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COMPLEX IP PSEUDO-RIEMANNIAN ALGEBRAIC CURVATURE TENSORS

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1. Introduction. The Riemann curvature tensor contains a great deal of information about the geometry of the underlying pseudo-Riemannian manifold; pseudo-Riemannian geometry is to a large extent the study of this tensor and its covariant derivatives. It is often convenient to work in a purely algebraic setting. We shall say that a tensor is an algebraic curvature tensor if it satisfies the symmetries of the Riemann curvature tensor. The Riemann curvature tensor defines an algebraic curvature tensor at each point of the manifold; conversely every algebraic curvature tensor is locally geometrically realizable. Thus algebraic curvature tensors are an integral part of certain questions in differential geometry.

The skew-symmetric curvature operator is a natural object of study; there are other natural operators and we refer to [4] for a survey of this area. In the present paper, we examine when the (complex) Jordan normal form of the skew-symmetric curvature operator is constant either in the real or complex settings. In Section 2, we give the basic definitions and notational conventions we shall need. We also review the basic results in the real setting. In Section 3, we discuss a natural generalization to the complex setting.

2. IP algebraic curvature tensors, Let (M, g) be a connected pseudo-Riemannian manifold of signature (p, q) and dimension $m = p+q$. Let ∇ be the Levi-Civita connection. The curvature operator and associated curvature tensor are defined by:

$$\begin{aligned}(1)\qquad & {}^gR(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]},\\ & {}^gR(X,Y,Z,W) := g({}^gR(X,Y)Z,W).\end{aligned}$$

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This tensor has the following symmetries:

$$ {}^gR(X,Y,Z,W) = -{}^gR(Y,X,Z,W), \tag{2} $$

$$ {}^gR(X,Y,Z,W) = {}^gR(Z,W,X,Y), \text{ and} \tag{3} $$

$$ {}^gR(X,Y,Z,W) + {}^gR(Y,Z,X,W) + {}^gR(Z,X,Y,W) = 0. \tag{4} $$

Let V be a finite dimensional real vector space which is equipped with a non-degenerate inner product $(\cdot,\cdot)$ of signature (p,q). We say that a 4 tensor $R \in \otimes^4 V$ is an *algebraic curvature tensor* if R satisfies the symmetries given in equations (2-4). Note that we do not impose the second Bianchi identity; algebraic curvature tensors measure second order phenomena. We say that (M,g) is a *geometric realization* of an algebraic curvature tensor at a point P if there is an isometry $\Psi : T_PM \to V$ so that ${}^gR_P = \Psi^* R$.

Let $\{e_1, e_2\}$ be an oriented basis for a 2 plane $\pi \subset V$ and let $h_{ij} := (e_i, e_j)$ describe the restriction of the inner product $(\cdot,\cdot)$ to π. We shall say that π is *non-degenerate* if h_{ij} is non-degenerate, i.e. $\det(h) := h_{11}h_{22} - h_{12}^2 \neq 0$. We say that π is *timelike*, *mixed*, or *spacelike* according as the quadratic form h_{ij} is negative definite, indefinite, or positive definite, respectively. We may decompose the Grassmannian of all oriented non-degenerate 2 planes as the disjoint union of the oriented timelike, mixed, and spacelike 2 planes.

Let R be an algebraic curvature tensor on V. Let $\{e_1, e_2\}$ be an oriented basis for a non-degenerate spacelike 2 plane $\pi \subset V$. The *skew-symmetric curvature operator*

$$ R(\pi) := \det(h)^{-1/2} R(e_i, e_j) $$

is independent of the particular oriented basis for π which was chosen. We say that R is *spacelike Jordan IP* if the complex Jordan normal form of the operator $R(\pi)$ is constant on the Grassmannian $\mathrm{Gr}^+_{0,2}(V)$ of oriented spacelike 2 planes. The notions of *timelike Jordan IP* and *mixed Jordan IP* are defined similarly using the Grassmannians $\mathrm{Gr}^+_{2,0}(V)$ and $\mathrm{Gr}^+_{1,1}(V)$ respectively. In the Riemannian setting $(p = 0)$, this is equivalent to assuming that the eigenvalues of $R(\pi)$ are constant on $\mathrm{Gr}^+_{(0,2)}(V)$. In the pseudo-Riemannian setting $(p > 0)$ a bit more care must be taken as there are examples where $R(\pi)$ has only the zero eigenvalue but where the rank of $R(\pi)$ varies with π; we refer to [7] for details. We note that there are examples of algebraic curvature tensors which are spacelike Jordan IP but not timelike Jordan IP; again see [7]. We say that a pseudo-Riemannian manifold (M,g) is *spacelike Jordan IP* if the associated curvature tensor ${}^gR$ is spacelike Jordan IP at every point of M; the eigenvalues and Jordan normal form are allowed to vary with the point.

Let $p = 0$. The study of the skew-symmetric curvature tensor was initiated in this context by Stanilov and Ivanova [16]; we also refer to related work by Ivanova [11]-[15]. Subsequently, Ivanov and Petrova [10] classified the spacelike Jordan IP metrics in the Riemannian setting for $m = 4$; for this reason the notation 'IP' has been used by later authors. The classification in [10] was later extended by P. Gilkey, J. Leahy, and H. Sadofsky [5] and by Gilkey [2] to the cases $m \geq 5$ and $m \neq 7$. We refer to Gilkey and Semmelman [6] for some partial results if $m = 7$. Zhang [17] has extended these results to the Lorentzian setting $(p = 1)$; see also Gilkey and Zhang [8] for related work.

We say that R has *spacelike rank* r if $\text{Rank}\,(R(\pi)) = r$ for every spacelike 2 plane π. The following theorem, which shows that $r = 2$ in many cases, was proved using topological methods [5, 17].

THEOREM 1. *Let R be an algebraic curvature tensor of spacelike rank r on a vector space of signature (p, q).*

1) *Let $p \leq 1$. Let $q = 5$, $q = 6$, or $q \geq 9$. Then $r = 2$.*
2) *Let $p = 2$. Let $q \geq 10$. Assume neither q nor $q + 2$ are powers of 2. Then $r = 2$.*

In light of Theorem 1, we shall focus our attention on the algebraic curvature tensors of rank 2. Let V be a vector space of signature (p, q). We say that a linear map ϕ of V is *admissible* if ϕ satisfies the following two conditions:

1. ϕ is self-adjoint and $\phi^2 = \text{Id}$, or $\phi^2 = -\text{Id}$, or $\phi^2 = 0$.
2. If $\phi(x) = 0$, then $(x, x) \leq 0$, i.e. $\ker\phi$ contains no spacelike vectors.

Let ϕ be admissible. If $\phi^2 = \text{Id}$, then ϕ is an isometry, i.e. $(\phi x, \phi y) = (x, y)$ for all $x, y \in V$. If $\phi^2 = -\text{Id}$, then ϕ is a *para-isometry*, i.e. $(\phi x, \phi y) = (-x, -y)$ for all $x, y \in V$; necessarily $p = q$ in this setting. If $\phi^2 = 0$, then the range of ϕ is *totally isotropic*, i.e. $(\phi x, \phi y) = 0$ for all $x, y \in V$. We define:

$$R_\phi(x, y)z := (\phi y, z)\phi x - (\phi x, z)\phi y, \text{ and}$$
$$R_\phi(x, y, z, w) := (\phi y, z)(\phi x, w) - (\phi x, z)(\phi y, w).$$

We showed in [7] that R_ϕ is an algebraic curvature tensor with $range(R_\phi(\pi)) = \phi\pi$ if π is a spacelike 2 plane. The eigenvalue structure of $R_\phi(\pi)$ is given by:

1. Suppose that $\phi^2 = \pm\text{Id}$. Then $R_\phi(\pi)$ is a rotation through an angle of 90 degrees on the spacelike ($\phi^2 = \text{Id}$) or timelike ($\phi^2 = -\text{Id}$) 2 plane $\phi\pi$, $R_\phi(\pi)$ vanishes on $\phi\pi^\perp$, and R_ϕ has two non-trivial complex eigenvalues $\pm\sqrt{-1}$.
2. Suppose that $\phi^2 = 0$. Since $\ker\phi$ contains no spacelike vectors, $R_\phi(\pi)$ has rank 2. The 2 plane $\phi\pi$ is totally isotropic. We have $\{R_\phi(\pi)\}^2 = 0$.

Thus CR_ϕ is a spacelike rank 2 Jordan IP algebraic curvature tensor for any $C \neq 0$. Conversely, we have the following classification result [7]:

THEOREM 2. *Let V be a vector space of signature (p, q), where $q \geq 5$. A tensor R is a spacelike rank 2 Jordan IP algebraic curvature tensor on V if and only if there exists a non-zero constant C and an admissible ϕ so that $R = CR_\phi$.*

3. Almost complex Jordan IP algebraic curvature tensors. Algebraic curvature tensors have been studied by many authors in the complex setting; we refer to Falcitelli, Farinola, and Salamon [1] and Gray [9] for further details concerning almost Hermitian geometry.

Let $J : V \to V$ be a real linear map with $J^2 = -\text{Id}$. We use J to provide V with a complex structure: $(a+\sqrt{-1}b)v = av+bJv$. Thus a real linear map S of V is *complex* if and only if $SJ = JS$. We shall assume that J is *pseudo-Hermitian*, i.e. $(Jv_1, Jv_2) = (v_1, v_2)$; necessarily both p and q are even.

A 2 plane π is called a *complex line* if $J\pi \subset \pi$. If π is non-degenerate, then π is either spacelike or timelike, there are no mixed complex lines.

An algebraic curvature tensor R is said to be *almost complex* if $JR(x, Jx) = R(x, Jx)J$ for all x in V, i.e. $R(x, Jx)$ is complex linear. Such an R is said to be *almost complex spacelike Jordan IP* if $R(\pi)$ (regarded as a complex linear map) has constant Jordan normal form for every spacelike complex line; the notion of *almost complex timelike Jordan IP* is defined similarly.

Theorem 2 controls the eigenvalue structure of a spacelike Jordan IP algebraic curvature tensor in the real setting. There is a similar result in the complex setting which we describe as follows. Let $p = 0$ and let R be an almost complex spacelike Jordan IP algebraic curvature tensor. The operator $JR(\pi)$ is a self-adjoint complex linear map and is therefore diagonalizable. Let $\{\lambda_i, \mu_i\}$ be the eigenvalues and multiplicities of $JR(\cdot)$, where $\mu_0 \geq \mu_1 \geq ... \geq \mu_\ell$; $\lambda_i \in \mathbb{R}$ and $q = 2(\mu_0 + ... + \mu_\ell)$. We refer to [3] for the proof of the following result which controls the eigenvalue structure in the Riemannian setting; it is not known if a similar result holds in the higher signature setting.

Theorem 3. *Let R be an almost complex spacelike Jordan IP algebraic curvature tensor on a Riemannian vector space of signature $(0, q)$. Let $\{\lambda_i, \mu_i\}$ be the eigenvalues and multiplicities of $JR(\cdot)$, where $\mu_0 \geq ... \geq \mu_\ell > 0$. Suppose $\ell \geq 1$. If $m \equiv 2$ mod 4, then $\ell = 1$ and $\mu_1 = 1$. If $m \equiv 0$ mod 4, then either $\ell = 1$ and $\mu_1 \leq 2$ or $\ell = 2$ and $\mu_1 = \mu_2 = 1$.*

We say that (ϕ, J) is an *admissible pair* if ϕ is admissible, if $\phi J = \pm J\phi$, and if J is a pseudo-Hermitian almost complex structure on V.

Theorem 4. *Let V be a vector space of signature (p, q). If (ϕ, J) is an admissible pair, then R_ϕ is an almost complex spacelike Jordan IP algebraic curvature tensor.*

Proof. By Theorem 2, R_ϕ is a spacelike rank 2 Jordan IP algebraic curvature tensor. Since $J\phi = \varepsilon\phi J$ for $\varepsilon = \pm 1$, we may compute:

$$\begin{aligned}
JR_\phi(x, Jx)z &= (\phi Jx, z)J\phi x - (\phi x, z)J\phi Jx \\
R_\phi(x, Jx)Jz &= (\phi Jx, Jz)\phi x - (\phi x, Jz)\phi Jx \\
&= -(J\phi Jx, z)\phi x + (J\phi x, z)\phi Jx \\
&= \varepsilon^2(\phi Jx, z)J\phi x - \varepsilon^2(\phi x, z)J\phi Jx = JR_\phi(x, Jx)z.
\end{aligned}$$

This shows that R is almost complex. ∎

Let V be a vector space of signature (p, q). We say that (ϕ_1, ϕ_2, J) is an *admissible triple* if the following conditions are satisfied:

1. ϕ_1 and ϕ_2 are admissible and either $\phi_1^2 \neq 0$ or $\phi_2^2 \neq 0$;
2. J is a pseudo-Hermitian almost complex structure on V;
3. $J\phi_1 = \phi_1 J$, $J\phi_2 = -\phi_2 J$, and $\phi_2\phi_1 + \phi_1\phi_2 = 0$.

Lemma 5. *Let V be a vector space of signature (p, q). Let $\{\phi_1, \phi_2, J\}$ be an admissible triple on V.*

1. *If x is spacelike, then the set $\{\phi_1 x, \phi_1 Jx, \phi_2 x, \phi_2 Jx\}$ is orthogonal and linearly independent.*
2. *For any $x \in V$, we have that:*

$$R_{\phi_1}(x,Jx)R_{\phi_2}(x,Jx) = R_{\phi_2}(x,Jx)R_{\phi_1}(x,Jx) = 0.$$

Proof. To show that $\{\phi_1 x, \phi_1 Jx, \phi_2 x, \phi_2 Jx\}$ is an orthogonal set, we compute:

$$\begin{aligned}
&(\phi_1 x, \phi_1 Jx) = \varepsilon_1(\phi_1 x, J\phi_1 x) = -\varepsilon_1(J\phi_1 x, \phi_1 x) = -(\phi_1 Jx, \phi_1 x),\\
&(\phi_1 x, \phi_2 x) = (\phi_2\phi_1 x, x) = -(\phi_1\phi_2 x, x) = -(\phi_2 x, \phi_1 x),\\
&(\phi_1 x, \phi_2 Jx) = -(J\phi_2\phi_1 x, x) = \varepsilon_1\varepsilon_2(\phi_1\phi_2 Jx, x) = -(\phi_2 Jx, \phi_1 x),\\
&(\phi_1 Jx, \phi_2 x) = -(x, J\phi_1\phi_2 x) = \varepsilon_1\varepsilon_2(x, \phi_2\phi_1 Jx) = -(\phi_2 x, \phi_1 Jx),\\
&(\phi_1 Jx, \phi_2 Jx) = (\phi_2\phi_1 Jx, Jx) = -(\phi_1\phi_2 Jx, Jx) = -(\phi_2 Jx, \phi_1 Jx),\\
&(\phi_2 x, \phi_2 Jx) = \varepsilon_2(\phi_2 x, J\phi_2 x) = -\varepsilon_2(J\phi_2 x, \phi_2 x) = -(\phi_2 Jx, \phi_2 x).
\end{aligned}$$

Let $\pi := \operatorname{Span}\{x, Jx\}$. Then $\phi_1\pi$ and $\phi_2\pi$ are orthogonal 2 planes. If $\phi_1^2 = \pm\mathrm{Id}$, then $\phi_1\pi$ is non-degenerate so $\phi_2\pi \subset \phi_1\pi^\perp$ implies $\phi_1\pi \cap \phi_2\pi = \{0\}$ and assertion (1) follows; the argument is the same if $\phi_2^2 = \pm\mathrm{Id}$.

To prove assertion (2), we compute:

$$\begin{aligned}
R_{\phi_1}(\pi)R_{\phi_2}(\pi)z &= R_{\phi_1}(\pi)\{(\phi_2 Jx, z)\phi_2 x - (\phi_2 x, z)\phi_2 Jx\}\\
&= (\phi_2 Jx, z)\{(\phi_1 Jx, \phi_2 x)\phi_1 x - (\phi_1 x, \phi_2 x)\phi_1 Jx\}\\
&\quad -(\phi_2 x, z)\{(\phi_1 Jx, \phi_2 Jx)\phi_1 x - (\phi_1 x, \phi_2 Jx)\phi_1 Jx\} = 0.
\end{aligned}$$

We argue similarly to show that $R_{\phi_2}(\pi)R_{\phi_1}(\pi) = 0$. ■

The following is the main result of this paper. It shows the estimates of Theorem 3 are sharp and provides a large family of non-trivial new examples.

THEOREM 6. *Let V be a vector space of signature (p,q). Let (ϕ_1, ϕ_2, J) be an admissible triple on V. Let λ_i be real constants. Then $R := \lambda_1 R_{\phi_1} + \lambda_2 R_{\phi_2}$ is an almost complex spacelike Jordan IP algebraic curvature tensor.*

Proof. We assume $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$; otherwise the proof follows directly from Theorem 4. As the set of almost complex algebraic curvature tensors is a linear subspace of the set of all 4 tensors, Theorem 4 shows that R is an almost complex algebraic curvature tensor.

We complete the proof by discussing the complex Jordan form. Let $\{x, Jx\}$ be an orthonormal basis for a spacelike complex line π. We use Lemma 5 to see that $\phi_1\pi$ and $\phi_2\pi$ are orthogonal complex lines and that $\operatorname{Rank}(R(\pi)) = 4$. Also by Lemma 5 we have:

$$R(\pi)^2 = \lambda_1^2 R_{\phi_1}(\pi)^2 + \lambda_2^2 R_{\phi_2}(\pi)^2.$$

Suppose that $\phi_1^2 = \pm\mathrm{Id}$ and that $\phi_2^2 = \pm\mathrm{Id}$. The metric restricted to $\phi_1\pi \oplus \phi_2\pi$ is non-degenerate. Let

$$V_0 := (\phi_1\pi \oplus \phi_2\pi)^\perp.$$

We have an orthogonal direct sum decomposition

$$V = \phi_1\pi \oplus \phi_2\pi \oplus V_0$$

which is preserved by $R(\pi)$. The map $R(\pi)$ has 4 non-trivial purely-imaginary eigenvalues $\pm\lambda_i\sqrt{-1}$; $V_0 = \ker(R(\pi))$. The map $R(\pi)^2$ is diagonalizable. Thus R is almost complex spacelike Jordan IP because:

$$\begin{aligned}
R(\pi)^2 &= 0 \text{ on } V_0,\\
R(\pi)^2 &= -\lambda_1^2 \text{ on } \phi_1\pi, \text{ and}\\
R(\pi)^2 &= -\lambda_2^2 \text{ on } \phi_2\pi.
\end{aligned}$$

Suppose that $\phi_1^2 = \pm\mathrm{Id}$ and $\phi_2^2 = 0$; the argument is similar if $\phi_1^2 = 0$ and $\phi_2^2 = \pm\mathrm{Id}$. By assumption $\ker\phi_2$ contains no spacelike vectors. Let $V_0 := (\phi_1\pi)^\perp$. Then we have an orthogonal direct sum decomposition $V = \phi_1\pi \oplus V_0$ which is preserved by $R(\pi)$. The map $R(\pi)$ has two non-zero eigenvalues $\pm\lambda_1\sqrt{-1}$. The map $R(\pi)^2$ is diagonalizable;

$$R(\pi)^2 = 0 \text{ on } V_0 \text{ and } R(\pi)^2 = -\lambda_1^2 \text{ on } \phi_1\pi.$$

Thus R is almost complex spacelike Jordan IP. ■

We construct examples to show that all 8 cases of Theorem 6 can occur. Let

$$e_1 := \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } e_2 := \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix} \text{ on } \mathbb{R}^{(0,4)},$$

$$J_0 := \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} \qquad \text{and } \alpha := \begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix} \qquad \text{on } \mathbb{R}^{(0,2)},$$

$$\beta := \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} \qquad \text{and } \gamma := \begin{pmatrix} 1 & -1\\ 1 & -1 \end{pmatrix} \qquad \text{on } \mathbb{R}^{(1,1)},$$

be matrices satisfying the relations

$$\begin{array}{lllll}
e_1^* = e_1, & e_2^* = e_2, & e_1^2 = \mathrm{Id}\,, & e_2^2 = \mathrm{Id}\,, & e_1e_2 + e_2e_1 = 0,\\
J_0^* = -J_0, & J_0^2 = -\mathrm{Id}\,, & \alpha^* = \alpha, & \alpha^2 = \mathrm{Id}\,, & J_0\alpha = -\alpha J_0,\\
\beta^* = \beta, & \beta^2 = -\mathrm{Id}\,, & \gamma^* = \gamma, & \gamma^2 = 0 & \mathrm{Range}\,\gamma = \ker\gamma.
\end{array}$$

Define the matrix τ_i by:

$$\tau_i = \begin{cases} \mathrm{Id} \otimes \mathrm{Id} & \text{if } \delta_i = +1,\\ \beta \otimes \mathrm{Id} & \text{if } \delta_i = -1,\\ \mathrm{Id} \otimes \gamma & \text{if } \delta_i = 0. \end{cases}$$

We construct (ϕ_1, ϕ_2, J) admissible so

$$\phi_1^2 = \delta_1\mathrm{Id} \text{ and } \phi_2^2 = \delta_2\mathrm{Id}$$

by setting:

$$\begin{aligned}
\phi_1 &:= e_1 \otimes \mathrm{Id} \otimes \tau_1,\\
\phi_2 &:= e_2 \otimes \alpha \otimes \tau_2, \text{ and}\\
J &:= \mathrm{Id} \otimes J_0 \otimes \mathrm{Id}\,.
\end{aligned}$$

The tensor

$$R = \lambda_1 R_{\phi_1} + \lambda_2 R_{\phi_2}$$

is then both almost complex spacelike Jordan IP and almost complex timelike Jordan IP.

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References

[1] M. Falcitelli, A. Farinola, and S. Salamon, *Almost-Hermitian geometry*, Diff. Geom. Appl. 4 (1994), 259-282.

[2] P. Gilkey, *Riemannian manifolds whose skew symmetric curvature operator has constant eigenvalues II*, in: Differential Geometry and Applications (eds. Kolar, Kowalski, Krupka, and Slovak), Publ. Masaryk University, Brno, 1999, 73–87.

[3] P. Gilkey, *Bundles over projective spaces and algebraic curvature tensors*, J. Geometry 71 (2001), 54–67.

[4] P. Gilkey and R. Ivanova, *Geometric consequences of some algebraic properties of the curvature tensor*, Bull. Math. Soc. Math. Roumanie 45 (93) (2000), 255–265.

[5] P. Gilkey, J. V. Leahy, and H. Sadofsky, *Riemannian manifolds whose skew-symmetric curvature operator has constant eigenvalues*, Indiana Univ. Math. J. 48 (1999), 615-634.

[6] P. Gilkey and U. Semmelman, *Spinors, self-duality, and IP algebraic curvature tensors of rank* 4, in: Proc. Symposium on Contemporary Mathematics devoted to the 125 anniversary of Faculty of Mathematics in Belgrade, Faculty Math. Univ. Belgrade, 2000, ed. Neda Bokan, 1–12.

[7] P. Gilkey and T. Zhang, *Algebraic curvature tensors whose skew-symmetric curvature operator has constant rank* 2, Periodica Math. Hungar. 44 (2002), 7–26.

[8] P. Gilkey and T. Zhang, *Algebraic curvature tensors for indefinite metrics whose skew-symmetric curvature tensor has constant Jordan normal form*, Houston Math. J. 28 (2002), 311–328.

[9] A. Gray, *Curvature identities for Hermitian and almost Hermitian manifolds*, Tohoku Math. J. 28 (1976), 601–612.

[10] S. Ivanov and I. Petrova, *Riemannian manifold in which the skew-symmetric curvature operator has pointwise constant eigenvalues*, Geom. Dedicata 70 (1998) 269–282.

[11] R. Ivanova, *About a classification of some Einstein manifolds*, in: Mathematics and Education in Mathematics, Proc. XXIII Spring Conf. of the Union of Bulgarian Mathematicians, Stara Zagora, 1994, 175–182.

[12] R. Ivanova, *Point-wise constancy of the skew-symmetric curvature operator's characteristical coefficients*, Tensor (N.S.) 57 (1996), 97–104.

[13] R. Ivanova, *An orthogonal tangential group of transformations of a 4-dimensional Riemannian manifold*, Math. Balkanica 11 (1997), 53–63.

[14] R. Ivanova, *4-dimensional Riemannian manifolds characterized by a skew-symmetric curvature operator*, Tensor (N.S.) 60 (1998), 293–302.

[15] R. Ivanova, *Generalization of a property of some* 4 *dimensional Einstein manifolds*, Tensor (N.S.) 60 (1998), 303–308.

[16] R. Ivanova and G. Stanilov, *A skew-symmetric curvature operator in Riemannian geometry*, in: Symposia Gaussiana (1994), Conf. A: Mathematics, eds. Behara, Fritsch and Lintz, 1995, 391–395.

[17] T. Zhang, *Manifolds with indefinite metrics whose skew-symmetric curvature operator has constant eigenvalues*, Ph.D. Thesis, Univ. of Oregon, 2000.