equations

$$P_u : \quad \langle x, u \rangle = h(u), \quad x \in R^{n+1}, \quad u \in S^n. \tag{1}$$

If the origin $\mathcal{O}$ of the Cartesian coordinate system in R^{n+1} is inside the convex body bounded by F then $h(u)$ = distance from $\mathcal{O}$ to the hyperplane P_u. The hypersurface F

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[1] All concepts and results presented in this paper are also valid in R^2 with some nonessential modifications. We restrict the discussion to the case of $n \geq 2$ only for convenience of presentation.

can be recovered explicitly from h as

$$r(u) = \nabla h(u) + h(u)u, \quad u \in S^n. \tag{2}$$

Here, u is considered as a unit vector in R^{n+1} and a point on S^n and ∇h is the gradient of h in the standard metric e of S^n.

For smooth convex hypersurfaces the expression (2) defines F as the inverse of the Gauss map $F \to S^n$ and (2) is convenient in problems in which the Gauss map is involved. It also allows to express conveniently in terms of support function and its derivatives many important geometric quantities, such as width, volume, principal radii of curvature and others.

In many problems of ray tracing and geometrical optics it is necessary to consider "reflector maps" that arise from applications of Snell's law of reflection. The corresponding inverse problems in which a surface must be determined from pre-specified scattering properties requires solution of a system of PDE's involving the Jacobian of the reflector map [1, 2, 3]. For such problems it would be extremely useful to have an analogue of the support function because then the problem of solving a system of PDE's would be reduced to a single PDE for an appropriate scalar function. And though the solutions to these problems are usually sought among convex surfaces, in contrast to problems involving the Gauss map, the classical notion of support function is not useful in these problems.

The purpose of this paper is to describe one such problem in which the required scalar function can be constructed. The main guiding idea behind our geometric constructions is to determine a suitable family of hypersurfaces which, similar to (1) connected with the Gauss map, is connected with the reflector map (to be defined below). Because reflecting properties of quadric surfaces, such as ellipsoids, paraboloids and hyperboloids are well understood, these surfaces are natural candidates for the desired families and below we show how families of confocal paraboloids of revolution can be used to construct reflectors of general shape. The results presented here are based partly on our paper [4] where the proofs in many cases were omitted. Other related results on the "reflector" problem can be found in our papers [5, 6, 7, 8] and in papers referenced there. In contrast to these papers, here, we emphasize the geometric aspects of the reflector map and *focal* function (defined below) which plays a role analogous to the support function. Local differential geometry of reflector surfaces has been studied also in [9, 10, 11].

2. Families of confocal paraboloids. Fix in R^{n+1} a Cartesian coordinate system with origin $\mathcal{O}$ and let, as before, S^n be the unit sphere centered at $\mathcal{O}$. Let Ω be an arbitrary set on S^n containing at least two distinct points; the case when $\Omega = S^n$ is not excluded. Let $p(y), y \in \Omega$, be a positive function. For a $y \in \Omega$ denote by $P(y)$ a paraboloid of revolution with focus at $\mathcal{O}$ and axis y. Everywhere in this paper we assume that the axes of paraboloids are directed towards their openings. The polar radius of $P(y)$ is given by

$$\rho(m) = \frac{p(y)}{1 - \langle m, y \rangle}, \quad m \in S^n \setminus \{y\}. \tag{3}$$

The number $p(y)$ is called the focal parameter of $P(y)$. The solid closed convex set bounded by $P(y)$ is denoted by $B(y)$.

DEFINITION 1. Let

$$B = \bigcap_{y \in \Omega} B(y), \quad R = \partial B.$$

We call R a (*convex*) *reflector* (with the light source at $\mathcal{O}$).

The set B is a closed convex body with interior points and, since $p > 0$ on Ω, the origin $\mathcal{O}$ is strictly inside B. The closed convex hypersurface R is star-shaped relative to $\mathcal{O}$. The set of closed convex reflectors with the light source at $\mathcal{O}$ will be denoted by $\mathcal{R}$.

Let $R \in \mathcal{R}$ and let B_{int} be the interior of the convex body bounded by R. Let $P(y)$ be a paraboloid of revolution with focus at $\mathcal{O}$ and axis y for some $y \in S^n$. It may happen that $P(y) \cap R = \emptyset$. However, if

$$B_{int} \cap P(y) = \emptyset, \text{ while } P(y) \cap R \neq \emptyset,$$

then we call $P(y)$ a *supporting paraboloid*. It follows from the definition that at each point of R there exists at least one supporting paraboloid from the family $\{P(y), y \in \Omega\}$.

Also, since B is compact, for every $y \in S^n$ there exists a paraboloid of revolution with focus at $\mathcal{O}$ and axis of direction y, supporting for R. In order to see that this property is true, consider a paraboloid of revolution with focus at $\mathcal{O}$ and axis y whose focal parameter is large enough so that the convex body bounded by this paraboloid contains the body B. Then, shrink this paraboloid homothetically until it becomes supporting for the reflector R. Of course, if $y \notin \Omega$ this paraboloid is not a member of the family defined by the function p.

For any unit vector $m \in S^n$ we consider a ray originating at $\mathcal{O}$ in direction m. Let

$$\rho(m) = max\{\lambda \geq 0 \mid \lambda m \in B\} \tag{4}$$

be the radial function of the reflector R. Consider the point $r(m) = \rho(m)m$ on reflector R. Let $P(y)$ be a supporting paraboloid at $r(m)$. It follows from the well known reflection property of parabolas that the ray of direction m reflects off $P(y)$ at $r(m)$ in direction y. If u is the exterior unit normal to the paraboloid $P(y)$ at $r(m)$ then, according to Snell's law,

$$y = m - 2\langle m, u\rangle u. \tag{5}$$

We associate with the reflector R a map $\gamma : S^n \to S^n$ by setting

$$\gamma(m) = \bigcup \{y\}, \tag{6}$$

where y is the axis of a supporting paraboloid at $r(m)$ and the union is taken over all such supporting paraboloids. The map γ is, in general, multivalued. Also, it maps S^n onto itself, since, as it was mentioned earlier, for any $y \in S^n$ there exists a supporting paraboloid with axis y. It is natural to call the map γ the *reflector map* and this justifies the name reflector for the hypersurface R.

Let $y \in \Omega$ and $P(y)$ be the paraboloid supporting for R at $r(m)$ for some $m \in S^n$. Consider the hyperplane

$$\alpha(y) = \{x \in R^{n+1} \mid \langle x, -y\rangle = p(y)\}.$$

Using (3), we find that

$$dist\ (r(m),\ \alpha(y)) = \left| \frac{p(y)\langle m, -y\rangle}{1-\langle m, y\rangle} - p(y) \right| = \rho(m). \tag{7}$$

Thus, the hyperplane $\alpha(y)$ is the directrix hyperplane of the paraboloid $P(y)$ and $p(y)$ is the distance from the focus $\mathcal{O}$ to the hyperplane $\alpha(y)$.

A point on a reflector R is *singular* if at that point there exists more than one supporting paraboloid. Since each tangent hyperplane for a supporting paraboloid at the point of contact with R will also be a supporting hyperplane for R, any singular point on R will also be singular in the sense of convexity theory, that is, at such a point there are more than one supporting hyperplane for R. Consequently, by a theorem of Reidemeister [12], section 2.2, the set of singular points on R has n-dimensional Hausdorff measure equal zero.

Consider a special case when the set Ω consists of a finite number of distinct points, that is, let $\Omega = \{y_1, y_2, \dots, y_N\}$, $N > 1$, and let $p_1, p_2, \dots, p_N$ be positive numbers. Denote by P_i, $i = 1, 2, \dots, N$, the corresponding paraboloids of revolution with the common focus $\mathcal{O}$ and focal parameters $p_1, p_2, \dots, p_N$ and let R be the reflector defined by this family of paraboloids. We shall refer to R as a *parabolic polytope* or, simply, *P-polytope.* For any paraboloid P supporting for R the set $P \cap R$ is called a *face.* In general, for a reflector $R \in \mathcal{R}$, if P is supporting for R, a face may have any dimension from 0 to n.

We return now to the general case and consider a reflector $R \in \mathcal{R}$. The original function p defining R may be modified in two ways. First of all it may be extended to the entire sphere S^n. Indeed, by one of the properties stated earlier, for any $y \in S^n$ there exists a paraboloid of revolution $P(y)$ with axis y and focus $\mathcal{O}$, supporting for R. Thus, for $y \notin \Omega$ we let $p(y)$ be the focal parameter of $P(y)$. Further, if for some $y \in \Omega$ the corresponding paraboloid $P(y)$ is not supporting for R we replace the original value $p(y)$ by the focal parameter of a paraboloid homothetic to $P(y)$ which is supporting for R. Thus, we can always associate with R some function $p(y)$ defined on the entire S^n and such that each of the paraboloids $P(y)$ defined by $p(y), y \in S^n$, is supporting. In this case such a function p will be called a *focal function of the reflector* R. Note that $p(y) > 0$ $\forall y \in S^n$, since the origin $\mathcal{O}$ is strictly inside the convex body B bounded by R.

An equivalent definition of the focal function of a reflector can be formulated as follows. Let B be the convex body bounded by a reflector R as in definition 1 and p the focal function of R. Obviously,

$$B = \bigcap_{y \in S^n} B(y), \tag{8}$$

where

$$B(y) = \{x \in R^{n+1} \mid |x| - \langle x, y\rangle \le p(y)\}.$$

It follows that the focal function of the reflector R is given by

$$p(y) = max_{x \in B}(|x| - \langle x, y\rangle), \quad y \in S^n. \tag{9}$$

Next, we state the following useful property of confocal paraboloids [4].

Proposition 2. *The intersection of two confocal paraboloids in* $R^{n+1}, n \ge 2$, *consists of at most one connected component. For confocal parabolas in* R^2 *this statement has to* *be modified; in this case, if the parabolas do not coincide, the intersection is either empty or consists of two points.*

Proof. If the intersection has at least two connected components then each of the paraboloids cuts off the other one a compact "cup". We rescale one of the paraboloids homothetically with respect to the focus $\mathcal{O}$ so that the corresponding cup on this paraboloid shrinks to a point. Denote this point by X. At X the paraboloids are tangent to each other. Let u be their common unit normal at X and let m be the direction of the ray from $\mathcal{O}$ to X. Then by (5) we must have

$$y_1 = m - 2\langle m, u\rangle u = y_2,$$

where y_1 and y_2 are the unit vectors in directions of the axes of the two paraboloids. Hence, the paraboloids must coincide. Thus, initially, prior to rescaling, the intersection of the paraboloids consisted of at most one connected component.

The following are a few examples of simple reflectors and their focal functions.

1. The sphere $S^n(c/2), c = const > 0$, centered at the origin and of radius $c/2$ is a reflector. It is an envelope of the family of confocal paraboloids tangent to $S^n(c/2)$. Its focal function $p(y) = c$. For a point source of light at the center $\mathcal{O}$ of $S^n(c/2)$ the light ray of direction m is reflected in direction $y = -m$. The reflector map in this case is $m \to -m$.

2. Consider two unit vectors y_1 and y_2, $y_1 \neq y_2$, and two confocal paraboloids of revolution $P_1 = P(y_1)$ and $P_2 = P(y_2)$ with focal parameters $p_1 > 0$ and $p_2 > 0$, respectively. Let $\rho_1(m) = p_1/(1 - \langle m, y_1\rangle)$, $m \in S^n \setminus \{y_1\}$, and $\rho_2(m) = p_2/(1 - \langle m, y_2\rangle)$, $m \in S^n \setminus \{y_2\}$, be the polar radii of P_1 and P_2. Consider the set

$$U = \{m \in S^n | \rho_1(m) = \rho_2(m)\}.$$

By proposition 2, U has only one connected component and we have

$$\left\langle \frac{y_2}{p_2} - \frac{y_1}{p_1}, m \right\rangle = \frac{1}{p_2} - \frac{1}{p_1}.$$

This implies that U is an $(n-1)$-sphere in S^n. If we denote by S_1^n and S_2^n the two parts of S^n determined by U and corresponding to P_1 and P_2, respectively, then the reflector R defined by P_1 and P_2 is given by the polar radius

$$\rho(m) = \begin{cases} \rho_1(m) & \text{if } m \in S_1^n, \\ \rho_2(m) & \text{if } m \in S_2^n. \end{cases}$$

To describe the focal function p in this case, consider a unit vector $y \in S^n, y \neq y_1, y_2$. A paraboloid $P(y)$ with axis y can be supporting for R only for some $\bar{m} \in U$. Then $p(y) = \rho(\bar{m}) - \langle \rho(\bar{m})\bar{m}, y\rangle$. It follows from (7) that for all y for which $P(y)$ is supporting at $\rho(\bar{m})\bar{m}$, $dist(\rho(\bar{m})\bar{m}, \alpha(y)) = \rho(\bar{m})$.

3. Let $p(y) = c + \langle \xi, y\rangle,\ y \in S^n$, where $c = const > 1$ and ξ a fixed unit vector. The reflector in this case is an ellipsoid of revolution with axis ξ given by the equation

$$r(y) = \xi + \frac{c^2 - 1}{2(c + \langle \xi, y\rangle)} y.$$

This expression is obtained with the use of formula (11) to be established below.

Next, we give some necessary conditions that a focal function of a reflector $R \in \mathcal{R}$ must satisfy.

PROPOSITION 3. *Let $R \in \mathcal{R}$ be a reflector and p its focal function. Extend p to arbitrary vectors $y \in R^{n+1}$ by setting*

$$p(y) = max_{B_R}(|x||y| - \langle x, y\rangle). \tag{10}$$

Then p has the following properties:

1. $p(\lambda y) = \lambda p(y) \quad \forall \lambda \geq 0$.
2. $p(y + y') \leq p(y) + p(y')$.

COROLLARY 4. *The focal function p is the support function of a compact convex body with the origin $\mathcal{O}$ strictly inside. This convex body is defined by directrix hyperplanes of the supporting paraboloids of R.*

Proof of Proposition 3. Property 1 is obvious. Property 2 follows from the inequality

$$|x||y + y'| - \langle x, y + y'\rangle \leq |x||y| - \langle x, y\rangle + |x||y'| - \langle x, y'\rangle.$$

The corollary is the classical theorem of Minkowski [12].

In order to simplify the terminology we will refer to the convex hypersurface bounding the convex body with the support function p as the *directrix* of the reflector R. It will be denoted by $D(R)$. Note that the convexity of p implies that the focal function is continuous and twice differentiable almost everywhere in R^{n+1}.

There is some analogy between the *directrix* and parallel hypersurface in convexity theory [12], p. 134. Let $R \in \mathcal{R}$. By definition, R is star-shaped relative to the common focus of paraboloids defining R. Therefore, the position vector of R can be represented as $r(m) = \rho(m)m, m \in S^n$, where ρ is the radial function of the reflector R. Fix some $m \in S^n$ and let $r(m)$ be a point on R. For any supporting paraboloid $P_m(y)$ at $r(m)$ consider the vector $r(m) - \rho(m)y$ and introduce the set

$$D_m(R) = r(m) - \bigcup_{\{P_m(y)\}} \rho(m)y,$$

where the union is taken over all paraboloids supporting for R at $r(m)$. Obviously,

$$D(R) = \bigcup_{m \in S^n} D_m(R).$$

The map $m \to D_m(R)$ is, in general, multivalued.

It will be useful to describe the geometric structure of the directrix $D(R)$ of a reflector when it is a P-polytope. Over an n-dimensional face f_n of R (which is a piece of a supporting paraboloid), the corresponding set on $D(R)$ is a piece of a hyperplane, the directrix hyperplane of that paraboloid. This hyperplane is given by the equation

$$\langle x, -y\rangle = p,$$

where y and p are, respectively, the axis and focal parameter of the paraboloid containing the face f_n.

A face of R of dimension $n-1$ is formed by the intersection of exactly two supporting paraboloids. Let f_{n-1} be one such $n-1$-dimensional face of R and $m \in S^n$ be such that

$r(m) = \rho(m)m$ is in the interior (in the topology of f_{n-1}) of f_{n-1}. It follows from (7) that the directrix hyperplanes for all supporting paraboloids at $r(m)$ are all at the same distance $\rho(m)$ from $r(m)$. Consequently, the part of the directrix over the point $r(m)$ is a portion of a circle with center at $r(m)$ and radius $\rho(m)$. This circle is lying in the 2-plane containing the axes of the paraboloids forming f_{n-1} and orthogonal to the $n-1$-plane tangent to f_{n-1} at $r(m)$. For all points which are interior of f_{n-1} the part of the directrix over f_{n-1} is a union of circular arcs in 2-planes perpendicular to f_{n-1}.

Similarly, for a face f_{n-2} of R of dimension $n-2$ the corresponding part of the directrix is a union of spherical sectors in 3-planes perpendicular to f_{n-2}. For a face of R which reduces to a point the corresponding part of the directrix is a sector of an n-sphere with center at that point.

These arguments obviously apply not only to a P-polytope but to an arbitrary reflector from $\mathcal{R}$. Consequently, it follows from preceding observations that at every point of $D(R)$ there is a unique supporting hyperplane. This fact and the known results on differentiability of convex functions (see, for example, [12], pp. 30-31) imply the following

THEOREM 5. *The directrix $D(R)$ of a closed convex reflector R is a convex hypersurface of class C^1.*

It would be interesting to establish sufficient conditions under which a given positive convex function on R^{n+1} is a focal function of a reflector from $\mathcal{R}$. Convexity by itself is not sufficient, as follows from the preceding theorem. Below, we give such a condition under the additional assumption of differentiability of the given function.

Let $y = y(u^1, \dots, u^n)$ be a smooth parametrization of S^n and $p : S^n \to (0, \infty)$, $p \in C^1(S^n)$, the focal function of a reflector $R \in \mathcal{R}$. The function p is the support function of the directrix $D(R)$ and the field of reflected directions determined by the reflector R is the field of oriented inward unit normals on $D(R)$. Then it is known ([12], p. 40) that the position vector of $D(R)$ is given by

$$d(y) = -\nabla p(y) - p(y)y,\ y \in S^n.$$

By definition of the reflector R, a point r of R with the reflected direction y is characterized by the property that $|r| =$ distance from r to the supporting hyperplane on $D(R)$ with the inward normal y. Therefore,

$$r(y) = d(y) + |r(y)|y.$$

This implies

$$-r(y) = \nabla p(y) + p(y)y - s(y)y, \tag{11}$$

where $s(y) = \frac{|\nabla p(y)|^2 + p^2(y)}{2p(y)}$. In a different way this formula was derived in [13]; see also [9].

PROPOSITION 6. *Let $R \in \mathcal{R}$ and suppose that its focal function $p \in C^1(S^n)$. Then each supporting paraboloid to R has only one common point with R.*

Proof. Since the directrix $D(R)$ is convex and its support function $p \in C^1(S^n)$, it follows from the Corollary 1.7.3 in [12], p. 40, that $D(R)$ for each $y \in S^n$ has only one point in the supporting hyperplane with normal y. On the other hand, if a supporting paraboloid $P(y)$ for R is such that the set $P(y) \cap R$ contains at least two distinct points, then such two points will produce two distinct points on $D(R)$ at which the supporting hyperplanes will have the common normal y, which contradicts the above cited result in [12].

Assume now that the focal function $p \in C^2(S^n)$. Then (see [13])

$$-\partial_i r = [\nabla_{ij} p + (p-s)e_{ij}]e^{jk}[\partial_k y - (1/p)\partial_k p\, y], \quad i,j,k = 1,2,\ldots,n, \tag{12}$$

where ∇_{ij} denotes the second covariant derivative in the metric e, $[e_{ij}]$ is the coefficient matrix of e, and $[e^{ij}] = [e_{ij}]^{-1}$.

THEOREM 7. *Let $p \in C^2(S^n)$ and $p > 0$. Suppose that everywhere on S^n*

$$p_{\alpha\alpha} + p - s > 0, \tag{13}$$

where the differentiation is performed along the arc length of any large circle of S^n and $s = \frac{|\nabla p|^2 + p^2}{2p}$. Then the map (11) defines a closed convex reflector in $\mathcal{R}$ of class C^1.

Proof. The map (11) is an immersion if the rank of the Jacobian of (11) is n, that is, if

$$det[\langle \partial_i r, \partial_j r\rangle] \neq 0.$$

The latter is true, since by (13) the matrix $[\nabla_{ij} p + (p-s)e_{ij}]$ is positive definite everywhere on S^n. Thus, the map (11) is an immersion.

Denote by R the hypersurface in R^{n+1} defined by (11). This hypersurface is of class C^1 and the unit normal vector field u on R is given by

$$u = -\frac{\nabla p + py}{\sqrt{|\nabla p|^2 + p^2}}. \tag{14}$$

Since

$$\langle r, u\rangle = (1/2)\sqrt{|\nabla p|^2 + p^2}, \tag{15}$$

we conclude that the normal u is in outward direction. Also, differentiating covariantly (in the metric e) (14), we get

$$-\partial_j u = (\nabla_{jk} p + p e_{jk})e^{ki}[(1/2)(|\nabla p|^2 + p^2)\partial_i p\, u + \sqrt{|\nabla p|^2 + p^2}\ \partial_i y].$$

The coefficients of the second fundamental form of R are given by

$$-\langle \partial_i r, \partial_j u\rangle = -\frac{[\nabla_{il} p + (p-s)e_{il}]e^{lk}(\nabla_{kj} p + p e_{kj})}{\sqrt{|\nabla p|^2 + p^2}}. \tag{16}$$

The matrix $[\langle \partial_i r, \partial_j u\rangle]$ is positive definite and therefore R is locally convex (recall that u is the outward normal). Since R is also compact, it is convex.

To show that R satisfies the reflecting property (5) consider a ray of direction $m = r/|r|$ emanating from the origin $\mathcal{O}$. Then,

$$r/|r| - 2\langle r/|r|, u\rangle u = y.$$

The proof of the theorem is now complete.

Remark. It can be seen from the proof of the theorem that for reflectors which are strictly convex and whose focal function is twice differentiable the condition (13) is also necessary.

3. The energy function. In this section we define the notion of the energy function associated with the reflector map (5) and (6). But first, we state several useful properties of the reflector map.

Let $R \in \mathcal{R}$ and ω a subset of S^n. Let

$$\tau(\omega) = \{x \in R \mid \exists y \in \omega \text{ such that } P(y) \text{ is supporting for } R \text{ at } x\}. \tag{17}$$

Thus, we have a map $\tau : S^n \to R$ given by $y \to \tau(y)$. Since R is star-shaped relative to the origin $\mathcal{O}$, we may also consider the radial projection of $\tau(\omega)$ by rays from $\mathcal{O}$ on S^n.

Definition 8. Let $R \in \mathcal{R}$ and $\omega \subset S^n$. The *visibility set* $V(\omega)$ (for the reflector R) is the radial projection of $\tau(\omega)$ on S^n.

The following Theorems 9 and 16 were first stated in [4] without proofs.

Theorem 9. *Let $\mathcal{B}$ be the σ-algebra of Borel subsets of S^n. Let $R \in \mathcal{R}$. For any set $\omega \in \mathcal{B}$ the sets $V(\omega)$ and $\tau(\omega)$ are measurable relative to the standard Lebesgue n-dimensional measure σ on S^n and the Lebesgue n-dimensional measure μ on R, respectively. In addition, $\sigma(V(\omega))$ and $\mu(\tau(\omega))$ are completely additive measures on $\mathcal{B}$.*

Proof. The proof of this theorem is divided into several lemmas. Note that because the radial projection of R onto S^n is a homeomorphism the measure theories on R and S^n are equivalent. Therefore, it suffices to consider only one of the cases. We will consider the case of the map $\omega \to V(\omega)$, $\omega \in \mathcal{B}$.

Lemma 10. *For any closed set $\omega \subset S^n$ the set $V(\omega)$ is closed.*

Proof. Let $\omega \subset S^n$ be a closed set and $\{m_i \in V(\omega)\}$, $i = 1, 2, \ldots$, a sequence converging to some $m \in S^n$. Let $r(m_i)$ be the image of m_i on R under radial projection of S^n from $\mathcal{O}$ onto R. Let P_i be a supporting paraboloid at $r(m_i)$ with axes $y_i \in \omega$. All of these paraboloids are confocal and for each of them the convex body B_i defined by P_i contains the convex body B bounded by R. Consequently, any convergent subsequence of $\{P_i\}$, $i = 1, 2, \ldots$, converges to a paraboloid $P(y)$ supporting at $r(m)$. Since ω is closed, $y \in \omega$. Thus, $m \in V(\omega)$.

Lemma 11. *Let $\omega_1, \omega_2 \subset S^n$ and $\omega_1 \cap \omega_2 = \emptyset$. Then $\sigma(V(\omega_1) \cap V(\omega_2)) = 0$.*

Proof. If $m \in V(\omega_1) \cap V(\omega_2)$ then at $r(m)$ there exist at least two distinct supporting paraboloids. That is, $r(m)$ is a singular point on R. But the set of singular points has measure zero on R and, consequently, the image of this set on S^n under radial projection is also of measure zero. If, on the other hand, $V(\omega_1) \cap V(\omega_2) = \emptyset$ the statement of the lemma is obviously true. The lemma is proved.

Lemma 12. *If*

$$\omega = \bigcup_{i=1}^{\infty} \omega_i \textit{ then } V(\omega) = \bigcup_{i=1}^{\infty} V(\omega_i).$$

Consequently, if $V(\omega_i)$ is measurable for each i then $V(\omega)$ is also measurable.

Proof. If $m \in V(\omega)$ then at $r(m)$ there exists a supporting paraboloid whose axis is in ω_i for some i. Hence, $m \in \bigcup_{i=1}^{\infty} V(\omega_i)$. Conversely, if $m \in V(\omega_i)$ for some i then at $r(m)$ there exists a supporting paraboloid with axis in ω_i. Therefore, $m \in V(\omega)$.

LEMMA 13. *Let ω be a subset of S^n such that $V(\omega)$ is measurable. Then $V(S^n \setminus \omega)$ is also measurable.*

Proof. Obviously,

$$V(S^n \setminus \omega) = (V(S^n) \setminus V(\omega)) \cup ((V(S^n \setminus \omega) \cap V(\omega)).$$

But

$$(S^n \setminus \omega) \cap \omega = \emptyset$$

and $\sigma(V(S^n \setminus \omega) \cap V(\omega)) = 0$. The set $V(S^n)$ is measurable because S^n is closed and $V(\omega)$ is measurable by assumption. Hence, $V(S^n \setminus \omega)$ is also measurable. This implies, in particular, that for any open $\omega \subset S^n$ the set $V(\omega)$ is measurable.

LEMMA 14. *Let $\omega = \bigcap_{i=1}^{\infty} \omega_i$, where $V(\omega_i)$ is measurable for each i. Then $V(\omega)$ is measurable.*

Proof. Since

$$\bigcap_{i=1}^{\infty} \omega_i = S^n \setminus \bigcup_{i=1}^{\infty} (S^n \setminus \omega_i),$$

it follows from Lemmas 12 and 13 that $V(\omega)$ is measurable.

The first part of theorem 9 follows now from Lemmas 10–14. It remains to prove the complete additivity. Again, we will consider only the case of $\sigma(V(\omega))$.

Let ω_1 and ω_2 be two sets from $\mathcal{B}$ such that $\omega_1 \cap \omega_2 = \emptyset$. By Lemma 11, $\sigma(V(\omega_1 \cap \omega_2)) = 0$. Hence, $\sigma(V(\omega_1 \cup \omega_2)) = \sigma(V(\omega_1)) + \sigma(V(\omega_2))$. This implies finite additivity of $\sigma(V(\omega))$.

To prove complete additivity it suffices now to establish the following continuity property: if $\{\omega_i\}_{i=1}^{\infty}$ is a sequence of sets on S^n such that $V(\omega_i)$ is measurable for each i, $\omega_{i+1} \subset \omega_i$, and $\bigcap_{i=1}^{\infty} \omega_i = \emptyset$, then $lim_{i \to \infty} \sigma(V(\omega_i)) = 0$. If $\bigcap_{i=1}^{\infty} V(\omega_i) = \emptyset$ the statement is obvious. Suppose there exists $m \in \bigcap_{i=1}^{\infty} V(\omega_i)$. At $r(m)$ there must exist at least two distinct supporting paraboloids; otherwise, the condition $\bigcap_{i=1}^{\infty} \omega_i = \emptyset$ cannot be true. But then $r(m)$ is a singular point and the measure of such points on R is zero. Consequently, $lim_{i \to \infty} \sigma(V(\omega_i)) = 0$. The proof of Theorem 9 is now complete.

Note that the measure $\sigma(V(\omega))$, $\omega \in \mathcal{B}$, is, in general, not absolutely continuous. This would be the case, for example, when R is a P-polytope.

Now we can introduce the "energy" function that measures the energy of the source $\mathcal{O}$ transferred by a reflector R in a given set of reflected directions.

DEFINITION 15. Let I be a non-negative integrable function on S^n and $R \in \mathcal{R}$. The *energy* of the source reflected by R in directions defined by a set $\omega \in \mathcal{B}$ is the function

$$G(R, \omega) = \int_{V(\omega)} I(m) d\sigma(m).$$

THEOREM 16. *The function $G(R, \omega)$ is a non-negative and completely additive measure on Borel sets of S^n. Furthermore, if $R_i, i = 1, 2, \ldots,$ is a sequence of reflectors in $\mathcal{R}$*

converging to a closed convex hypersurface R in the Hausdorff metric then the hypersurface R is also a reflector with the source $\mathcal{O}$ and the measures $G(R_i, \omega)$ converge weakly to $G(R, \omega)$.

Proof. The non-negativity of $G(R, \omega)$ is obvious and the complete additivity follows from Theorem 9.

Fix some $r \in R$ and choose a sequence of points $r_i \in R_i,\ i = 1, 2, \dots,$ converging to r as $i \to \infty$. Let $\{P_i\}$ be the corresponding sequence of supporting paraboloids at the points r_i. We can select from $\{P_i\}$ a subsequence of paraboloids converging to some paraboloid P passing through r. Since all of the paraboloids $P_i,\ i = 1, 2, \dots,$ are confocal with focus at $\mathcal{O}$, the paraboloid P has the same focus. Furthermore, since P_i is supporting for R_i, the convex set B bounded by R is contained in the convex body bounded by P. Hence, P is supporting to R at r. Since r was an arbitrary point of R, we conclude that at every point of R there exists a supporting paraboloid with focus at $\mathcal{O}$, that is, $R \in \mathcal{R}$.

Let γ_i and γ be the reflector maps for R_i and R, respectively. Let η_i and η be the sets on S^n where the maps γ_i and γ are not single valued and put

$$\beta = \eta \cup \Big(\bigcup_{i=1}^{\infty} \eta_i \Big).$$

The sets of points on R_i and R corresponding (under radial projection) to points in η_i and η are singular and have measure zero. Then, $\sigma(\eta_i) = 0$ and $\sigma(\eta) = 0$. Thus, $\sigma(\beta) = 0$.

For any function $f \in C(S^n)$ we may now consider the integrals

$$\int_{S^n} f(y) dG_i(y) = \int_{V(S^n)} f(\gamma_i(m)) I(m) d\sigma(m),$$
$$\int_{S^n} f(y) dG(y) = \int_{V(S^n)} f(\gamma(m)) I(m) d\sigma(m).$$

Note that $S^n = V(S^n)$ and the functions $f(\gamma_i(m))$ and $f(\gamma(m))$ are defined everywhere on $S^n \setminus \beta$. Let $m \in S^n \setminus \beta$. The sequence of paraboloids $P_{r_1(m)},\ i = 1, 2, \dots,$ each of which is supporting for the corresponding R_i at the point $r_i(m)$ converges to the paraboloid $P_{r(m)}$ supporting for R at $r(m)$ and the respective sequence of paraboloid axes $y_i,\ i = 1, 2, \dots,$ converges to the axis y of $P_{r(m)}$. Therefore, $f_i(m) \to f(m)$ almost everywhere. Then,

$$\int_{S^n} f(y) dG_i(y) \to \int_{S^n} f(y) dG(y)$$

for any $f \in C(S^n)$. The theorem is proved.

REMARK 1. If a reflector $R^k \in \mathcal{R}$ is a P-polytope formed by k confocal paraboloids of revolution $P_1, P_2, \dots, P_k,\ k \geq 2$, and $y_1, y_2, \dots, y_k$ are their respective axes, then it is clear from the definition that the energy function in this case is zero everywhere on $S^n \setminus \{y_1, y_2, \dots, y_k\}$. More precisely, for any $\omega \subset S^n$

$$G(R^k, \omega) = \sum_{y_i \in \omega} G(R^k, y_i).$$

For each y_i $G(R^k, y_i)$ is the weighted (with weight I) n-volume of the radial projection on the sphere S^n of the n-dimensional face of R^k formed by the paraboloid P_i.

REMARK 2. Let $R \in \mathcal{R}$ and suppose that the reflector map (5) is a diffeomorphism of S^n onto S^n. Assume further that the focal function of R is of class C^2. Using local formulas at the end of section 2 we obtain

$$m(y) = \gamma^{-1}(y) = r(y)/|r(y)| = r(y)/s, \tag{18}$$

$$V(\omega) = \gamma^{-1}(\omega),\ \omega \in \mathcal{B}, \tag{19}$$

and

$$G(R,\omega) = \int_{V(\omega)} I(m)d\sigma(m) = \int_{\omega} I(\gamma^{-1}(y))J(\gamma^{-1}(y))d\sigma(y),\ \omega \in \mathcal{B}, \tag{20}$$

where J denotes the Jacobian. The latter can be computed as

$$J(\gamma^{-1}) = \frac{\sqrt{det[\langle \partial_i(r/s), \partial_j(r/s)\rangle]}}{\sqrt{det[e_{ij}]}},\ i,j = 1,2,\ldots,n.$$

Using equation (12) and noting that

$$\partial_i s = [\nabla_{ij}p + (p-s)e_{ij}]e^{jk}(1/p)\partial_k p, \quad i,j,k = 1,2,\ldots,n,$$

we obtain

$$J(\gamma^{-1}) = \frac{det[Hess(p) + (p-s)e]}{s^n det(e)} \equiv M(p), \tag{21}$$

where $Hess(p)$ is the matrix of second covariant derivatives computed in the metric e. It follows from Theorem 7 that $M(p)$ is elliptic on smooth focal functions of reflectors from $\mathcal{R}$.

4. Generalized Minkowski problem. The classical Minkowski problem consists in finding a closed convex hypersurface in R^{n+1} such that at the point with the unit normal u the reciprocal of the Gauss curvature is a positive function prescribed in advance on the unit sphere S^n [14]. In the weak formulation of the problem the n-volume of a measurable set A on the hypersurface is prescribed as a measure on the gaussian image (defined appropriately) of A on the sphere S^n. In this section we describe a generalization of the Minkowski problem based on the constructions in sections 2 and 3. This generalization is motivated by a problem in geometrical optics [1, 3, 2], which is, of course, a problem in R^3. However, all considerations are valid in R^{n+1} and in order not to introduce new notation we will continue with the exposition for an arbitrary n. The purpose of this section is to provide a brief outline of an important practical problem in which the methods described in preceding sections have been applied successfully.

4.1. *Creating a prescribed intensity pattern in the far-field.* Consider a non-isotropic point source of light $\mathcal{O}$ and denote by $I(m)$ the intensity of this source in direction m, $|m| = 1$. Let D be a domain on S^n, - the "input aperture". Suppose that a light ray emitted by the source $\mathcal{O}$ in direction $m \in D$ is incident on some smooth, convex, and star-shaped with respect to $\mathcal{O}$ surface R at some point $r(m)$ and reflects off it in direction y. The reflector map $\gamma : m \to y$ maps the input aperture $D \subset S^n$ onto some set $T \subset S^n$ called the "output aperture"; see Fig. 1, where for convenience the input aperture D and the set of reflected directions T are shown on the same unit sphere S^n. If γ is a

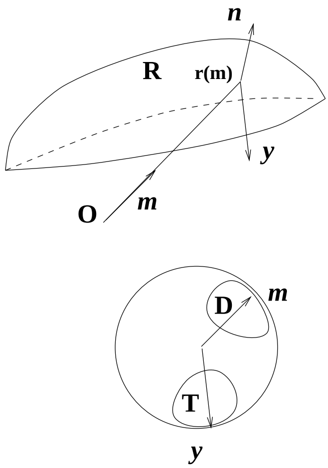

Fig. 1. Formulation of the beam shaping problem

smooth diffeomorphism then the intensity of the light reflected in direction $y = \gamma(m)$ is given by

$$\frac{I(m)}{J(\gamma(m))}.$$

Suppose now that the domain D and a compact set T on S^n are given as well as positive functions $I(m)$ on D and $L(y)$ on T. Consider the problem of finding a piece of a closed convex hypersurface R, star-shaped relative to $\mathcal{O}$, and such that the map γ defined by R maps $\bar{D}$ onto $\bar{T}$ and satisfies the equation

$$L(y) = I(\gamma^{-1}(y))|J(\gamma^{-1}(y))|, \quad y \in T. \tag{22}$$

Note that the last equation is formulated on the output aperture T rather than on the input aperture D. One could also set up this problem on D. For our purposes, the formulation on T is more convenient because the geometric constructions in such setting are more transparent and fit into the scheme in sections 2 and 3. We will refer to this problem as the "reflector problem"; see [9].

For smooth hypersurfaces the considerations at the end of section 2 and in Remark 2 at the end of section 3 are local and we can use the formulas there to obtain an equation of the reflector problem. Thus, if p is a smooth positive function on the set T and (11) defines a smooth hypersurface with non-degenerate reflector map γ, then the equation (22) becomes

$$L(y) = I(p, \nabla p)M(p) \quad \text{on } T. \tag{23}$$

If the equation (23) can be solved, then the reflector can be recovered as the map (11). If it is also a piece of a smooth convex hypersurface then it is the required solution of the reflector problem. The map (11) plays here the role of a generalized envelope in the sense defined in [15, 16], and it is analogous to the map (2).

4.2. *Weak formulation of the reflector problem.* In the *weak formulation* of the reflector problem the reflector R and a weak solution are defined by considering closed convex reflectors as defined in section 2. That is, R is not assumed smooth and the reflector map γ is defined by (5) and (6).

It is convenient to assume that the given function I defined on the input aperture D is extended to the entire sphere S^n by setting it equal to zero on $S^n \setminus D$. In order not to introduce more notation we will keep the same notation I.

We also assume that instead of the given function L on T we are given a non-negative completely additive measure F defined on all Borel subsets of S^n. A closed convex reflector $R \in \mathcal{R}$ is a *weak solution* of the reflector problem (in the weak formulation) if

$$G(R, \omega) = F(\omega) \quad \text{for any Borel set} \quad \omega \subset S^n. \tag{24}$$

THEOREM 17. *A weak solution to the reflector problem exists if and only if*

$$\int_{S^n} I(m) d\sigma(m) = F(S^n). \tag{25}$$

Two solutions may differ only by a homothetic transformation relative to $\mathcal{O}$.

The existence part was established in [4] in the case where the measure

$$F(\omega) = \int_\omega L(y) d\sigma(y)$$

for some non-negative integrable L on S^n (if L is defined only on a subset T of S^n, it is extended to the entire S^n by setting it equal to zero on $S^n \setminus T$), but the proof in the general case is essentially the same.

In [4] it was shown that the weak solution to the reflector problem can be obtained as a limit of a sequence of "discrete" problems that can be described as follows. We consider the same source $\mathcal{O}$ with intensity $I(m)$ and approximate the measure F by a sequence of discrete measures

$$F_k = \sum_{i=1}^{k} f_i \delta(y_i), \quad f_i > 0,$$

concentrated at some distinct points $y_1, \dots, y_k \in S^n$. The "discrete" version of the equation (24) consists of constructing a P-polytope $R^k \in \mathcal{R}$ defined by a finite number of confocal paraboloids which reflect the incident rays in directions $y_1, \cdots, y_k$ in such a way that for each $i = 1, \dots, k$ $G(R^k, y_i) = f_i$. Further details can be found in [4] and [8]

The weak solution to the reflector problem with a given input and output apertures which are sub-domains of S^n is obtained by finding a closed convex reflector and then deleting from it the part which projects radially onto $S^n \setminus \bar{D}$. Of course, for physical reasons it is natural to require that $\bar{D} \cap \bar{T} = \emptyset$.

Uniqueness of the weak solution was established in [4] for the case when reflectors are convex P-polytopes. The general case is considered in [17] and by a different method in [18]. Regularity of solutions was studied in [17] and [19].

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