Abstract. Consider the problem of time-periodic strong solutions of the Stokes system modelling viscous incompressible fluid flow past a rotating obstacle in the whole space \( \mathbb{R}^3 \). Introducing a rotating coordinate system attached to the body yields a system of partial differential equations of second order involving an angular derivative not subordinate to the Laplacian. In a recent paper [2] the author proved \( L^q \)-estimates of second order derivatives uniformly in the angular and translational velocities, \( \omega \) and \( k \), of the obstacle, whereas the transport terms fails to have \( L^q \)-estimates independent of \( \omega \). In this paper we clarify this unexpected behavior and prove weighted \( L^q \)-estimates of first order terms independent of \( \omega \).

1. Introduction. Consider the Navier-Stokes equations modelling viscous flow past a rotating body \( K \subset \subset \mathbb{R}^3 \) with axis of rotation \( \omega = \tilde{\omega} e_3 = \tilde{\omega}(0,0,1)^T, \tilde{\omega} = |\omega| > 0 \), and with velocity \( u_\infty = ke_3, k > 0 \), at infinity. Then the velocity \( v \) and the pressure \( p \) satisfy the system

\[
\begin{align*}
  v_t - \nu \Delta v + v \cdot \nabla v + \nabla p &= \tilde{f} & \text{in } \Omega(t), t > 0, \\
  \text{div } v &= 0 & \text{in } \Omega(t), t > 0, \\
  v(y,t) &= \omega \wedge y & \text{on } \partial \Omega(t), t > 0, \\
  v(y,t) &\to u_\infty \neq 0 & \text{as } |y| \to \infty,
\end{align*}
\]

with an initial value \( v(y,0) = v_0(y) \), with constant viscosity \( \nu > 0 \) and external force \( \tilde{f} \) in the time-dependent exterior domain \( \Omega(t) = O_\omega(t) \Omega \); here \( O_\omega(t) \) denotes the orthogonal matrix

\[
O_\omega(t) = \begin{pmatrix}
  \cos |\omega|t & -\sin |\omega|t & 0 \\
  \sin |\omega|t & \cos |\omega|t & 0 \\
  0 & 0 & 1
\end{pmatrix}.
\]

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Introducing the new independent and dependent variables
\[ x = O_w^2(t)y, \quad u(x, t) = O_w^2(t)(v(y, t) - u_\infty), \quad p(x, t) = q(y, t), \]
and linearizing, \((u, p)\) will satisfy the modified Stokes system
\[
\begin{align*}
  u_t - \nu \Delta u + k \partial_3 u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p &= f, \\
  \text{div } u &= 0,
\end{align*}
\]
in a time-independent exterior domain \(\Omega \subset \mathbb{R}^3\) together with the initial-boundary condition
\(u(x, t) = \omega \wedge x - u_\infty\) for \(x \in \partial\Omega\), \(u(x, 0) = u_0\), \(u \to 0\) as \(|x| \to \infty\). In the stationary whole space case to be analyzed in this paper we are led to the elliptic equation
\[
-\nu \Delta u + k \partial_3 u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p = f, \quad \text{div } u = 0 \quad \text{in } \mathbb{R}^3
\]
in which the term \((\omega \wedge x) \cdot \nabla u\) is not subordinate to \(-\nu \Delta u\). Note that a stationary solution \((u, p)\) of (2) is related to a time-periodic solution of (1).

In [2] the author proved \textit{a priori} estimates of strong solutions \((u, p)\) of (2) which are found in the homogeneous Sobolev spaces \(\tilde{W}^{2, q}(\mathbb{R}^3)^3 \times \tilde{W}^{1, q}(\mathbb{R}^3)\) where
\[
\tilde{W}^{k, q}(\Omega) = \{ u \in L^1_{\text{loc}}(\Omega)/\Pi_{k-1} : \partial^\alpha u \in L^q(\Omega) \quad \text{for all } \alpha \in \mathbb{N}_0^n, |\alpha| = k \}.
\]
Here \(\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}\) for a multi-index \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n\) and \(\Pi_{k-1}\) denotes the set of all polynomials on \(\mathbb{R}^n\) of degree \(\leq k - 1\). The space \(\tilde{W}^{k, q}(\Omega)\) consists of equivalence classes of \(L^1_{\text{loc}}\)-functions being unique only up to elements from \(\Pi_{k-1}\) and is equipped with the norm \(\| \partial^\alpha u \|_q\), where \(\| \cdot \|_q\) denotes the \(L^q\)-norm. However, sometimes being less careful, we will consider \(v \in \tilde{W}^{k, q}(\Omega)\) as a function (representative) rather than an equivalence class of functions, i.e., \(v \in L^1_{\text{loc}}(\Omega)\) such that \(\partial^\alpha v \in L^q(\Omega)\) for every multi-index \(\alpha\) with \(|\alpha| = k\). For further results on similar problems we refer to [4], [8], [9], [10], [11], [12] and [13].

**Theorem 1.** (1) Let \(1 < q < \infty\), \(\nu > 0\), \(k > 0\), \(\omega = (0, 0, \omega)^T \in \mathbb{R}^3 \setminus \{0\}\), and let \(f \in L^q(\mathbb{R}^3)^3\). Then the linear problem (2) has a solution \((u, p)\) in \(\tilde{W}^{2, q}(\mathbb{R}^3)^3 \times \tilde{W}^{1, q}(\mathbb{R}^3)\) satisfying the \textit{a priori} estimate
\[
\| \nu \nabla^2 u \|_q + \| \nabla p \|_q \leq c \| f \|_q
\]
with a constant \(c\) independent of \(\nu, k\) and \(\omega\).

(2) Moreover,
\[
\| k \partial_3 u \|_q + \| (\omega \wedge x) \cdot \nabla u - \omega \wedge u \|_q \leq c \left( 1 + \frac{k^4}{\nu^2 |\omega|^2} \right) \| f \|_q
\]
with a constant \(c > 0\) independent of \(\nu, k\) and \(\omega\).

(3) Let \(1 < q < 4\), \(f \in L^q(\mathbb{R}^3)^3\) and let \((u, p)\) in \(\tilde{W}^{2, q}(\mathbb{R}^3)^3 \times \tilde{W}^{1, q}(\mathbb{R}^3)\) be the solution of (2). Then there exists \(\beta \in \mathbb{R}\) such that
\[
\nabla'(u - \beta \omega \wedge x) \in L^r(\mathbb{R}^3)^6 \quad \text{for all} \quad r > 1, \quad \frac{1}{r} = \frac{1}{q} - \left[ \frac{1}{4}, \frac{1}{3} \right],
\]
and there exists a constant \(C = C(\nu, k, \omega; r) > 0\) such that
\[
\| \nabla'(u - \beta \omega \wedge x) \|_r \leq C \left( \| f \|_q + \| \nu \nabla g + (\omega \wedge x) g - k \ge_3 \|_q \right).
\]
The proof of Theorem 1(1), see [2], is based on an explicit representation of \( u \) the estimate of which uses Fourier transforms, Hardy-Littlewood decomposition methods and maximal operators. Estimate (4) shows a surprising and crucial dependence on \( \frac{1}{|\omega|} \) via the term \( \frac{k^4}{\nu^2|\omega|^2} \). On the one hand, it is not at all clear that the terms \( k \partial_3u \) and \( (\omega \wedge x) \cdot \nabla u - \omega \wedge u \) can be estimated in \( L^q(\mathbb{R}^3) \) separately from each other. On the other hand, the dependence on \( \frac{1}{|\omega|} \) seems to be unnatural. Note that, as \( |\omega| \to 0 \), problem (2) converges formally to Oseen’s equation

\[-\nu \Delta u + k \partial_3 u + \nabla p = f, \quad \text{div} u = 0,\]

the solutions of which satisfy the estimate \( \|k \partial_3 u\|_q \leq c\|f\|_q \), see [1, 5]. To be more precise, a sequence of solutions \( (u_\omega) \) converges weakly in \( \dot{W}^{2,q}(\mathbb{R}^3)^3 \) to the solution of Oseen’s equation as \( |\omega| \to 0 \), cf. [2] Remark 1.3(1).

Concerning Theorem 1(3) note that the solutions of the homogeneous system (2) are given by \( \beta \omega \wedge x + \alpha e_3, \alpha, \beta \in \mathbb{R} \). Hence, with \( u \) also \( u - \beta \omega \wedge x \) is a solution of (2). The proof of Theorem 1(3) uses an improved Sobolev embedding theorem exploiting the fact that besides \( \nabla^2 u \) also \( k \partial_3 u \) lies in \( L^q \), cf. [1]. However, the dependence on \( 1/|\omega| \) in (4) implies that the constant \( C \) in (5) also depends on \( 1/|\omega| \). Due to this dependence the above-mentioned weak convergence of \( (u_\omega) \) in \( \dot{W}^{2,q}(\mathbb{R}^3)^3 \), i.e. of second order derivatives in \( L^q \), seems not to extend to \( k \partial_3 u_\omega \) in \( L^q \).

The aim of this paper is to clarify these unusual features. In Theorem 2 below we present an improvement of (4) and simplify the proof given in [2]. Then, for small \( q \), we prove a weighted \( L^q \)-estimate of \( k \partial_3 u \) and of \( (\omega \wedge x) \cdot \nabla u - \omega \wedge u \) independent of \( |\omega| \), see Theorem 4. An example and a heuristic argument will show that the dependence in (4) on \( \frac{k}{\sqrt{\nu|\omega|}} \) is not a weakness of the proof.

**Theorem 2.** Let \( 1 < q < \infty, \nu > 0, k \in \mathbb{R}, \omega = (0,0,\tilde{\omega})^T \in \mathbb{R}^3 \setminus \{0\} \) and let \( f \in L^q(\mathbb{R}^3)^3 \). Then the solution \( u \in \dot{W}^{2,q}(\mathbb{R}^3)^3 \) satisfies the a priori estimate

\[
\|k \partial_3 u\|_q + \| (\omega \wedge x) \cdot \nabla u - \omega \wedge u \|_q \leq c \left( 1 + \frac{k^2}{\nu|\omega|} \right)^{2 \max(1/q, 1 - 1/q) + \varepsilon} \|f\|_q
\]

with a constant \( c > 0 \) independent of \( \nu, k \) and \( \omega \); here \( \varepsilon > 0 \) can be chosen arbitrarily small and \( \varepsilon = 0 \) if \( q = 2 \).

**Example 3.** In Section 2 we will show that the term \( \frac{k^2}{\nu^2|\omega|^2} \) is needed in the \( L^2 \)-case. However, it is not clear whether the exponent \( 2 \max(1/q, 1 - 1/q) + \varepsilon > 1 \) is necessary if \( q \neq 2 \). We note that in [13] dealing with the nonlinear problem in exterior domains no dependence of a priori estimates on \( \frac{1}{|\omega|} \) occurs; the reason is that the author uses strong and weak a priori \( L^q \)-estimates of \( u \) by assuming that even \( f \in L^{6/5}(\mathbb{R}^3) \subset \dot{W}^{1,2}(\mathbb{R}^3) \). Using results of a forthcoming paper [3] dealing with the weak \( L^q \)-theory of (2) it is obvious that \( \|\nu \nabla u\|_q \) is bounded uniformly w.r.t. \( \omega \) and \( k \) by suitable norms of \( f \).

**Theorem 4.** (1) Let \( 1 < q < 2, \nu > 0, k > 0, \omega = (0,0,\tilde{\omega})^T \in \mathbb{R}^3 \setminus \{0\} \), and let \( f \in L^q(\mathbb{R}^3)^3 \). Then (2) has a solution \( u \in \dot{W}^{2,q}(\mathbb{R}^3)^3 \) satisfying the a priori estimates

\[
\left\| \frac{\nabla u}{|x'|} \right\|_q \leq c \nu \|f\|_q
\]
\[
\left\| \frac{(\omega \wedge x) \cdot \nabla u - \omega \wedge u}{1 + |x'|} \right\|_q \leq c \left( 1 + \frac{k}{\nu} \right) \| f \|_q
\]
(8)

with a constant \( c > 0 \) independent of \( k, \nu, \omega \); here, for \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) the term \( |x'| \) denotes the Euclidean length \( \sqrt{x_1^2 + x_2^2} \) of \( x' = (x_1, x_2) \).

(2) If \( 1 < q < 3 \), then
\[
\left\| \frac{\nabla u}{|x|} \right\|_q \leq \frac{1}{\nu} \| f \|_q
\]
(9)

and
\[
\left\| \frac{(\omega \wedge x) \cdot \nabla u - \omega \wedge u}{1 + |x|} \right\|_q \leq c \left( 1 + \frac{k}{\nu} \right) \| f \|_q
\]
(10)

with a constant \( c > 0 \) independent of \( k, \nu, \omega \).

(3) For all \( 1 < q < \infty \) the third component \( u_3 \) of the solution \( u \) satisfies the a priori estimate
\[
\left\| k\partial_3 u_3 \right\|_q + \left\| \frac{(\omega \wedge x) \cdot \nabla u_3}{1 + |x'|} \right\|_q \leq c \left( 1 + \frac{k}{\nu} \right) \| f \|_q.
\]
(11)

At the end of Section 2 we present a heuristic argument why \( L^q \)-estimates of \( k\partial_3 u \) and of \((\omega \wedge x) \cdot \nabla u - \omega \wedge u\) independent of \( \frac{k^2}{\nu |\omega|} \) are unlikely to hold and why weighted estimates with the weight \( \frac{1}{1+|x'|} \) will help.

2. Preliminaries and proofs. From [2] we recall the calculation of the explicit representation of the solution \( u \) of (2). First, we eliminate the pressure term by applying \( \text{div} \) to (2)\(_1\). Then \( \nabla p \) is seen to be the unique weak solution of the equation \( \Delta p = \text{div} f \) satisfying the a priori estimate
\[
\| \nabla p \|_q \leq c \| f \|_q.
\]
(12)

Hence \( u \) is the solenoidal solution of the equation
\[
-\nu \Delta u + k\partial_3 u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u = f - \nabla p
\]
(13)

where \( f - \nabla p \) is solenoidal. For simplicity, we will write \( f \) instead of \( f - \nabla p \) and divide by \( \tilde{\omega} = |\omega| > 0 \) to get
\[
-\frac{\nu}{|\omega|} \Delta u + \frac{k}{|\omega|} \partial_3 u - (e_3 \wedge x) \cdot \nabla u + e_3 \wedge u = \frac{1}{|\omega|} f.
\]
(14)

Next use cylindrical coordinates \( (r, \theta, x_3) \in \mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R}, \ r = |x'| = \sqrt{x_1^2 + x_2^2}, \) for \( x = (x_1, x_2, x_3)^T \) and observe that
\[
\partial_\theta u = (e_3 \wedge x) \cdot \nabla u = (-x_2, x_1) \cdot \nabla' u.
\]

Moreover, we introduce the Fourier transform
\[
\mathcal{F}u(\xi) = \hat{u}(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} u(x) \, dx, \quad \xi \in \mathbb{R}^3.
\]
For the Fourier variable $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ we also use cylindrical coordinates $(s, \varphi, \xi_3) \in \mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R}, \ s = |\xi'| = \sqrt{\xi_1^2 + \xi_2^2}$, and note that

$$\partial_{\theta} u = \partial_{\varphi} \hat{u}.$$

Thus $\hat{u}$ satisfies the equation

$$\frac{1}{|\omega|} (\nu |\xi|^2 + ik\xi_3) \hat{u} - \partial_{\varphi} \hat{u} + e_3 \wedge \hat{u} = \frac{1}{|\omega|} \hat{f}.$$  \hspace{1cm}(15)

This inhomogeneous, linear ordinary differential equation of first order with respect to $\varphi$ has a unique $2\pi$-periodic solution. An elementary calculation leads to the representation

$$\hat{u}(\xi) = \frac{1}{|\omega|} \int_{0}^{2\pi} e^{-(\nu |\xi|^2 + ik\xi_3)t/|\omega|} O^T(t) \hat{f}(O(t)\xi) \ dt,$$  \hspace{1cm}(16)

where

$$D(\xi) = 1 - e^{-2\nu(\nu |\xi|^2 + ik\xi_3)/|\omega|}.$$  \hspace{1cm}(17)

Moreover, using the geometric series and the $2\pi/|\omega|$ periodicity of $t \mapsto O^T(t) \hat{f}(O(t)\xi)$ w.r.t. $t$, we get the second representation

$$\hat{u}(\xi) = \int_{0}^{\infty} e^{-(\nu |\xi|^2 + ik\xi_3)t} O^T(t) \hat{f}(O(t)\xi) \ dt.$$  \hspace{1cm}(18)

Note that in $x$-space (18) leads to the identity

$$u(x) = \int_{0}^{\infty} E_t * O^T(t) f(O(t) - k\nu e_3)(x) \ dt,$$  \hspace{1cm}(19)

where $E$ denotes the heat kernel $E_t(x) = \frac{1}{(4\pi t)^{3/2}} \ e^{-|x|^2/4t}$ in $\mathbb{R}^3$.

**Proof of Theorem 2.** We start with the case $q = 2$ in which we may use the Theorem of Plancherel to estimate $\|k\partial_3 u\|_2$. By (16), (17), the inequality of Cauchy-Schwarz and Fubini’s Theorem,

$$\int |k\xi \hat{u}|^2 \ d\xi = \int \frac{k^2 \xi_3^2 / |\omega|^2}{|D(\xi)|^2} \left( \int_{0}^{2\pi} e^{-(\nu |\xi|^2 + ik\xi_3)t/|\omega|} O^T(t) \hat{f}(O(t)\xi) \ dt \right)^2 \ d\xi$$

$$\leq \int \frac{k^2 \xi_3^2 / |\omega|^2}{|D(\xi)|^2} \left( \int_{0}^{2\pi} e^{-\nu |\xi|^2 t/|\omega|} |\hat{f}(O(t)\xi)|^2 \ dt \right) \left( \int_{0}^{2\pi} e^{-\nu |\xi|^2 t/|\omega|} |\hat{f}(O(t)\xi)|^2 \ d\xi \right).$$

In the inner integral the change of variable formula implies that the term $|\hat{f}(O(t)\xi)|^2$ may be replaced by $|\hat{f}(\xi)|^2$. Then a further application of Fubini’s theorem yields

$$\int |k\xi \hat{u}|^2 \ d\xi = \int |\hat{f}(\xi)|^2 \frac{k^2 \xi_3^2 / |\omega|^2}{|D(\xi)|^2} \left( \frac{1 - e^{-2\nu \nu |\xi|^2 / |\omega|}}{\nu |\xi|^2 / |\omega|} \right) \left( \int_{0}^{2\pi} e^{-\nu |\xi|^2 t/|\omega|} \ d\xi \right) \ dt,$$  \hspace{1cm}(18)

$$\int |\hat{f}(\xi)|^2 \frac{k^2 \xi_3^2 / |\omega|^2}{|D(\xi)|^2} \left( \frac{1 - e^{-2\nu \nu |\xi|^2 / |\omega|}}{\nu^2 |\xi|^4 / |\omega|^2} \right) \ d\xi.$$
Hence it suffices to consider the 'multiplier function'

\[
m(\xi) = \frac{1 - e^{-\frac{2\pi \nu |\xi|^2}{|\omega|}}}{\nu |\xi|^2/|\omega|} \frac{k\xi_3/|\omega|}{D(\xi)}
\]  
(20)

and to prove the estimate

\[
|m(\xi)| \leq c(1 + \frac{k^2}{\nu |\omega|}), \quad \xi \in \mathbb{R}^3.
\]  
(21)

If \( \frac{\nu |\xi|^2}{|\omega|} > 1 \), then \( |D(\xi)| \) is bounded below by a positive constant and

\[
|m(\xi)| \leq c \frac{k\xi_3/|\omega|}{\nu |\xi|^2/|\omega|} \leq c \frac{k}{\nu |\xi|} \leq c \frac{k}{\sqrt{\nu |\omega|}}.
\]

If \( \frac{\nu |\xi|^2}{|\omega|} \leq 1 \), the first factor in the definition of \( m(\xi) \) is uniformly bounded. To estimate the remaining term

\[
m_0(\xi) = \frac{k\xi_3/|\omega|}{D(\xi)}
\]

we partition the ball \( \frac{\nu |\xi|^2}{|\omega|} \leq 1 \) into infinitely many slices \( S_n = \{ \xi \in \mathbb{R}^3 : \frac{\nu |\xi|^2}{|\omega|} \leq 1, \ |k\xi_3/|\omega| - n| \leq \frac{1}{4}, \ n \in \mathbb{Z} \} \), and the remaining part \( S' \) where \( \text{dist}(\frac{k\xi_3}{|\omega|}, \mathbb{Z}) \geq \frac{1}{4} \) and consequently \( |D(\xi)| \geq 1 \). Hence,

\[
|m_0(\xi)| \leq \frac{|k\xi_3|}{|\omega|} \leq \frac{k}{\sqrt{\nu |\omega|}} \quad \text{on} \quad S'.
\]

For \( \xi \in S_n, \ n \in \mathbb{Z} \), Taylor’s expansion of \( 1 - e^{-z} \) yields the lower bound

\[
|D(\xi)| = \left| 1 - e^{-2\pi \nu (\nu |\xi|^2/|\omega| + i(k\xi_3/|\omega| - n))} \right| \geq c_0 \left| \frac{\nu |\xi|^2}{|\omega|} + i \left( \frac{k\xi_3}{|\omega|} - n \right) \right|
\]

with a constant \( c_0 > 0 \) independent of all variables \( \nu, \xi, k, \omega, n \). Hence for \( \xi \in S_0 \) we get the estimate \( |m_0(\xi)| \leq \frac{1}{c_0} \). If \( \xi \in S_n, \ n \neq 0 \), then

\[
|m_0(\xi)| \leq \frac{|k\xi_3|/|\omega|}{\nu |\xi|^2/|\omega|} \leq \frac{k}{\nu |\xi|} \leq \frac{4}{3} \frac{k^2}{\nu |\omega|},
\]

since \( \frac{\nu |\xi|^2}{|\omega|} \geq \frac{3}{4} \). Now (21) is proved and implies the estimate

\[
\int |i k\xi_3 \hat{u}|^2 d\xi \leq c \left( 1 + \frac{k^2}{\nu |\omega|} \right)^2 \int |f|^2 d\xi.
\]

Then the Theorem of Plancherel completes the proof in the case \( q = 2 \).

For the case \( q \neq 2 \) we write (16) in the form

\[
\hat{u}(\xi) = \frac{1}{|\omega|} \int_0^{2\pi} \frac{1}{D(\xi)} e^{-\nu |\xi|^2} O^T(t) (\mathcal{F} f(O(t) \cdot - kte_3/|\omega|))(\xi) dt
\]

using that \( e^{-itk\xi_3} \in \mathcal{S}'(\mathbb{R}^3) \) is the Fourier transform of the shift operator \( f \mapsto f(\cdot - kte_3) \) on \( \mathcal{S}'(\mathbb{R}^3) \). Hence in \( x \)-space,

\[
k\partial_3 u(x) = \int_0^{2\pi} T_t F(t, \cdot)(x) dt
\]
\[ F(t, \cdot) = O^T(t) f(O(t) \cdot - k t e_3/|\omega|) \]

and the operator family \( T_t, 0 < t < 2\pi, \) is defined by its multiplier
\[ m_t(\xi) = \frac{i k \xi_3/|\omega|}{D(\xi)} e^{-\nu|\xi|^2 t/|\omega|}, \] (22)
i.e., \( T_t = F^{-1}(m_t(\xi) \cdot) \). Note that \( \| F(t, \cdot) \|_q \leq \| f \|_q \) for all \( t \in (0, 2\pi) \).

Below we will prove the multiplier estimate
\[ \max_\alpha \sup_{\xi \neq 0} |\xi^\alpha D_\xi^\alpha m_t(\xi)| \leq c \left( 1 + \frac{k}{\sqrt{\nu|\omega|} t} + \frac{k^2}{\nu|\omega|} \right) \cdot \left( 1 + \frac{k^2}{\nu|\omega|} \right) \] (23)
with a constant \( c > 0 \) independent of \( \nu, \omega, k \) and \( t \); here \( \alpha \in \mathbb{N}_0^3 \) runs through the set of all multi-indices \( \alpha \in \{0, 1\}^3 \). Then Marcinkiewicz’ multiplier theorem [14] implies that in the operator norm \( \| \cdot \|_q \) on \( L^q \)
\[ \| T_t \|_q \leq c \left( 1 + \frac{k}{\sqrt{\nu|\omega|} t} + \frac{k^2}{\nu|\omega|} \right) \cdot \left( 1 + \frac{k^2}{\nu|\omega|} \right). \]

Hence
\[ \| k \partial_3 u \|_q \leq c \int_0^{2\pi} \| T_t \|_q \| F(t, \cdot) \|_q dt \leq c \left( 1 + \frac{k^4}{\nu^2|\omega|^2} \right) \| f \|_q \] (24)
with a constant \( c > 0 \) independent of \( \nu, \omega, k \).

Now the \( L^2 \)-result and (24) are combined by using complex multiplier theory to prove (6). Given \( 1 < q < 2 \) we formally interpolate between the \( L^2 \)-result and the \( L^1 \)-result (24) using \( \theta = \frac{2}{q} - 1 \) such that \( \frac{1}{q} = \frac{1 - \theta}{2} + \frac{\theta}{4} \). Then
\[ \| \partial_3 u \|_q \leq c \left( 1 + \frac{k^2}{\nu|\omega|} \right)^{2-2/q} \left( 1 + \frac{k^4}{\nu^2|\omega|^2} \right)^{2/q-1} \| f \|_q \]
yielding \( \| \partial_3 u \|_q \leq c (1 + \frac{k^2}{\nu|\omega|})^{2/q} \| f \|_q \). Since no estimate (24) holds for \( L^1 \), we have to interpolate between \( L^2 \) and \( L^p \) for \( p > 1 \) arbitrarily close to 1. Therefore, the additional exponent \( \varepsilon > 0 \) in (6) occurs. If \( 2 < q < \infty \), we formally interpolate between \( L^2 \) and \( L^\infty \) to get (6) with the exponent \( 2(1 - 1/q) \). Since again there is no \( L^\infty \)-result available, we choose \( p \) arbitrarily large instead of \( p = \infty \) and have to add the exponent \( \varepsilon > 0 \) in (6).

Finally we prove (23), start with \( m_t \) itself and distinguish between the cases \( \frac{\nu|\xi|^2}{|\omega|} > 1 \) and \( \frac{\nu|\xi|^2}{|\omega|} \leq 1 \). In the first case \( |D(\xi)| \) is bounded below by a positive constant yielding
\[ |m_t(\xi)| \leq \frac{k |\xi_3|}{|\omega|} e^{-\nu|\xi|^2 t/|\omega|} \leq c \frac{k}{\sqrt{\nu|\omega|} t}. \]
If \( \frac{\nu|\xi|^2}{|\omega|} \leq 1 \), we may neglect the term \( e^{-\nu|\xi|^2 t/|\omega|} \) and conclude from the detailed estimates of \( m_0(\xi) \) in the \( L^2 \)-case above that
\[ |m_t(\xi)| \leq |m_0(\xi)| \leq c \left( 1 + \frac{k^2}{\nu|\omega|} \right). \]
Hence $|m_t(\xi)| \leq c(1 + \frac{k}{\sqrt{|\nu|t}} + \frac{k^2}{|\nu|}|\xi^2|)$ proving (23) for $m_t$. Next consider

$$\xi_3 \partial_3 m_t(\xi) = m_t(\xi) - \frac{2\nu \xi^2_3 t}{|\nu|} m_t(\xi) - \xi_3 \frac{\partial D(\xi)}{D(\xi)} m_t(\xi)$$

where

$$\xi_3 \frac{\partial D(\xi)}{D(\xi)} = 2\pi \frac{(2\nu \xi^2_3 + ik\xi_3)/|\nu|}{D(\xi)} e^{-2\pi(|\nu|\xi^2 + ik\xi_3)/|\nu|}.$$  

Writing the exponential function $e^{-|\nu|\xi^2 t/|\nu|}$ as $e^{-|\nu|\xi^2 t/|\nu|} \cdot e^{-|\nu|\xi^2 t/|\nu|}$, we see that the second term on the right-hand side of (25) may be estimated as $m_t$ itself. In the third term note—due to properties of $D(\xi)$ proved above—that $\frac{2\nu \xi^2_3}{|\nu|} e^{-2\pi(|\nu|\xi^2 + ik\xi_3)/|\nu|}$ is uniformly bounded. Moreover, the estimate of $\frac{k \xi_3}{|\nu|} e^{-2\pi(|\nu|\xi^2 + ik\xi_3)/|\nu|}$ is similar to the estimate of the multiplier function $m(\xi)$ in (20), (21) yielding

$$\frac{k |\xi_3|/|\nu|}{D(\xi)} e^{2\pi(|\nu|\xi^2 + ik\xi_3)/|\nu|} \leq c \left(1 + \frac{k^2}{|\nu|}\right).$$

Combining the previous estimates we get (23) for $\xi_3 \partial_3 m_t$. Concerning $\xi_1 \partial_1 m_t(\xi)$ note that (26) must be replaced by

$$\xi_1 \frac{\partial D(\xi)}{D(\xi)} = 2\pi \frac{2\nu \xi^2_1}{|\nu|} e^{-2\pi(|\nu|\xi^2 + ik\xi_3)/|\nu|}.$$ 

Obviously, looking at the properties of $D(\xi)$ proved above, the modulus of this term is uniformly bounded requiring no further power of $k^2/|\nu|$. Since the same assertion holds for the derivatives $\xi_2 \partial_2$ and $\xi_1 \partial_1 \xi_2 \partial_2$ of $m_t$, (23) is completely proved. ■

**Example 3.** For fixed $k > 0$, $\nu > 0$ we will construct a sequence of solenoidal forces $(f) = (f_0) \subset L^2(\mathbb{R}^3)^3, |\omega| \to 0$, such that the corresponding sequence of solutions $(u) = (u_\omega)$ will satisfy

$$\|k \partial_3 u\|_q \geq c \frac{k^2}{|\nu|} \|f\|_q$$

with $c > 0$ independent of $k, \nu, \omega$. Given $k > 0, \nu > 0$ choose $|\omega|$ small enough such that

$$\frac{\nu |\omega|}{k^2} < \frac{1}{16}.$$ 

Then define $f = (f', 0) \in L^2(\mathbb{R}^3)^3$ such that in Fourier space

$$\hat{f}'(\xi) = i \left\{ \begin{array}{ll} \xi^\perp, & 0 < \varphi < \pi \\ -\xi^\perp, & \pi < \varphi < 2\pi \end{array} \right., \quad \text{when} \quad \left| \frac{k |\xi'|}{|\omega|} - 1 \right| < \frac{1}{2}, \quad \left| \frac{k |\xi_3|}{|\omega|} - 1 \right| < \frac{1}{2},$$

but $\hat{\hat{f}}'(\xi) = 0$ elsewhere; here, as usual, $\varphi$ is the angular part of $\xi$. Since $\hat{\hat{f}}(\xi) = \hat{\hat{f}}(-\xi)$, the vector field $f$ is real-valued and obviously solenoidal. By (16)

$$\hat{\hat{f}}(\xi) = \frac{e^{i|\nu|\xi^2 + ik\xi_3}\varphi/|\omega|}{D(\xi)|\omega|} O(\varphi) \int_{\varphi}^{\pi + 2\pi} e^{-(3|\nu|\xi^2 + ik\xi_3)t/|\omega|} O^T(t) \hat{f}(O(t)e_1) \, dt.$$
Since $O^T(t)\hat{f}(O(t)e_1) = +e_2$ and $-e_2$ when $0 < \varphi < \pi$ and $\pi < \varphi < 2\pi$, resp., a simple integration leads to the formula
\[
  ik\xi_3\hat{u}(\xi) = \hat{f}(\xi) \frac{ik\xi_3/|\omega|}{(\nu|\xi|^2 + ik\xi_3)/|\omega|} \left(1 - \frac{2e^{-\pi(\nu|\xi|^2 + ik\xi_3)/|\omega|}}{1 + e^{-\pi(\nu|\xi|^2 + ik\xi_3)/|\omega|}}\right),
\]
when $0 < \varphi < \pi$; for $\pi < \varphi < 2\pi$ the exponential term $e^{-\pi(\nu|\xi|^2 + ik\xi_3)/|\omega|}$ must be replaced by $e^{-(\pi-\varphi)(\nu|\xi|^2 + ik\xi_3)/|\omega|}$. The assumptions on $k, \nu, \omega$ imply for $\xi \in \text{supp} \hat{f}$ that $\frac{|k\xi_3/|\omega|}{(\nu|\xi|^2 + ik\xi_3)/|\omega|} \sim \frac{\nu|\omega|}{k^2}$; consequently, we have $|\nu|\xi|^2 + ik\xi_3)/|\omega| \sim |\xi_3/\omega| \sim 1$ and $e^{-(\pi-\varphi)(\nu|\xi|^2 + ik\xi_3)/|\omega|} \sim 1$. Finally, the crucial term is
\[
  |1 + e^{-\pi(\nu|\xi|^2 + ik\xi_3)/|\omega|}| \sim \frac{\nu|\xi|^2}{|\omega|} \sim \frac{\nu|\omega|}{k^2} \quad \text{for} \quad \xi \in \text{supp} \hat{f}.
\]
Hence
\[
  |\xi_3\hat{u}(\xi)| \sim \frac{k^2}{\nu|\omega|} |\hat{f}(\xi)| \quad \text{for} \quad \xi \in \text{supp} \hat{f}.
\]
Since all similarity estimates $\sim$ can be made precise by using positive constants independent of $k, \nu, \omega$, (27) is proved.

**Proof of Theorem 4.** (i) Given a solution $u \in \dot{W}^{2,q}(\mathbb{R}^3)^3$, i.e. a function $u$ with $\nabla^2 u \in L^q(\mathbb{R}^3)$, Theorems 1 and 2 yield $\beta \in \mathbb{R}$ such that $\nabla'(u - \beta(\omega \wedge x)) \in L^r(\mathbb{R}^3)^6$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{4}$, and $\partial_\omega(u - \beta(\omega \wedge x)) \in L^q(\mathbb{R}^3)^3$. Since also $u - \beta(\omega \wedge x)$ solves (2), assume without loss of generality that $\beta = 0$ implying for a.a. $x_3 \in \mathbb{R}$ that
\[
  \int_{\mathbb{R}^2} |\nabla'(x', x_3)|^r \, dx' < \infty, \quad \int_{\mathbb{R}^2} |\partial_3 u(x', x_3)|^q \, dx' < \infty.
\]
Then classical arguments show the existence of a sequence of radii $(R_j) \subset \mathbb{R}_+$ such that
\[
  \int_0^{2\pi} |\nabla'(R_j, \theta, x_3)|^r \, d\theta = o(R_j^{-2}), \quad \int_0^{2\pi} |\partial_3 u(R_j, \theta, x_3)|^q \, d\theta = o(R_j^{-2}) \quad (28)
\]
as $j \to \infty$.

On the other hand, since $1 < q < 2$ and $\nabla^2 u \in L^q(\mathbb{R}^3)$, Theorem II5.1 in [6] yields for a.a. $x_3 \in \mathbb{R}$ a matrix $A(x_3) \in \mathbb{R}^{3,3}$ such that
\[
  \left( \int_{\mathbb{R}^2} \frac{\nabla u(x', x_3) - A(x_3)}{|x'|^q} \, dx' \right)^{1/q} \leq \frac{q}{2 - q} \left( \int_{\mathbb{R}^2} |\nabla'\nabla u(x', x_3)|^q \, dx' \right)^{1/q}. \quad (29)
\]
Note that Theorem II5.1 in [6] is stated only for exterior domains; however, since the constant $q/(2 - q)$ does not depend on the 'inner radius' of the exterior domain, the estimate holds for the whole space $\mathbb{R}^2$ as well. Moreover, by Lemma 5.2 in [6],
\[
  \int_0^{2\pi} |\nabla u(R, \theta, x_3) - A(x_3)|^q \, d\theta = o(R^{-2}) \quad (30)
\]
as $R \to \infty$. Now (28), (30) show that $A(x_3) = 0$; hence (29) and (3) yield (7). Then (8) is an easy consequence of (7) and of (2).

(ii) If $1 < q < 3$, Theorem II5.1 in [6] yields the estimate
\[
  \left( \int_{\mathbb{R}^3} \frac{|\nabla u(x) - A|^q}{|x|} \, dx \right)^{1/q} \leq \frac{q}{3 - q} \left( \int_{\mathbb{R}^3} |\nabla^2 u(x)|^q \, dx \right)^{1/q}.
\]
with a constant matrix $A \in \mathbb{R}^{3,3}$. Moreover, by Lemma 5.2 in [6],
\[
\int_{|y|=1} |\nabla u(Ry)|^q \, do(y) = o(R^{q-3})
\]
as $R \to \infty$, where $\int_{|y|=1} \ldots \, do(y)$ denotes the surface integral on the unit sphere of $\mathbb{R}^3$. Since $\nabla' u \in L^q(\mathbb{R}^3)^6$ and $\partial_3 u \in L^q(\mathbb{R}^3)^3$, arguments as above imply that $A$ vanishes. Now (9) and (10) are easy consequences.

(iii) By (2) $u_3$ solves the problem $-\nu \Delta u_3 + k \partial_3 u_3 - (\omega \wedge x) \cdot \nabla u_3 = f_3$. Since $(\omega \wedge x) \cdot \nabla u_3 = |\omega| \partial_3 u_3$, an integration w.r.t. $\theta \in (0,2\pi)$ yields for the $\theta$-independent function $U_3(x) := \frac{1}{2\pi} \int_0^{2\pi} u_3(|x'|, \theta', x_3) \, d\theta'$ the equation
\[
-\nu \Delta U_3 + k \partial_3 U_3 = \frac{1}{2\pi} \int_0^{2\pi} f_3 \, d\theta'.
\]
Applying Fourier transforms and using Marcinkiewicz’ multiplier theorem we get that $U_3$ satisfies the estimate
\[
\|k \partial_3 U_3\|_q \leq c\|f\|_q
\]
with a constant $c > 0$ independent of $f$, $k$, $\nu$, cf. the analysis of the related Oseen problem [1], [5], [6]. By Wirtinger’s inequality there exists a constant $c > 0$ such that for a.a. $r = |x'| > 0$ and $x_3 \in \mathbb{R}$
\[
\left\| \partial_3 u_3(r, \cdot, x_3) \right\|_{L^q(0,2\pi)} \leq c \left( \left\| \partial_\theta \partial_3 u_3(r, \cdot, x_3) \right\|_{L^q(0,2\pi)} + \left\| \partial_3 U_3(r, \cdot, x_3) \right\|_{L^q(0,2\pi)} \right).
\]
Now divide by $1 + r$ and integrate w.r.t. $r \, dr$, $r > 0$, and $dx_3$, $x_3 \in \mathbb{R}$, to get that
\[
\left\| \frac{\partial_3 u_3}{1 + |x'|} \right\|_q \leq c \left( \left\| \frac{\partial_\theta \partial_3 u_3}{1 + |x'|} \right\|_q + \left\| \frac{\partial_3 U_3}{1 + |x'|} \right\|_q \right).
\]
Since the second term on the right-hand side is bounded by $\|\partial_3 \nabla' u_3\|_q \leq (c/\nu)\|f\|_q$ and since the third term is bounded by $\|\partial_3 U_3\|_q \leq (c/k)\|f\|_q$, we get (11).

**Remark.** The ideas of the proof of Theorem 4 (iii) do not apply to $u_1$ and $u_2$, since the term $\omega \wedge u$ does not vanish when applying the integration $\int_0^{2\pi} \ldots \, d\theta'$. Also the identity $(\omega \wedge x) \cdot \nabla u - \omega \wedge u = |\omega|O(\theta) \partial_\theta (O^T(\theta)u)$ will not help, since no a priori estimates of $\partial_3 \partial_\theta (O^T(\theta)u)$ are available except for the case when $1 < q < 2$.

**Heuristic argument.** Let us motivate why estimates of $(\omega \wedge x) \cdot \nabla u - \omega \wedge u$ and of $k \partial_3 u$ cannot be expected to be independent of $k^2/\nu|\omega|$. For simplicity ignore the terms $\omega \wedge u$ and $p$, recall that $(\omega \wedge x) \cdot \nabla u = |\omega| \partial_\theta u$ and let us perform a simple scaling analysis. Define the non-dimensional quantities $\tilde{u} = |\omega|u/A$, where $A \in \mathbb{R}$ is a characteristic acceleration of the flow, and $\tilde{x} = x\sqrt{|\omega|}/\nu$. Then, dividing (2) by $A$ and omitting $\gamma$, (2) simplifies to the non-dimensional equation
\[
-\Delta \tilde{u} + \frac{k}{\sqrt{\nu|\omega|}} \partial_3 \tilde{u} - \partial_\theta \tilde{u} = f \quad \text{in} \quad \mathbb{R}^3.
\]
Note that $\frac{k}{\sqrt{\nu|\omega|}}$ is a new non-dimensional characteristic number of the flow. For fixed $r = |x'|$ let us interpret $\frac{k}{\sqrt{\nu|\omega|}} \partial_3 \tilde{u} - \partial_\theta \tilde{u}$ as a directional derivative defined by the unit
vector
\[ d_\omega(x') = \frac{1}{\sqrt{r^2 + k^2/\nu \omega}} \left( \frac{k}{\sqrt{\nu |\omega|}} e_3 - (-x_2, x_1, 0)^T \right) \in \mathbb{R}^3 \]
which is tangential to the cylinder \( C_r = \{ x \in \mathbb{R}^3 : |x'| = r \} \). Hence, defining the curve
\[ \gamma_\omega(s) = \left( -r \cos s, -r \sin s, \frac{k}{\sqrt{\nu |\omega|}} s \right)^T \]
on \( C_r \) with tangential vector \( \sqrt{r^2 + k^2/\nu \omega} d_\omega \), we get that
\[ \frac{d}{ds} u(\gamma_\omega(s)) = (\sqrt{r^2 + k^2/\nu \omega} d_\omega \cdot \nabla u)(\gamma_\omega(s)). \]
Obviously \( d_\omega \) converges to the third unit vector \( e_3 \), whereas the curve \( \gamma_\omega(s) \) has no reasonable limit on the cylinder \( C_r \). In this sense, the information on the directional derivative \( d_\omega \cdot \nabla u \) on \( C_r \) is lost in the limit \( \omega = 0 \). This discrepancy vanishes for \( r = 0 \), but gets larger as \( r \to \infty \). Therefore, the weight \( \frac{1}{1+|x'|} \) has to occur in Theorem 1.4.

References
