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STOKES EQUATIONS IN ASYMPTOTICALLY FLAT LAYERS

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Abstract. We study the generalized Stokes resolvent equations in asymptotically flat layers, which can be considered as compact perturbations of an infinite (flat) layer $\Omega_0 = \mathbb{R}^{n-1} \times (-1,1)$. Besides standard non-slip boundary conditions, we consider a mixture of slip and non-slip boundary conditions on the upper and lower boundary, respectively. We discuss the results on unique solvability of the generalized Stokes resolvent equations as well as the existence of a bounded H_∞-calculus for the associated Stokes operator and some of its consequences, which also yields an application to a free boundary value problem.

1. Introduction. Throughout this contribution $\Omega_\gamma \subset \mathbb{R}^n$, $n \geq 2$, denotes an asymptotically flat layer with $C^{1,1}$-boundary, which is a domain bounded by two surfaces $\partial\Omega_\gamma^+$ and $\partial\Omega_\gamma^-$ that get "close" to two parallel hyper-planes at infinity. More precisely,

$$\Omega_\gamma = \{(x', x_n) \in \mathbb{R}^n : \gamma^+(x') < x_n < \gamma^-(x')\},$$

where $\gamma^\pm \in C^{1,1}$ with $\gamma^\pm \to \pm 1$ and $\nabla\gamma^\pm, \nabla^2\gamma^\pm \to 0$ as $|x'| \to \infty$ and $\partial\Omega_\gamma^\pm = \{(x', \gamma^\pm(x')) : x' \in \mathbb{R}^{n-1}\}$.

We consider the generalized Stokes resolvent equations

$$(\lambda - \Delta)u + \nabla p = f \quad \text{in } \Omega_\gamma, \tag{1.1}$$

$$\operatorname{div} u = g \quad \text{in } \Omega_\gamma, \tag{1.2}$$

$$T_j^+(u,p) = a^+ \quad \text{on } \partial\Omega_\gamma^+, \tag{1.3}$$

$$u|_{\partial\Omega_\gamma^-} = 0 \quad \text{on } \partial\Omega_\gamma^- \tag{1.4}$$

with two kinds of boundary conditions, $j = 0$ or $j = 1$, where

$$T_0^+(u,p) = u|_{\partial\Omega_\gamma^+}, \quad T_1^+(u,p) = (\nu \cdot S(u) - \nu p)|_{\partial\Omega_\gamma^+}, \quad S(u) = \nabla u + (\nabla u)^T,$$

and $\lambda \in \Sigma_\delta \cup \{0\}$.

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The case $j = 0$ corresponds to standard non-slip boundary conditions. The mixed case $j = 1$ is important for application to free boundary value problems, see Section 6 below.

The structure of the article is as follows: First we present the results on unique solvability of (1.1)-(1.4) in Section 2. Second we discuss the existence of a bounded H_∞-calculus of the (reduced) Stokes operator associated to (1.1)-(1.4). Since the corresponding proofs use the calculus of pseudodifferential boundary value problems, we introduce the reduced Stokes equations in Section 3, which is an equivalent system to (1.1)-(1.4) and fits well in the latter calculus. Moreover, in Section 4, we give a short introduction to the theory of pseudodifferential boundary value problems, which was introduced by Boutet de Monvel [12] and extended by Grubb [18] to a parameter-dependent calculus and by Abels [6, 8, 9] to the case of non-smooth symbols. Then, in Section 5, we present the main ideas and the structure of the proof that the (reduced) Stokes operators admit a bounded H_∞-calculus. It is based on an explicit construction of a parametrix to (1.1)-(1.4), which is an "solution operator modulo lower order terms". Finally, in Section 6, we discuss some consequences of the bounded H_∞-calculus, which are the maximal regularity of the Stokes operators and the characterization of the domains of fractional powers. Moreover, the maximal regularity is the basis for the proof of the short-time existence of the motion of viscous surface waves in L^q-Sobolev spaces as discussed by Beale [11] and others in an L^2-setting.

2. Solvability of the Stokes resolvent equations. In the following $W^m_q(\Omega)$, $m \in \mathbb{N}_0$ denotes the standard Sobolev space based on the space of q-integrable functions $L^q(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is a domain. Moreover, $W^m_{q,0}(\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ in the $W^m_q(\Omega)$-norm. Because of the mixed boundary conditions in the case $j = 1$, we also introduce ${}^0W^1_q(\Omega_\gamma) = \{u \in W^1_q(\Omega_\gamma) : u|_{\partial\Omega^+_\gamma} = 0\}$ and ${}_0W^{-1}_q(\Omega_\gamma) := ({}^0W^1_{q'}(\Omega_\gamma))'$ with $\frac{1}{q} + \frac{1}{q'} = 1$. In order to get estimates uniform in the spectral parameter λ, we have to use the following parameter-dependent norm of $a \in W_q^{m-\frac{1}{q}}(\partial\Omega^+_\gamma)$, $m \in \mathbb{N}$,

$$
\begin{aligned}
\|a\|^q_{m-\frac{1}{q},q,\lambda} := &\sum_{|\alpha|\leq m-1} (1+|\lambda|)^{\frac{(m-k)q}{2}-\frac{1}{2}} \|a\|^q_{L^q(\partial\Omega^+_\gamma)} \\
&+ \sum_{|\alpha|=m-1} \int_{\partial\Omega^+_\gamma}\int_{\partial\Omega^+_\gamma} \frac{|D^\alpha a(x) - D^\alpha a(y)|^q}{|x-y|^{n-1+\frac{q}{q'}}} d\sigma(x) d\sigma(y),
\end{aligned}
$$

where $d\sigma$ denotes the surface measure on $\partial\Omega^+_\gamma$. Finally,

$$
\dot{W}^1_q(\Omega) = \left\{u \in L^q_{loc}(\overline{\Omega}) : \nabla u \in L^q(\Omega)\right\}
$$

and $\dot{W}^{-1}_{q,0}(\Omega) := (\dot{W}^1_{q'}(\Omega))'$.

THEOREM 2.1. *Let $1 < q < \infty$ and $\lambda \in \mathbb{C} \setminus (-\infty, 0)$. Then for every $(f, g, a^+) \in L^q(\Omega_\gamma)^n \times W^1_q(\Omega_\gamma) \times W_q^{2-j-\frac{1}{q}}(\partial\Omega^+_\gamma)^n$ with $g \in \dot{W}^{-1}_{q,0}(\Omega_\gamma)$ if $j = 0$ there is a unique solution $(u, p) \in W^2_q(\Omega_\gamma)^n \times \dot{W}^1_q(\Omega_\gamma)$ of (1.1)-(1.4). Moreover,*

$$
\begin{aligned}
&(1+|\lambda|)\|u\|_q + \|\nabla^2 u\|_q + \|\nabla p\|_q \\
&\leq C_\delta(\|f\|_q + \|\nabla g\|_q + (1+|\lambda|)\|g\|_{\dot{W}^{-1}_{q,0}(\Omega_\gamma)} + \|a^+\|_{2-\frac{1}{q},q,\lambda}) \qquad (2.1)
\end{aligned}
$$

if $j = 0$ *and*

$$\begin{aligned}&(1+|\lambda|)\|u\|_q + \|\nabla^2 u\|_q + \|\nabla p\|_q + \|p|_{\partial\Omega_\gamma^+}\|_{1-\frac{1}{q},q,\lambda}\\ &\leq C_\delta(\|f\|_q + \|\nabla g\|_q + (1+|\lambda|)\|g\|_{{}_0W_q^{-1}} + \|a^+\|_{1-\frac{1}{q},q,\lambda})\end{aligned} \tag{2.2}$$

if $j = 1$ *both uniformly in* $\lambda \in \Sigma_\delta \cup \{0\}$*, where* $\Sigma_\delta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| < \delta\}$.

The theorem is proved in [5] and [8] in detail. The proof consists of the following three parts:

First the unique solvability in an infinite layer $\Omega_0 = \mathbb{R}^{n-1} \times (-1, 1)$ is proved. This was done by Abels and Wiegner [10] in the case $j = 0$ and by Abels [3] in the case $j = 1$. More precisely, in [10] it is shown that for $j = 0$, (1.1)-(1.4) is uniquely solvable even for every $\lambda \in \mathbb{C} \setminus (-\infty, -\frac{\pi^2}{4}]$ with some modified estimates for λ close to $-\frac{\pi^2}{4}$. Moreover, Abe and Shibata [1, 2] proved Theorem 2.1 in the case $\Omega_0 = \mathbb{R}^{n-1} \times (-1, 1)$, $j = 0$, and $g = 0$. Because of the special geometry the solution of (1.1)-(1.4) can be calculated explicitly using partial Fourier transformation, i.e., Fourier transformation in the tangential variable $x' = (x_1, \dots, x_{n-1})$. Then it remains to estimate the solution operator, which acts like a Fourier multiplier operator in x' and like an integral operator in $x_n \in (-1, 1)$. This is done using the well-known Mikhlin multiplier theorem, cf. [3, 10] for details.

Second the first part implies the unique solvability for asymptotically flat layers which are "sufficiently" close to an infinite layer. This is done by a similar perturbation argument as in Farwig and Sohr [14, Section 3].

Finally, by definition, an arbitrary asymptotically flat layers can be decomposed as $\Omega_\gamma = \Omega_{\gamma_0} \cup \Omega_b$, where Ω_{γ_0} is sufficiently close to Ω_0 and Ω_b is bounded, cf. Figure 1.

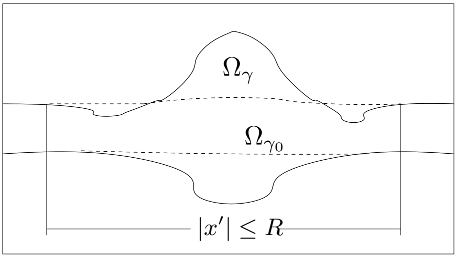

Fig. 1. Choice of Ω_{γ_0}

Using this decomposition the Fredholm solvability of (1.1)-(1.4) and uniqueness of the solutions in Ω_γ can be shown by means of standard cut-off techniques as presented in [14]. Finally, the parametrix construction done in [6] and discussed below implies the unique solvability of (1.1)-(1.4) and the estimates of the solution for $\lambda \in \Sigma_\delta$, $|\lambda| \geq R > 0$. Because of the continuity of the Fredholm index, this proves Theorem 2.1.

3. Reduced Stokes equations. The basis of the parametrix construction to (1.1)-(1.4) using the calculus of pseudodifferential boundary value problems is the following reduction, which goes back to Grubb and Solonnikov [20]. It is proved in [5, Section 3] that (1.1)-(1.4) are uniquely solvable (with the restriction $\lambda \neq 0$ if $j = 0$) if and only if the reduced Stokes resolvent equations

$$(\lambda - \Delta)u + G_{j0}u = f_r \quad \text{in } \Omega_\gamma, \tag{3.1}$$

$$T_j'^{+}u = a_r^+ \quad \text{on } \partial\Omega_\gamma^+, \tag{3.2}$$

$$u|_{\partial\Omega_\gamma^-} = 0 \quad \text{on } \partial\Omega_\gamma^-, \tag{3.3}$$

where

$$G_{00} = \nabla K_N \nu \cdot (\Delta - \nabla \operatorname{div})u|_{\partial\Omega_\gamma}, \; G_{10}u = \nabla K_{DN} \begin{pmatrix} 2\partial_\nu u_\nu|_{\partial\Omega_\gamma^+} \\ \nu \cdot (\Delta - \nabla \operatorname{div})u|_{\partial\Omega_\gamma^-} \end{pmatrix},$$

$$T_0'^{+}u = u|_{\partial\Omega_\gamma^+}, \quad (T_1'^{+}u)_\tau = (\nu \cdot S(u))_\tau|_{\partial\Omega_\gamma^+}, \quad (T_1'^{+}u)_\nu = \operatorname{div} u|_{\partial\Omega_\gamma^+},$$

are uniquely solvable in suitable L^q-Sobolev spaces, see [5, Section 3] for details. Here τ denotes the tangential component and K_N and K_{DN} denote the Poisson operators for the Laplace equation with Neumann and mixed Dirichlet-Neumann boundary conditions, resp., i.e., $\Delta K_N a = \Delta K_{DN} a = 0$, $\partial_\nu K_N a|_{\partial\Omega_\gamma} a = a$, $K_{DN} a|_{\partial\Omega_\gamma^+} = a^+$, and $\partial_\nu K_{DN} a|_{\partial\Omega_\gamma^-} = a^-$. Moreover, the a priori estimates (2.1)-(2.2) are equivalent to

$$(1 + |\lambda|)\|u\|_q + \|\nabla^2 u\|_q \leq C_{q,\delta}(\|f_r\|_q + \|a_r^+\|_{2-j-\frac{1}{q},q,\lambda}). \tag{3.4}$$

We will only present the main idea of the reduction in the case $j = 1$. Applying div and $\nu \cdot .|_{\partial\Omega_\gamma^-}$ to the equation (1.1) and using the normal component of (1.3), the pressure solves

$$\begin{aligned} \Delta p &= \operatorname{div} f - (\lambda - \Delta)g && \text{in } \Omega_\gamma, \\ p|_{\partial\Omega_\gamma^+} &= 2\partial_\nu u_\nu|_{\partial\Omega^+} - a_\nu^+ && \text{on } \partial\Omega_\gamma^+, \\ \partial_\nu p|_{\partial\Omega_\gamma^-} &= \nu \cdot (\Delta - \nabla \operatorname{div})u|_{\partial\Omega_\gamma^-} + \nu \cdot f|_{\partial\Omega_\gamma^-} + \partial_\nu g|_{\partial\Omega_\gamma^-} && \text{on } \partial\Omega_\gamma^-. \end{aligned}$$

Hence we can split $p = p_1 + p_2$ such that p_1 depends only on u and p_2 depends only on (f, g, a^+). More precisely,

$$p_1 = K_{DN} \begin{pmatrix} -a_\nu^+ \\ \nu \cdot f|_{\partial\Omega_\gamma^-} + \partial_\nu g|_{\partial\Omega_\gamma^-} \end{pmatrix}.$$

Hence u solves (3.1)-(3.3) with $f_r = f - \nabla p_2$ and $\nu \cdot a_r^+ = \operatorname{div} g|_{\partial\Omega_\gamma^+}$, $(a_r^+)_\tau = a_\tau$, if (u, p) is a solution of (1.1)-(1.4). Conversely, if u is a solution of the reduced system (3.1)-(3.3) with (f_r, a_r) as above,

$$\begin{aligned} (\lambda - \Delta) \operatorname{div} u &= \operatorname{div} f_r && = (\lambda - \Delta)g && \text{in } \Omega_\gamma, \\ \operatorname{div} u|_{\partial\Omega_\gamma^+} &= \nu \cdot a_r && = \operatorname{div} g|_{\partial\Omega_\gamma^+} && \text{on } \partial\Omega_\gamma^+, \\ \partial_\nu \operatorname{div} u|_{\partial\Omega_\gamma^-} &= \nu \cdot f_r|_{\partial\Omega_\gamma^-} && = \partial_\nu g|_{\partial\Omega^-} && \text{on } \partial\Omega_\gamma^-. \end{aligned}$$

Hence $\operatorname{div} u = g$ since the latter system is uniquely solvable.

To (3.1)-(3.3) we naturally associate the *reduced Stokes operator* $A_{j0} = -\Delta + G_{j0}$ on $L^q(\Omega_\gamma)^n$ with domain

$$\mathcal{D}(A_{j0}) := \{u \in W_q^2(\Omega)^n : T_j'^{+}u = 0, u|_{\partial\Omega_\gamma^- = 0}\}.$$

An important fact is that $A_{00}|_{J_{q,0}(\Omega)} = A_q$, where $A_q = -P_q\Delta$ is the (usual) Stokes operator with domain $\mathcal{D}(A_q) = W^2_q(\Omega_\gamma)^n \cap W^1_{q,0}(\Omega_\gamma)^n \cap J_{q,0}(\Omega)$, $P_q\colon L^q(\Omega_\gamma)^n \to J_{q,0}(\Omega_\gamma)$ is the Helmholtz projection and

$$J_{q,0}(\Omega_\gamma) = \{u \in L^q(\Omega_\gamma)^n : \operatorname{div} u = 0, \nu \cdot u|_{\partial\Omega_\gamma} = 0\},$$

cf. [5, 8] for the corresponding results in Ω_γ.

Finally, we mention that the reduced system (3.1)-(3.3) fits well into the general calculus of parameter-dependent pseudodifferential boundary value problems, cf. Section 4 below. This was used in [20] to solve the nonstationary Navier-Stokes equations in anisotropic L^2-Sobolev spaces in bounded smooth domains locally in time for various kinds of boundary conditions. Later this result was extended to L^q-Sobolev spaces, cf. [17], and smooth exterior domains, cf. [19].

4. Pseudodifferential boundary value problems. In order to explain the parametrix construction used in the proof of the bounded H_∞-calculus, cf. Theorem 5.1 below, we give a short introduction to the calculus of pseudodifferential boundary value problems, also called Boutet de Monvel calculus.

Recall that by definition a smooth function $p\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ belongs to the symbol class $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}^n$, if and only if for every $\alpha, \beta \in \mathbb{N}^n_0$

$$|\partial^\beta_x \partial^\alpha_\xi p(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|} \tag{4.1}$$

uniformly in $x, \xi \in \mathbb{R}^n$. For given $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ the associated pseudodifferential operator is defined as

$$p(x, D_x)f \equiv OP(p)f = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^n),$$

where $\hat{f}(\xi) = \mathcal{F}_{x\mapsto\xi}[f]$ and $\mathcal{F}$ denotes the Fourier transformation. It is easy to observe that most of the basic results on pseudodifferential operators as for example presented in [21, Section 2] generalize to the case of *operator-valued* pseudodifferential symbols $p\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(X_0, X_1)$, where X_j, $j = 0, 1$ are arbitrary Banach spaces. The proofs carry over literally except for the results on mapping properties in Bessel potential and Sobolev spaces then X_j have to be Hilbert spaces. In the following $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; X)$ denotes the corresponding symbol classes defined as in (4.1) by replacing $|\cdot|$ by the norm of a Banach space X.

The calculus of pseudodifferential operators was introduced by Boutet de Monvel [12] and consists of operators modeling elliptic differential boundary value problems on manifolds with boundary and their solution operators. It was extended by Grubb [18] to parameter-dependent problems. In the following we will for simplicity restrict ourselves to the case that the manifold is $\mathbb{R}^n_+$. Then the calculus consists of Green operator $a(x, D_x)$ of the form

$$\begin{pmatrix} p(x,D_x)_+ + g(x',D_x) & k(x',D_x) \\ t(x',D_x) & s(x',D_x) \end{pmatrix} : \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}^n_+)^N \\ \times \\ \mathcal{S}(\mathbb{R}^{n-1})^M \end{matrix} \to \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}^n_+)^{N'} \\ \times \\ \mathcal{S}(\mathbb{R}^{n-1})^{M'} \end{matrix},$$

where $p(x, D_x)_+ = r^+ p(x, D_x)e^+$ is a *truncated* pseudo-differential operator satisfying the transmission condition, $r^+f = f|_{\mathbb{R}^n_+}$, and e^+g denotes the extension by 0 of a function g defined on $\mathbb{R}^n_+$ to $\mathbb{R}^n$. Moreover, $k(x', D_x)$, $t(x', D_x)$, and $g(x', D_x)$ are Poisson, trace, singular Green operators, resp., and $s(x', D_x)$ is an $(n-1)$-dimensional pseudodifferential operator. It is useful to look at $a(x, D_x)$ as an operator-valued pseudodifferential operator $a(x, D_x) = \mathrm{OP}'(a(x, \xi', D_n))$, where OP' denotes the operator associated to a pseudodifferential symbol in $x', \xi' \in \mathbb{R}^{n-1}$ and

$$a(x, \xi', D_n) =$$

$$\begin{pmatrix} p(x, \xi', D_n)_+ + g(x', \xi', D_n) & k(x', \xi', D_n) \\ t(x', \xi', D_n) & s(x', \xi') \end{pmatrix} : \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+)^N \\ \times \\ \mathbb{C}^M \end{matrix} \to \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+)^{N'} \\ \times \\ \mathbb{C}^{M'} \end{matrix}$$

is called boundary symbol operator. We refer to [6, 18] for the precise definitions. For the convenience of the reader we mention that

$$t(x', \xi', D_n)f = \int_0^\infty \tilde{t}(x', \xi', y_n) f(y_n) dy_n \qquad k(x', \xi', D_n)a = \tilde{k}(x', \xi', x_n)a$$

$$g(x', \xi', D_n)f = \int_0^\infty \tilde{g}(x', \xi', x_n, y_n) f(y_n) dy_n, \quad a \in \mathbb{C}^M, f \in \mathcal{S}(\overline{\mathbb{R}}_+)^N,$$

where $\tilde{k} \in S^{m-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $\tilde{t} \in S^m_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, and $\tilde{g} \in S^{m-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+) \hat{\otimes} \mathcal{S}(\overline{\mathbb{R}}_+))$, $m \in \mathbb{R}$. Here

$$\begin{aligned} &\tilde{f}(x', \xi', x_n) \in S^d_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+)) \\ &:\Leftrightarrow x_n^l \partial_n^{l'} \tilde{f}(x', \xi', x_n) \in S^{d+\frac{1}{2}-l+l'}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; L^2(\mathbb{R}_+)) \quad \text{for all } l, l' \in \mathbb{N}_0 \end{aligned}$$

and $S^d_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+) \hat{\otimes} \mathcal{S}(\overline{\mathbb{R}}_+))$ is defined similarly.

The reduced Stokes system fits naturally in this calculus. It is modeled by the boundary symbol operators

$$a_{j,\lambda}(\xi', D_n) = \begin{pmatrix} \lambda + |\xi'|^2 - \partial_n^2 + k_j(\xi', D_n) t_j(\xi', D_n) \\ t'_j(\xi', D_n) \end{pmatrix}, \quad j = 0, 1,$$

where

$$\begin{aligned} &k_0(\xi', D_n)a = -e^{-|\xi'|x_n} \begin{pmatrix} \frac{i\xi'}{|\xi'|} \\ -1 \end{pmatrix} i\xi'^T a, \quad k_1(\xi', D_n)a = e^{-|\xi'|x_n} \begin{pmatrix} i\xi' \\ -|\xi'| \end{pmatrix} a, \\ &t_0(\xi', D_n)u = \partial_n u'(0), \quad t_1(\xi', D_n)u = 2\partial_n u_n(0), \\ &t'_0(\xi', D_n)u = u(0), \qquad t'_1(\xi', D_n)u, = \begin{pmatrix} i\xi' u_n(0) + \partial_n u'(0) \\ i\xi' \cdot u'(0) + \partial_n u_n(0) \end{pmatrix}. \end{aligned}$$

Here $u' = (u_1, \ldots, u_{n-1})$. (To be precise, $\xi' \mapsto |\xi'|$ has to be modified suitably in a neighborhood of 0 in order to get a smooth dependence on $\xi' \in \mathbb{R}^{n-1}$.) It was shown by Grubb and Solonnikov [20, Theorem 6.1] that $a_{j,\lambda}(\xi', D_n)$ is parameter-elliptic for $\lambda = \mu^2 e^{i\theta}$, $\mu \in \mathbb{R}$, and arbitrary $\theta \in (-\pi, \pi)$ in the sense of [18, Definition 3.1.2.]. This result implies that there is a $c_0 > 0$ such that

$$a_{j,\lambda}(\xi', D_n)\colon H^2_2(\mathbb{R}_+)^n \to L^2(\mathbb{R}_+)^n \times \mathbb{C}^n$$

is bijective for all $|\xi'| + |\lambda|^{\frac{1}{2}} \geq c_0$, $\lambda \in \Sigma_\delta$, $\delta \in (0, \pi)$, $c_0 = c_0(\delta)$; the same is true if $H^2_2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_+)$ are replaced by $\mathcal{S}(\overline{\mathbb{R}_+})$. Moreover, $a_{j,\lambda}(\xi',\mu,D_n)^{-1}$ is again a boundary symbol operator belonging to the calculus.

In order to construct a parametrix to (1.1)-(1.4) in a curved half-space $\mathbb{R}^n_\gamma = \{(x',x_n) \in \mathbb{R}^n : x_n > \gamma(x')\}$, one has to analyze how the operators behave if conjugated with a coordinate transformation. It is easy to observe that, if $p(x,D_x)$ is a differential operator with principal part $p_0(x,D_x)$ and $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism and $(F^*u)(x) := u(F(x))$, then the principal part of $F^*p(x,D_x)F^{*,-1}$ is $q_0(x,D_x)$, where $q_0(x,\xi) = p_0(F(x), D_xF(x)^{-T}\xi)$. This statement generalizes to pseudodifferential operators, cf. [21, Section 2, Theorem 6.3]. Similarly, it can be shown that, using $F\colon \mathbb{R}^n_+ \to \mathbb{R}^n_\gamma : x \mapsto (x', x_n + \gamma(x'))$, the reduced system on $\mathbb{R}^n_\gamma$ carries over to a system on $\mathbb{R}^n_+$, the principal part of which is given by

$$\underline{a}_{j,\lambda}(x',\xi',D_n) = U(x')^T a_{j,\lambda}(A'(x')\xi', c(x')D_n + d(x'))U(x'). \tag{4.2}$$

Here $a_{j,\lambda}(\xi',D_n)$ is the boundary symbol operator of the reduced Stokes system on $\mathbb{R}^n_+$ defined above, $U(x')$ is an orthogonal matrix mapping the exterior normal on $\partial\mathbb{R}^n_\gamma$ onto the exterior normal on $\partial\mathbb{R}^n_+$, and $A'(x'), c(x'), d(x')$ are related to the Jacobian of F, cf. [6, Section 5.3] for details.

Because of (4.2) and the invertibility of $a_{j,\lambda}(\xi',D_n)$, $\underline{a}_{j,\lambda}(x',\xi',D_n)$ is invertible for $|\xi'| + |\lambda|^{\frac12} \geq c_0 > 0$, $\lambda \in \Sigma_\delta$. Moreover, the results on the composition of parameter-dependent Green operators tell us that the composition of two Green operators coincides with the operator obtained by the product of their boundary symbol operators modulo lower order terms. More precisely, this yields that, if $\underline{b}_{j,\lambda}(x',\xi',D_n) = \underline{a}_{j,\lambda}(x',\xi',D_n)^{-1}$ for $|\xi'| + |\lambda|^{\frac12} \geq c_0 > 0$, then

$$\underline{a}_{j,\lambda}(x',D_x)\underline{b}_{j,\lambda}(x',\xi',D_n) = OP'(\underline{a}_{j,\lambda}(x',\xi',D_n)\underline{b}_{j,\lambda}(x',\xi',D_n)) + R_\lambda = I + R'_\lambda,$$

where $\|(R_\lambda, R'_\lambda)\|_{\mathcal{L}(L^q(\mathbb{R}^n_+))} \leq C_\delta(1+|\lambda|)^{-\varepsilon}$, $\varepsilon > 0$, $\lambda \in \Sigma_\delta$, $\delta \in (0,\pi)$. Hence we can use

$$R_{j,\lambda} f := F^{*,-1}\underline{a}_{j,\lambda}(x',D_x)F^* \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{4.3}$$

as parametrix to the reduced Stokes system in a curved half-space $\mathbb{R}^n_\gamma$.

Finally, we note that the usual operator classes defined in [12, 18] require smooth symbols. In particular, using the latter calculus, $F\colon \mathbb{R}^n_+ \to \mathbb{R}^n_\gamma$ and therefore $\partial\mathbb{R}^n_\gamma$ have to be smooth. But in order to construct the parametrix assuming only a $C^{1,1}$-boundary, the calculus has to be generalized to symbols, which are Hölder continuous in the space variable x. This was done partially in [6, 8], as far as needed to prove Theorem 5.1 below, and in [9] in a more general treatment.

5. H_∞-calculus of the reduced Stokes operator. Using the calculus of pseudodifferential boundary value problems, it was proved in [6] that the usual Stokes operator in the Dirichlet case and the reduced Stokes operator in the mixed case admit a bounded H_∞-calculus in the sense of McIntosh [22].

THEOREM 5.1. *Let $1 < q < \infty$ and let $\delta \in (0,\pi)$. Moreover, let A_q and $A_{10} = -\Delta + G_{10}$ be the (reduced) Stokes operators as defined in Section 3. Then A_q and A_{10} admit a* *bounded H_∞-calculus with respect to δ on $X = J_{q,0}(\Omega_\gamma)$ and $X = L^q(\Omega_\gamma)^n$, resp. , i.e.,*

$$h(A) = \frac{1}{2\pi i}\int_\Gamma h(-\lambda)(\lambda + A)^{-1}d\lambda, \qquad A = A_q, A_{10}, \tag{5.1}$$

is a bounded operator on X and

$$\|h(A)\|_{\mathcal{L}(X)} \le C_\delta \|h\|_\infty \tag{5.2}$$

for every $h \in H_\infty(\delta)$, where $H_\infty(\delta)$ denotes the algebra of all bounded holomorphic functions $h\colon \Sigma_{\pi-\delta} \to \mathbb{C}$ and Γ is the negatively oriented boundary of Σ_δ.

REMARK 5.2. The existence of bounded imaginary powers of the Stokes operator A_q on an infinite layer $\Omega_0 = \mathbb{R}^{n-1} \times (-1, 1)$ was already proved in [7]. Because of the flat boundary, the proof can be done only using Mikhlin multipliers, i.e., constant coefficient pseudodifferential operators. Although the proof in [7] is formulated only for bounded imaginary powers, it directly carries over to the more general case of a bounded H_∞-calculus.

Sketch of the proof. First of all, we note that $(\lambda + A_{00})^{-1}|_{J_{q,0}(\Omega_\gamma)} = (\lambda + A_q)^{-1}$. Hence in order to prove the theorem it is sufficient to construct an approximate resolvent $R_{j0,\lambda}$ such that

$$(\lambda + A_{j0})^{-1} = R_{j0,\lambda} + S_{j,\lambda}, \qquad j = 0, 1, \tag{5.3}$$

where $\|S_{j,\lambda}\|_{\mathcal{L}(X)} \le C(1 + |\lambda|)^{-1-\varepsilon}$, $\varepsilon > 0$, and

$$\left\| \int_\Gamma h(-\lambda) R_{j0,\lambda} d\lambda \right\|_{\mathcal{L}(X)} \le C_\delta \|h\|_\infty, \qquad h \in H_\infty(\delta). \tag{5.4}$$

Here $R_{j0,\lambda}$ is constructed explicitly with the aid of the calculus of parameter-dependent pseudodifferential boundary value problems introduced in Section 4. More precisely,

$$R_{j0,\lambda} = \psi_+ R^+_{j,\lambda}\varphi_+ + \psi_- R^-_{0,\lambda}\varphi_- + \psi_0 P_\lambda \varphi_0,$$

where $R^\pm_{j,\lambda}$, $j = 0, 1$, is the parametrix to the resolvent of the reduced Stokes operator in the *curved* half-spaces $\mathbb{R}^n_{\gamma\pm} = \{x \in \mathbb{R}^n : \pm x_n > \pm\gamma^\pm(x')\}$ as defined in (4.3), $P_\lambda = \mathrm{OP}((\lambda + |\xi|^2)^{-1})$ is the resolvent of $-\Delta$ on $\mathbb{R}^n$, $\{\varphi_+, \varphi_-, \varphi_0\}$ is a partition of unity such that $\varphi_\pm = 1$ in a neighborhood of $\partial\Omega^\pm_\gamma$, and ψ_* are suitable smooth functions such that $\psi_* = 1$ on $\operatorname{supp}\varphi_*$, $* = +, -, 0$. Then the statements on compositions and the mapping properties of Green operators (with Hölder continuous coefficients) yield that

$$(\lambda + A_{j0})R_{j0,\lambda} = I + S'_{j,\lambda},$$

where $\|S'_{j,\lambda}\|_{\mathcal{L}(X)} \le C(1 + |\lambda|)^{-\varepsilon}$, $\varepsilon > 0$, which implies (5.3). All remainder terms due to localization and coordinate transformation are of lower order and do not change the principal part. – In particular, this shows that $(\lambda + A_{j0})^{-1}$ exists if $|\lambda| \ge R > 0$, $\lambda \in \Sigma_\delta$, which is used in the proof of Theorem 2.1.

Finally, it remains to prove (5.4), which can be reduced to appropriate estimates on the boundary symbol operators of $R^\pm_{j,\lambda}$, cf. [6, Section 5.4] for details.

6. Consequences of theorem 5.1 and applications. Theorem 5.1 implies the existence of *bounded imaginary powers* A^{iy}, $y \in \mathbb{R}$, of the Stokes and the reduced Stokes operator, resp., since $h_y(z) = z^{iy} \in H_\infty(\delta)$. More precisely,

$$\|A^{iy}\|_{\mathcal{L}(X)} \leq C_\delta e^{(\pi-\delta)y}, \qquad A = A_q, A_{10},$$

where $\delta \in (0, \pi)$ is arbitrary. Hence the well-known result due to Dore and Venni [13, Theorem 3.2] and its extension by Giga and Sohr [16, Theorem 2.1] gives an important application of this abstract property:

THEOREM 6.1. *Let $1 < p, q < \infty$, $0 < T \leq \infty$, $f \in L^p(0, T; X_q)$, and let A and $X = X_q$ be as in Theorem 5.1. Then the Cauchy problem*

$$u'(t) + Au(t) = f(t), \qquad 0 < t < T,$$
$$u(0) = 0$$

has a unique solution $u \in W^1_p(0, T; X_q) \cap L^p(0, T; \mathcal{D}(A))$. Moreover,

$$\|u'\|_{L^p(0,T;X_q)} + \|Au\|_{L^p(0,T;X_q)} \leq C\|f\|_{L^p(0,T;X_q)},$$

where C does not depend on T.

Therefore the Stokes operator A_q and the reduced Stokes operator A_{10} have *maximal regularity.*

Moreover, by [15, Proposition 6.1] the bounded imaginary powers of the (reduced) Stokes operators $A = A_q, A_{10}$ implies that the domain of fractional powers $\mathcal{D}(A^\alpha)$, $\alpha \in (0, 1)$, can be characterized as

$$\mathcal{D}(A_q^\alpha) = (J_{q,0}(\Omega), \mathcal{D}(A_q))_{[\alpha]}, \qquad \mathcal{D}(A_{10}^\alpha) = (L^q(\Omega), \mathcal{D}(A_{10}))_{[\alpha]}$$

where $(.,.)_{[\alpha]}$ denotes the complex interpolation functor.

Using Theorem 6.1, the initial value problem for the motion of an "infinite ocean" of a viscous fluid under the force of gravity, firstly studied in [11], can be solved in L^q-Sobolev spaces. According to this problem the motion of the fluid is described as the solution of the Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = -g_0 e_n \qquad \text{with } x \in \Omega(t), t \in (0, T), \tag{6.1}$$
$$\operatorname{div} u = 0 \qquad \text{with } x \in \Omega(t), t \in (0, T), \tag{6.2}$$
$$T_1^+(u, p) = -P_0 \nu \qquad \text{with } x \in \partial\Omega(t)^+, t \in (0, T), \tag{6.3}$$
$$u|_{\partial\Omega^-} = 0 \qquad \text{on } \partial\Omega(0)^- \times (0, T), \tag{6.4}$$
$$u|_{t=0} = u_0 \qquad \text{in } \Omega(0) \tag{6.5}$$

in a *layer-like domain* $\Omega(t) = \{(x', x_n) \in \mathbb{R}^n : \gamma^-(x', t) < x_n < \gamma^+(x', t)\}$ with a fixed bottom below, i.e., $\partial\Omega(t)^- = \partial\Omega(0)^-$, where $\partial\Omega(t)^\pm = \{(x', \gamma^\pm(x', t)) : x' \in \mathbb{R}^{n-1}\}$ and a free surface above. Here u_0 is a given initial velocity, $\Omega(0)$ is the initial domain filled by the fluid, $g_0 > 0$ is the constant due to the acceleration of gravity, e_n the n-th canonical unit vector, and P_0 is the atmospheric pressure which is assumed to be constant. Moreover, the velocity field u and the domain $\Omega(t)$ have to satisfy a kinematic relation: Let $X(\xi, t)$, $t > 0$, be the trajectory of the mass particle, i.e., $X(\xi, t)$ solves

$$\partial_t X(\xi, t) = u(X(\xi, t), t), \qquad \text{for } t > 0, \qquad X(\xi, 0) = \xi.$$

Then $X(., t)\colon \Omega(0) \to \Omega(t)$ is bijective for all $t > 0$. Finally, we note that the effect of surface tension is neglected.

Beale [11] proved the short time existence of a unique solution of (6.1)-(6.5) in L^2-Sobolev spaces. The proof is done by passing to *Lagrangian coordinates*, i.e., considering $v(\xi,t) := u(X(\xi,t),t)$ and $q(\xi,t) := p(X(\xi,t),t)$, and linearizing the transformed system of (6.1)-(6.5), which is the nonstationary (generalized) Stokes system on $\Omega(0)$, i.e., (1.1)-(1.4) with λ replaced by ∂_t and an additional initial condition. Then a fixed point argument yields the unique solvability of (6.1)-(6.5) locally in time.

Based on Theorem 6.1, the proof can be adapted to L^q-Sobolev spaces for $q > n$. This has the advantage that the regularity assumptions on the data can be reduced by using the embedding $W^1_q(\Omega) \hookrightarrow L^\infty(\Omega)$ for $q > n$ instead of $W^m_2(\Omega) \hookrightarrow L^\infty(\Omega)$ for $m > \frac{n}{2}$. Roughly speaking, unique solvability of (6.1)-(6.5) can be proved for $u_0 \in W_q^{2-\frac{2}{q}}(\Omega(0))^n, q > n$, satisfying the natural compatibility conditions, $\Omega(0)$ being an asymptotically flat layer with $\partial\Omega(0)^+ \in W_q^{1-\frac{1}{q}}$, and sufficiently small $T > 0$, cf. [4] for details. In [11] the assumption $u_0 \in H^{r-1}(\Omega(0))$ and $\partial\Omega(0)^+ \in H^{r-\frac{3}{2}}$ with $r > 3$ is needed.

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