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ESTIMATES OF LOWER ORDER DERIVATIVES OF VISCOUS FLUID FLOW PAST A ROTATING OBSTACLE

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Abstract. Consider the problem of time-periodic strong solutions of the Stokes system modelling viscous incompressible fluid flow past a rotating obstacle in the whole space $\mathbb{R}^3$. Introducing a rotating coordinate system attached to the body yields a system of partial differential equations of second order involving an angular derivative *not* subordinate to the Laplacian. In a recent paper [2] the author proved L^q-estimates of second order derivatives uniformly in the angular and translational velocities, ω and k, of the obstacle, whereas the transport terms fails to have L^q-estimates independent of ω. In this paper we clarify this unexpected behavior and prove weighted L^q-estimates of first order terms independent of ω.

1. Introduction. Consider the Navier-Stokes equations modelling viscous flow past a rotating body $K \subset\subset \mathbb{R}^3$ with axis of rotation $\omega = \tilde{\omega} e_3 = \tilde{\omega}(0,0,1)^T$, $\tilde{\omega} = |\omega| > 0$, and with velocity $u_\infty = ke_3$, $k > 0$, at infinity. Then the velocity v and the presure p satisfy the system

$$\begin{aligned} v_t - \nu\Delta v + v\cdot\nabla v + \nabla q &= \tilde{f} && \text{in } \Omega(t),\ t>0,\\ \operatorname{div} v &= 0 && \text{in } \Omega(t),\ t>0,\\ v(y,t) &= \omega\wedge y && \text{on } \partial\Omega(t),\ t>0,\\ v(y,t) &\to u_\infty \neq 0 && \text{as } |y|\to\infty, \end{aligned}$$

with an initial value $v(y,0) = v_0(y)$, with constant viscosity $\nu > 0$ and external force $\tilde{f}$ in the time-dependent exterior domain $\Omega(t) = O_\omega(t)\Omega$; here $O_\omega(t)$ denotes the orthogonal matrix

$$O_\omega(t) = \begin{pmatrix} \cos|\omega|t & -\sin|\omega|t & 0\\ \sin|\omega|t & \cos|\omega|t & 0\\ 0 & 0 & 1 \end{pmatrix}.$$

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Introducing the new independent and dependent variables

$$x = O_\omega^T(t)y, \quad u(x,t) = O_\omega^T(t)\big(v(y,t) - u_\infty\big), \quad p(x,t) = q(y,t),$$

and linearizing, (u,p) will satisfy the modified Stokes system

$$\begin{aligned} u_t - \nu\Delta u + k\partial_3 u - (\omega \wedge x)\cdot\nabla u + \omega\wedge u + \nabla p = f, \\ \operatorname{div} u = 0, \end{aligned} \tag{1}$$

in a time-independent exterior domain $\Omega \subset \mathbb{R}^3$ together with the initial-boundary condition $u(x,t) = \omega \wedge x - u_\infty$ for $x \in \partial\Omega$, $u(x,0) = u_0$, $u \to 0$ as $|x| \to \infty$. In the stationary whole space case to be analyzed in this paper we are led to the elliptic equation

$$-\nu\Delta u + k\partial_3 u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p = f, \quad \operatorname{div} u = 0 \text{ in } \mathbb{R}^3 \tag{2}$$

in which the term $(\omega\wedge x)\cdot\nabla u$ is *not* subordinate to $-\nu\Delta u$. Note that a stationary solution (u,p) of (2) is related to a time-periodic solution of (1).

In [2] the author proved *a priori* estimates of strong solutions (u,p) of (2) which are found in the homogeneous Sobolev spaces $\hat W^{2,q}(\mathbb{R}^3)^3 \times \hat W^{1,q}(\mathbb{R}^3)$ where

$$\hat W^{k,q}(\Omega) = \{u \in L^1_{\rm loc}(\overline{\Omega})/\Pi_{k-1} : \partial^\alpha u \in L^q(\Omega) \text{ for all } \alpha\in\mathbb{N}_0^n, |\alpha| = k\}.$$

Here $\partial^\alpha = \partial_1^{\alpha_1}\cdot\ldots\cdot\partial_n^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1,\ldots,\alpha_n)\in\mathbb{N}_0^n$ and Π_{k-1} denotes the set of all polynomials on $\mathbb{R}^n$ of degree $\le k-1$. The space $\hat W^{k,q}(\Omega)$ consists of equivalence classes of $L^1_{\rm loc}$-functions being unique only up to elements from Π_{k-1} and is equipped with the norm $\sum_{|\alpha|=k}\|\partial^\alpha u\|_q$, where $\|\cdot\|_q$ denotes the L^q-norm. However, sometimes being less careful, we will consider $v \in \hat W^{k,q}(\Omega)$ as a function (representative) rather than an equivalence class of functions, i.e., $v\in L^1_{\rm loc}(\Omega)$ such that $\partial^\alpha v \in L^q(\Omega)$ for every multi-index α with $|\alpha| = k$. For further results on similar problems we refer to [4], [8], [9], [10], [11], [12] and [13].

THEOREM 1. (1) *Let* $1 < q < \infty$, $\nu > 0$, $k > 0$, $\omega = (0,0,\tilde\omega)^T \in \mathbb{R}^3\backslash\{0\}$, *and let* $f \in L^q(\mathbb{R}^3)^3$. *Then the linear problem (2) has a solution* $(u,p)\in \hat W^{2,q}(\mathbb{R}^3)^3\times\hat W^{1,q}(\mathbb{R}^3)$ *satisfying the a priori estimate*

$$\|\nu\nabla^2 u\|_q + \|\nabla p\|_q \le c\|f\|_q \tag{3}$$

with a constant c *independent of* ν, k *and* ω.

(2) *Moreover,*

$$\|k\partial_3 u\|_q + \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_q \le c\left(1 + \frac{k^4}{\nu^2|\omega|^2}\right)\|f\|_q \tag{4}$$

with a constant $c > 0$ *independent of* ν, k *and* ω.

(3) *Let* $1 < q < 4$, $f \in L^q(\mathbb{R}^3)^3$ *and let* $(u,p)\in\hat W^{2,q}(\mathbb{R}^3)^3\times\hat W^{1,q}(\mathbb{R}^3)$ *be the solution of (2). Then there exists* $\beta\in\mathbb{R}$ *such that*

$$\nabla'(u - \beta\omega\wedge x) \in L^r(\mathbb{R}^3)^6 \quad \textit{for all} \quad r > 1,\ \frac{1}{r}\in\frac{1}{q} - \left[\frac{1}{4},\frac{1}{3}\right],$$

and there exists a constant $C = C(\nu,k,\omega;r) > 0$ *such that*

$$\|\nabla'(u-\beta\omega\wedge x)\|_r \le C\big(\|f\|_q + \|\nu\nabla g + (\omega\wedge x)g - kge_3\|_q\big). \tag{5}$$

The proof of Theorem 1(1), see [2], is based on an explicit representation of u the estimate of which uses Fourier transforms, Hardy-Littlewood decomposition methods and maximal operators. Estimate (4) shows a surprising and crucial dependence on $\frac{1}{|\omega|}$ via the term $\frac{k^4}{\nu^2|\omega|^2}$. On the one hand, it is not at all clear that the terms $k\partial_3 u$ and $(\omega \wedge x)\cdot\nabla u - \omega\wedge u$ can be estimated in $L^q(\mathbb{R}^3)$ separately from each other. On the other hand, the dependence on $\frac{1}{|\omega|}$ seems to be unnatural. Note that, as $|\omega| \to 0$, problem (2) converges formally to Oseen's equation

$$-\nu\Delta u + k\partial_3 u + \nabla p = f, \quad \operatorname{div} u = 0,$$

the solutions of which satisfy the estimate $\|k\partial_3 u\|_q \le c\|f\|_q$, see [1, 5]. To be more precise, a sequence of solutions (u_ω) converges weakly in $\hat{W}^{2,q}(\mathbb{R}^3)^3$ to the solution of Oseen's equation as $|\omega| \to 0$, cf. [2] Remark 1.3(1).

Concerning Theorem 1(3) note that the solutions of the homogeneous system (2) are given by $\beta\omega\wedge x + \alpha e_3, \alpha, \beta \in \mathbb{R}$. Hence, with u also $u - \beta\omega\wedge x$ is a solution of (2). The proof of Theorem 1(3) uses an improved Sobolev embedding theorem exploiting the fact that besides $\nabla^2 u$ also $k\partial_3 u$ lies in L^q, cf. [1]. However, the dependence on $1/|\omega|$ in (4) implies that the constant C in (5) also depends on $1/|\omega|$. Due to this dependence the above-mentioned weak convergence of (u_ω) in $\hat{W}^{2,q}(\mathbb{R}^3)^3$, i.e. of second order derivatives in L^q, seems not to extend to $k\partial_3 u_\omega$ in L^q.

The aim of this paper is to clarify these unusual features. In Theorem 2 below we present an improvement of (4) and simplify the proof given in [2]. Then, for small q, we prove a weighted L^q-estimate of $k\partial_3 u$ and of $(\omega\wedge x)\cdot\nabla u - \omega\wedge u$ independent of $|\omega|$, see Theorem 4. An example and a heuristic argument will show that the dependence in (4) on $\frac{k}{\sqrt{\nu|\omega|}}$ is not a weakness of the proof.

Theorem 2. *Let* $1 < q < \infty$, $\nu > 0$, $k \in \mathbb{R}$, $\omega = (0,0,\tilde{\omega})^T \in \mathbb{R}^3\setminus\{0\}$ *and let* $f \in L^q(\mathbb{R}^3)^3$. *Then the solution* $u \in \hat{W}^{2,q}(\mathbb{R}^3)^3$ *satisfies the a priori estimate*

$$\|k\partial_3 u\|_q + \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_q \le c\left(1 + \frac{k^2}{\nu|\omega|}\right)^{2\max(1/q,1-1/q)+\varepsilon} \|f\|_q \tag{6}$$

with a constant $c > 0$ *independent of* ν, k *and* ω*; here* $\varepsilon > 0$ *can be chosen arbitrarily small and* $\varepsilon = 0$ *if* $q = 2$.

Example 3. In Section 2 we will show that the term $\frac{k^2}{\nu|\omega|}$ is needed in the L^2-case. However, it is not clear whether the exponent $2\max(1/q, 1-1/q)+\varepsilon > 1$ is necessary if $q \ne 2$. We note that in [13] dealing with the nonlinear problem in exterior domains no dependence of *a priori* estimates on $\frac{1}{|\omega|}$ occurs; the reason is that the author uses strong and weak *a priori* L^2-estimates of u by assuming that even $f \in L^{6/5}(\mathbb{R}^3) \subset \hat{W}^{-1,2}(\mathbb{R}^3)$. Using results of a forthcoming paper [3] dealing with the weak L^q-theory of (2) it is obvious that $\|\nu\nabla u\|_q$ is bounded uniformly w.r.t. ω and k by suitable norms of f.

Theorem 4. (1) *Let* $1 < q < 2$, $\nu > 0$, $k > 0$, $\omega = (0,0,\tilde{\omega})^T \in \mathbb{R}^3\setminus\{0\}$, *and let* $f \in L^q(\mathbb{R}^3)^3$. *Then (2) has a solution* $u \in \hat{W}^{2,q}(\mathbb{R}^3)^3$ *satisfying the a priori estimates*

$$\left\|\frac{\nabla u}{|x'|}\right\|_q \le \frac{c}{\nu}\|f\|_q \tag{7}$$

and

$$\left\| \frac{(\omega \wedge x) \cdot \nabla u - \omega \wedge u}{1+|x'|} \right\|_q \le c\left(1+\frac{k}{\nu}\right)\|f\|_q \tag{8}$$

with a constant $c > 0$ *independent of* k, ν, ω; *here, for* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ *the term* $|x'|$ *denotes the Euclidean length* $\sqrt{x_1^2 + x_2^2}$ *of* $x' = (x_1, x_2)$.

(2) *If* $1 < q < 3$, *then*

$$\left\| \frac{\nabla u}{|x|} \right\|_q \le \frac{1}{\nu}\|f\|_q \tag{9}$$

and

$$\left\| \frac{(\omega \wedge x) \cdot \nabla u - \omega \wedge u}{1+|x|} \right\|_q \le c\left(1+\frac{k}{\nu}\right)\|f\|_q \tag{10}$$

with a constant $c > 0$ *independent of* k, ν, ω.

(3) *For all* $1 < q < \infty$ *the third component* u_3 *of the solution* u *satisfies the a priori estimate*

$$\left\| \frac{k\partial_3 u_3}{1+|x'|} \right\|_q + \left\| \frac{(\omega \wedge x) \cdot \nabla u_3}{1+|x'|} \right\|_q \le c\left(1+\frac{k}{\nu}\right)\|f\|_q. \tag{11}$$

At the end of Section 2 we present a heuristic argument why L^q-estimates of $k\partial_3 u$ and of $(\omega \wedge x) \cdot \nabla u - \omega \wedge u$ independent of $\frac{k^2}{\nu|\omega|}$ are unlikely to hold and why weighted estimates with the weight $\frac{1}{1+|x'|}$ will help.

2. Preliminaries and proofs. From [2] we recall the calculation of the explicit representation of the solution u of (2). First, we eliminate the pressure term by applying div to $(2)_1$. Then ∇p is seen to be the unique weak solution of the equation $\Delta p = \operatorname{div} f$ satisfying the *a priori* estimate

$$\|\nabla p\|_q \le c\|f\|_q. \tag{12}$$

Hence u is the solenoidal solution of the equation

$$-\nu\Delta u + k\partial_3 u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u = f - \nabla p \tag{13}$$

where $f - \nabla p$ is solenoidal. For simplicity, we will write f instead of $f - \nabla p$ and divide by $\tilde{\omega} = |\omega| > 0$ to get

$$-\frac{\nu}{|\omega|}\Delta u + \frac{k}{|\omega|}\partial_3 u - (e_3 \wedge x) \cdot \nabla u + e_3 \wedge u = \frac{1}{|\omega|} f. \tag{14}$$

Next use cylindrical coordinates $(r, \theta, x_3) \in \overline{\mathbb{R}_+} \times [0, 2\pi) \times \mathbb{R}$, $r = |x'| = \sqrt{x_1^2 + x_2^2}$, for $x = (x_1, x_2, x_3)^T$ and observe that

$$\partial_\theta u = (e_3 \wedge x) \cdot \nabla u = (-x_2, x_1) \cdot \nabla' u.$$

Moreover, we introduce the Fourier transform

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} u(x)\, dx, \quad \xi \in \mathbb{R}^3.$$

For the Fourier variable $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ we also use cylindrical coordinates $(s, \varphi, \xi_3) \in \overline{\mathbb{R}_+} \times [0, 2\pi) \times \mathbb{R}$, $s = |\xi'| = \sqrt{\xi_1^2 + \xi_2^2}$, and note that

$$\widehat{\partial_\theta u} = \partial_\varphi \hat{u}.$$

Thus $\hat{u}$ satisfies the equation

$$\frac{1}{|\omega|}\big(\nu|\xi|^2 + ik\xi_3\big)\hat{u} - \partial_\varphi \hat{u} + e_3 \wedge \hat{u} = \frac{1}{|\omega|}\,\hat{f}. \tag{15}$$

This inhomogeneous, linear ordinary differential equation of first order with respect to φ has a unique 2π-periodic solution. An elementary calculation leads to the representation

$$\hat{u}(\xi) = \frac{1/|\omega|}{D(\xi)} \int_0^{2\pi} e^{-(\nu|\xi|^2 + ik\xi_3)t/|\omega|} O^T(t)\, \hat{f}\big(O(t)\xi\big)\, dt, \tag{16}$$

where

$$D(\xi) = 1 - e^{-2\pi(\nu|\xi|^2 + ik\xi_3)/|\omega|}. \tag{17}$$

Moreover, using the geometric series and the $\frac{2\pi}{|\omega|}$–periodicity of $t \mapsto O_\omega^T(t)\hat{f}\big(O_\omega(t)\xi\big)$ w.r.t. t, we get the second representation

$$\hat{u}(\xi) = \int_0^\infty e^{-(\nu|\xi|^2 + ik\xi_3)t} O_\omega^T(t)\, \hat{f}\big(O_\omega(t)\xi\big)\, dt. \tag{18}$$

Note that in x-space (18) leads to the identity

$$u(x) = \int_0^\infty E_t * O_\omega^T(t)\, f\big(O_\omega(t)\cdot - kte_3\big)(x)\, dt \tag{19}$$

where E denotes the heat kernel $E_t(x) = \frac{1}{(4\pi\nu t)^{3/2}}\, e^{-|x|^2/4\nu t}$ in $\mathbb{R}^3$.

Proof of Theorem 2. We start with the case $q = 2$ in which we may use the Theorem of Plancherel to estimate $\|k\partial_3 u\|_2$. By (16), (17), the inequality of Cauchy-Schwarz and Fubini's Theorem,

$$\begin{aligned}
\int |ik\xi_3\hat{u}|^2\, d\xi &= \int \frac{k^2\xi_3^2/|\omega|^2}{|D(\xi)|^2} \left| \int_0^{2\pi} e^{-(\nu|\xi|^2 + ik\xi_3)t/|\omega|} O^T(t) \hat{f}(O(t)\xi)\, dt \right|^2 d\xi \\
&\le \int \frac{k^2\xi_3^2/|\omega|^2}{|D(\xi)|^2} \left(\int_0^{2\pi} e^{-\nu|\xi|^2 t/|\omega|} |\hat{f}(O(t)\xi)|^2\, dt \right) \left(\int_0^{2\pi} e^{-\nu|\xi|^2 t/|\omega|}\, dt \right) d\xi \\
&= \int_0^{2\pi} \left(\int e^{-\nu|\xi|^2 t/|\omega|}\, \frac{k^2\xi_3^2/|\omega|^2}{|D(\xi)|^2}\, \frac{1 - e^{-2\pi\nu|\xi|^2/|\omega|}}{\nu|\xi|^2/|\omega|} |\hat{f}(O(t)\xi)|^2\, d\xi \right) dt.
\end{aligned}$$

In the inner integral the change of variable formula implies that the term $|\hat{f}(O(t)\xi)|^2$ may be replaced by $|\hat{f}(\xi)|^2$. Then a further application of Fubini's theorem yields

$$\begin{aligned}
\int |ik\xi_3\hat{u}|^2\, d\xi &= \int |\hat{f}(\xi)|^2\, \frac{k^2\xi_3^2/|\omega|^2}{|D(\xi)|^2}\, \frac{1 - e^{-2\pi\nu|\xi|^2/|\omega|}}{\nu|\xi|^2/|\omega|} \int_0^{2\pi} e^{-\nu|\xi|^2 t/|\omega|}\, dt \\
&= \int |\hat{f}(\xi)|^2\, \frac{k^2\xi_3^2/|\omega|^2}{|D(\xi)|^2}\, \frac{(1 - e^{-2\pi\nu|\xi|^2/|\omega|})^2}{\nu^2|\xi|^4/|\omega|^2}\, d\xi.
\end{aligned}$$

Hence it suffices to consider the 'multiplier function'

$$m(\xi) = \frac{1-e^{-(2\pi\nu|\xi|^2/|\omega|}}{\nu|\xi|^2/|\omega|}\ \frac{k\xi_3/|\omega|}{D(\xi)} \tag{20}$$

and to prove the estimate

$$|m(\xi)| \le c\big(1+\frac{k^2}{\nu|\omega|}\big), \quad \xi\in\mathbb{R}^3. \tag{21}$$

If $\frac{\nu|\xi|^2}{|\omega|} > 1$, then $|D(\xi)|$ is bounded below by a positive constant and

$$|m(\xi)| \le c\frac{k\xi_3/|\omega|}{\nu|\xi|^2/|\omega|} \le c\frac{k}{\nu|\xi|} \le c\frac{k}{\sqrt{\nu|\omega|}}.$$

If $\frac{\nu|\xi|^2}{|\omega|} \le 1$, the first factor in the definition of $m(\xi)$ is uniformly bounded. To estimate the remaining term

$$m_0(\xi) = \frac{k\xi_3/|\omega|}{D(\xi)}$$

we partition the ball $\frac{\nu|\xi|^2}{|\omega|} \le 1$ into infinitely many slices $S_n = \{\xi\in\mathbb{R}^3 : \frac{\nu|\xi|^2}{|\omega|} \le 1,\ \big|\frac{k\xi_3}{|\omega|}-n\big| \le \frac14\},\ n\in\mathbb{Z}$, and the remaining part S' where $\operatorname{dist}\big(\frac{k\xi_3}{|\omega|},\mathbb{Z}\big) \ge \frac14$ and consequently $|D(\xi)|\ge 1$. Hence,

$$|m_0(\xi)| \le \frac{k|\xi_3|}{|\omega|} \le \frac{k}{\sqrt{\nu|\omega|}} \quad \text{on} \quad S'.$$

For $\xi\in S_n,\ n\in\mathbb{Z}$, Taylor's expansion of $1-e^{-z}$ yields the lower bound

$$|D(\xi)| = \big|1-e^{-2\pi(\nu|\xi|^2/|\omega|+i(k\xi_3/|\omega|-n))}\big| \ge c_0\left|\frac{\nu|\xi|^2}{|\omega|}+i\Big(\frac{k\xi_3}{|\omega|}-n\Big)\right|$$

with a constant $c_0>0$ independent of all variables ν,ξ,k,ω,n. Hence for $\xi\in S_0$ we get the estimate $|m_0(\xi)|\le\frac{1}{c_0}$. If $\xi\in S_n,\ n\ne 0$, then

$$|m_0(\xi)| \le \frac{k|\xi_3|/|\omega|}{\nu|\xi|^2/|\omega|} \le \frac{k}{\nu|\xi|} \le \frac43\frac{k^2}{\nu|\omega|},$$

since $\frac{k|\xi|}{|\omega|} \ge \frac{k|\xi_3|}{|\omega|} \ge \frac34$. Now (21) is proved and implies the estimate

$$\int |ik\xi_3\hat{u}|^2\,d\xi \le c\Big(1+\frac{k^2}{\nu|\omega|}\Big)^2\int|\hat{f}|^2\,d\xi.$$

Then the Theorem of Plancherel completes the proof in the case $q=2$.

For the case $q\ne 2$ we write (16) in the form

$$\hat{u}(\xi) = \frac{1}{|\omega|}\int_0^{2\pi}\frac{1}{D(\xi)}e^{-\nu|\xi|^2t/|\omega|}O^T(t)\big(\mathcal{F}f(O(t)\cdot-kte_3/|\omega|)\big)(\xi)dt$$

using that $e^{-itk\xi_3}\in\mathcal{S}'(\mathbb{R}^3)$ is the Fourier transform of the shift operator $f\mapsto f(\cdot-kte_3)$ on $\mathcal{S}'(\mathbb{R}^3)$. Hence in x-space,

$$k\partial_3u(x) = \int_0^{2\pi}T_t\,F(t,\cdot)(x)\,dt$$

where

$$F(t,\cdot) = O^T(t)f(O(t)\cdot - kte_3/|\omega|)$$

and the operator family T_t, $0 < t < 2\pi$, is defined by its multiplier

$$m_t(\xi) = \frac{ik\xi_3/|\omega|}{D(\xi)} e^{-\nu|\xi|^2 t/|\omega|}, \tag{22}$$

i.e., $T_t = \mathcal{F}^{-1}(m_t(\xi)\cdot\ )$. Note that $\|F(t,\cdot)\|_q \le \|f\|_q$ for all $t \in (0, 2\pi)$.

Below we will prove the multiplier estimate

$$\max_\alpha \sup_{\xi\neq 0} |\xi^\alpha D_\xi^\alpha m_t(\xi)| \le c\left(1 + \frac{k}{\sqrt{\nu|\omega|t}} + \frac{k^2}{\nu|\omega|}\right)\cdot\left(1 + \frac{k^2}{\nu|\omega|}\right) \tag{23}$$

with a constant $c > 0$ independent of ν, ω, k and t; here $\alpha \in \mathbb{N}_0^3$ runs through the set of all multi-indices $\alpha \in \{0,1\}^3$. Then Marcinkiewicz' multiplier theorem [14] implies that in the operator norm $\|\cdot\|_q$ on L^q

$$\|T_t\|_q \le c\left(1 + \frac{k}{\sqrt{\nu|\omega|t}} + \frac{k^2}{\nu|\omega|}\right)\cdot\left(1 + \frac{k^2}{\nu|\omega|}\right).$$

Hence

$$\|k\partial_3 u\|_q \le c\int_0^{2\pi} \|T_t\|_q \|F(t,\cdot)\|_q\, dt \le c\left(1 + \frac{k^4}{\nu^2|\omega|^2}\right)\|f\|_q \tag{24}$$

with a constant $c > 0$ independent of ν, ω and k.

Now the L^2-result and (24) are combined by using complex multiplier theory to prove (6). Given $1 < q < 2$ we formally interpolate between the L^2-result and the L^1-result (24) using $\theta = \frac{2}{q} - 1$ such that $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{1}$. Then

$$\|\partial_3 u\|_q \le c\left(1 + \frac{k^2}{\nu|\omega|}\right)^{2-2/q}\left(1 + \frac{k^4}{\nu^2|\omega|^2}\right)^{2/q-1}\|f\|_q$$

yielding $\|\partial_3 u\|_q \le c(1 + \frac{k^2}{\nu|\omega|})^{2/q}\|f\|_q$. Since no estimate (24) holds for L^1, we have to interpolate between L^2 and L^p for $p > 1$ arbitrarily close to 1. Therefore, the additional exponent $\varepsilon > 0$ in (6) occurs. If $2 < q < \infty$, we formally interpolate between L^2 and L^∞ to get (6) with the exponent $2(1 - 1/q)$. Since again there is no L^∞-result available, we choose p arbitrarily large instead of $p = \infty$ and have to add the exponent $\varepsilon > 0$ in (6).

Finally we prove (23), start with m_t itself and distinguish between the cases $\frac{\nu|\xi|^2}{|\omega|} > 1$ and $\frac{\nu|\xi|^2}{|\omega|} \le 1$. In the first case $|D(\xi)|$ is bounded below by a positive constant yielding

$$|m_t(\xi)| \le c\frac{k|\xi_3|}{|\omega|} e^{-\nu|\xi|^2 t/|\omega|} \le c\frac{k}{\sqrt{\nu|\omega|t}}.$$

If $\frac{\nu|\xi|^2}{|\omega|} \le 1$, we may neglect the term $e^{-\nu|\xi|^2 t/|\omega|}$ and conclude from the detailed estimates of $m_0(\xi)$ in the L^2-case above that

$$|m_t(\xi)| \le |m_0(\xi)| \le c\left(1 + \frac{k^2}{\nu|\omega|}\right).$$

Hence $|m_t(\xi)| \le c(1 + \frac{k}{\sqrt{\nu|\omega|t}} + \frac{k^2}{\nu|\omega|})$ proving (23) for m_t. Next consider

$$\xi_3 \partial_3 m_t(\xi) = m_t(\xi) - \frac{2\nu\xi_3^2 t}{|\omega|} m_t(\xi) - \xi_3 \frac{\partial_3 D(\xi)}{D(\xi)} m_t(\xi) \tag{25}$$

where

$$\xi_3 \frac{\partial_3 D(\xi)}{D(\xi)} = 2\pi \frac{(2\nu\xi_3^2 + ik\xi_3)/|\omega|}{D(\xi)} e^{-2\pi(\nu|\xi|^2 + ik\xi_3)/|\omega|}. \tag{26}$$

Writing the exponential function $e^{-\nu|\xi|^2 t/|\omega|}$ as $e^{-\nu|\xi|^2 t/2|\omega|} \cdot e^{-\nu|\xi|^2 t/2|\omega|}$, we see that the second term on the right-hand side of (25) may be estimated as m_t itself. In the third term note—due to properties of $D(\xi)$ proved above—that $\frac{2\nu\xi_3^2/|\omega|}{D(\xi)} e^{-2\pi(\nu|\xi|^2+ik\xi_3)/|\omega|}$ is uniformly bounded. Moreover, the estimate of $\frac{k\xi_3/|\omega|}{D(\xi)} e^{-2\pi(\nu|\xi|^2+ik\xi_3)/|\omega|}$ is similar to the estimate of the multiplier function $m(\xi)$ in (20), (21) yielding

$$\frac{k|\xi_3|/|\omega|}{D(\xi)}\, e^{2\pi(-\nu|\xi|^2+ik\xi_3)/|\omega|} \le c\Big(1 + \frac{k^2}{\nu|\omega|}\Big).$$

Combining the previous estimates we get (23) for $\xi_3\partial_3 m_t$. Concerning $\xi_1\partial_1 m_t(\xi)$ note that (26) must be replaced by

$$\xi_1 \frac{\partial_1 D(\xi)}{D(\xi)} = 2\pi \frac{2\nu\xi_1^2/|\omega|}{D(\xi)} e^{-2\pi(\nu|\xi|^2+ik\xi_3)/|\omega|}.$$

Obviously, looking at the properties of $D(\xi)$ proved above, the modulus of this term is uniformly bounded requiring no further power of $k^2/\nu|\omega|$. Since the same assertion holds for the derivatives $\xi_2\partial_2$ and $\xi_1\partial_1\xi_2\partial_2$ of m_t, (23) is completely proved. ∎

Example 3. For fixed $k > 0$, $\nu > 0$ we will construct a sequence of solenoidal forces $(f) = (f_\omega) \subset L^2(\mathbb{R}^3)^3$, $|\omega| \to 0$, such that the corresponding sequence of solutions $(u) = (u_\omega)$ will satisfy

$$\|k\partial_3 u\|_q \ge c \frac{k^2}{\nu|\omega|} \|f\|_q \tag{27}$$

with $c > 0$ independent of k, ν, ω. Given $k > 0$, $\nu > 0$ choose $|\omega|$ small enough such that

$$\frac{\nu|\omega|}{k^2} < \frac{1}{16}.$$

Then define $f = (f', 0) \in L^2(\mathbb{R}^3)^3$ such that in Fourier space

$$\hat f'(\xi) = i \begin{cases} \xi'^{\perp}, 0 < \varphi < \pi \\ -\xi'^{\perp}, \pi < \varphi < 2\pi \end{cases}, \quad \text{when} \quad \left|\frac{k|\xi'|}{|\omega|} - 1\right| < \frac{1}{2}, \ \left|\frac{k|\xi_3|}{|\omega|} - 1\right| < \frac{1}{2},$$

but $\hat f'(\xi) = 0$ elsewhere; here, as usual, φ is the angular part of ξ. Since $\overline{\hat f(\xi)} = \hat f(-\xi)$, the vector field f is real-valued and obviously solenoidal. By (16)

$$\hat u(\xi) = \frac{e^{\nu|\xi|^2+ik\xi_3)\varphi/|\omega|}}{D(\xi)|\omega|}\, O(\varphi) \int_\varphi^{\varphi+2\pi} e^{-(\nu|\xi|^2+ik\xi_3)t/|\omega|} O^T(t)\, \hat f\big(O(t)e_1\big)\, dt.$$

Since $O^T(t)\hat f(O(t)e_1) = +e_2$ and $= -e_2$ when $0<\varphi<\pi$ and $\pi<\varphi<2\pi$, resp., a simple integration leads to the formula

$$ik\xi_3\hat u(\xi) = \hat f(\xi)\frac{ik\xi_3/|\omega|}{(\nu|\xi|^2+ik\xi_3)/|\omega|}\left(1-\frac{2e^{-(\pi-\varphi)(\nu|\xi|^2+ik\xi_3)/|\omega|}}{1+e^{-\pi(\nu|\xi|^2+ik\xi_3)/|\omega|}}\right),$$

when $0<\varphi<\pi$; for $\pi<\varphi<2\pi$ the exponential term $e^{-(\pi-\varphi)(\nu|\xi|^2+ik\xi_3)/|\omega|}$ must be replaced by $e^{-(2\pi-\varphi)(\nu|\xi|^2+ik\xi_3)/|\omega|}$. The assumptions on k, ν, ω imply for $\xi\in\operatorname{supp}\hat f$ that $|\frac{k\xi_3}{|\omega|}|\sim 1$, $\frac{\nu|\xi|^2}{|\omega|}\sim\frac{\nu|\omega|}{k^2}$; consequently, we have $|\nu|\xi|^2+ik\xi_3|/|\omega|\sim|\frac{k\xi_3}{|\omega|}|\sim 1$ and $|e^{-(\pi-\varphi)(\nu|\xi|^2+ik\xi_3)/|\omega|}|\sim 1$. Finally, the crucial term is

$$|1+e^{-\pi(\nu|\xi|^2+ik\xi_3)/|\omega|}|\sim\frac{\nu|\xi|^2}{|\omega|}\sim\frac{\nu|\omega|}{k^2}\quad\text{for }\ \xi\in\operatorname{supp}\hat f.$$

Hence

$$|ik\xi_3\hat u(\xi)|\sim\frac{k^2}{\nu|\omega|}|\hat f(\xi)|\quad\text{for }\ \xi\in\operatorname{supp}\hat f.$$

Since all similarity estimates $\sim$ can be made precise by using positive constants independent of k,ν,ω, (27) is proved.

Proof of Theorem 4. (i) Given a solution $u\in\hat W^{2,q}(\mathbb{R}^3)^3$, i.e. a function u with $\nabla^2 u\in L^q(R^3)$, Theorems 1 and 2 yield $\beta\in\mathbb{R}$ such that $\nabla'(u-\beta(\omega\wedge x))\in L^r(\mathbb{R}^3)^6$, $\frac1r=\frac1q-\frac14$, and $\partial_3(u-\beta(\omega\wedge x))\in L^q(\mathbb{R}^3)^3$. Since also $u-\beta\omega\wedge x$ solves (2), assume without loss of generality that $\beta=0$ implying for a.a. $x_3\in\mathbb{R}$ that

$$\int_{\mathbb{R}^2}|\nabla' u(x',x_3)|^r\,dx'<\infty,\qquad\int_{\mathbb{R}^2}|\partial_3 u(x',x_3)|^q\,dx'<\infty.$$

Then classical arguments show the existence of a sequence of radii $(R_j)\subset\mathbb{R}_+$ such that

$$\int_0^{2\pi}|\nabla' u(R_j,\theta,x_3)|^r\,d\theta=o\big(R_j^{-2}\big),\qquad\int_0^{2\pi}|\partial_3 u(R_j,\theta,x_3)|^q\,d\theta=o\big(R_j^{-2}\big)\tag{28}$$

as $j\to\infty$.

On the other hand, since $1<q<2$ and $\nabla^2u\in L^q(\mathbb{R}^3)$, Theorem II5.1 in [6] yields for a.a. $x_3\in\mathbb{R}$ a matrix $A(x_3)\in\mathbb{R}^{3,3}$ such that

$$\left(\int_{\mathbb{R}^2}\frac{|\nabla u(x',x_3)-A(x_3)|^q}{|x'|^q}\,dx'\right)^{1/q}\le\frac{q}{2-q}\left(\int_{\mathbb{R}^2}|\nabla'\nabla u(x',x_3)|^q\,dx'\right)^{1/q}.\tag{29}$$

Note that Theorem II5.1 in [6] is stated only for exterior domains; however, since the constant $q/(2-q)$ does not depend on the 'inner radius' of the exterior domain, the estimate holds for the whole space $\mathbb{R}^2$ as well. Moreover, by Lemma 5.2 in [6],

$$\int_0^{2\pi}|\nabla u(R,\theta,x_3)-A(x_3)|^q\,d\theta=o\big(R^{q-2}\big)\tag{30}$$

as $R\to\infty$. Now (28), (30) show that $A(x_3)=0$; hence (29) and (3) yield (7). Then (8) is an easy consequence of (7) and of (2).

(ii) If $1<q<3$, Theorem II5.1 in [6] yields the estimate

$$\left(\int_{\mathbb{R}^3}\frac{|\nabla u(x)-A|^q}{|x|^q}\,dx\right)^{1/q}\le\frac{q}{3-q}\left(\int_{\mathbb{R}^3}|\nabla^2u(x)|^q\,dx\right)^{1/q}$$

with a constant matrix $A \in \mathbb{R}^{3,3}$. Moreover, by Lemma 5.2 in [6],

$$\int_{|y|=1} |\nabla u(Ry)|^q do(y) = o\big(R^{q-3}\big)$$

as $R \to \infty$, where $\int_{|y|=1} \dots do(y)$ denotes the surface integral on the unit sphere of $\mathbb{R}^3$. Since $\nabla' u \in L^r(\mathbb{R}^3)^6$ and $\partial_3 u \in L^q(\mathbb{R}^3)^3$, arguments as above imply that A vanishes. Now (9) and (10) are easy consequences.

(iii) By (2) u_3 solves the problem $-\nu\Delta u_3 + k\partial_3 u_3 - (\omega \wedge x)\cdot\nabla u_3 = f_3$. Since $(\omega \wedge x)\cdot \nabla u_3 = |\omega|\partial_\theta u_3$, an integration w.r.t. $\theta \in (0, 2\pi)$ yields for the θ-independent function $U_3(x) := \frac{1}{2\pi}\int_0^{2\pi} u_3(|x'|, \theta', x_3)\, d\theta'$ the equation

$$-\nu\Delta U_3 + k\partial_3 U_3 = \frac{1}{2\pi}\int_0^{2\pi} f_3 \, d\theta'. \tag{31}$$

Applying Fourier transforms and using Marcinkiewicz' multiplier theorem we get that U_3 satisfies the estimate

$$\|k\partial_3 U_3\|_q \le c\|f\|_q$$

with a constant $c > 0$ independent of f, k, ν, cf. the analysis of the related Oseen problem [1], [5], [6]. By Wirtinger's inequality there exists a constant $c > 0$ such that for a.a. $r = |x'| > 0$ and $x_3 \in \mathbb{R}$

$$\|\partial_3 u_3(r,\cdot,x_3)\|_{L^q(0,2\pi)} \le c\big(\|\partial_\theta\partial_3 u_3(r,\cdot,x_3)\|_{L^q(0,2\pi)} + \|\partial_3 U_3(r,\cdot,x_3)\|_{L^q(0,2\pi)}\big).$$

Now divide by $1 + r$ and integrate w.r.t. $r\,dr$, $r > 0$, and dx_3, $x_3 \in \mathbb{R}$, to get that

$$\Big\|\frac{\partial_3 u_3}{1+|x'|}\Big\|_q \le c\left(\Big\|\frac{\partial_\theta\partial_3 u_3}{1+|x'|}\Big\|_q + \Big\|\frac{\partial_3 U_3}{1+|x'|}\Big\|_q\right).$$

Since the second term on the right-hand side is bounded by $\|\partial_3\nabla' u_3\|_q \le (c/\nu)\|f\|_q$ and since the third term is bounded by $\|\partial_3 U_3\|_q \le (c/k)\|f\|_q$, we get (11). ∎

Remark. The ideas of the proof of Theorem 4 (iii) do not apply to u_1 and u_2, since the term $\omega \wedge u$ does not vanish when applying the integration $\int_0^{2\pi} \dots d\theta'$. Also the identity $(\omega \wedge x)\cdot\nabla u - \omega \wedge u = |\omega| O(\theta)\partial_\theta(O^T(\theta)u)$ will not help, since no *a priori* estimates of $\partial_3\partial_\theta(O^T(\theta)u)$ are available except for the case when $1 < q < 2$.

Heuristic argument. Let us motivate why estimates of $(\omega\wedge x)\cdot\nabla u - \omega\wedge u$ and of $k\partial_3 u$ cannot be expected to be independent of $k^2/\nu|\omega|$. For simplicity ignore the terms $\omega \wedge u$ and p, recall that $(\omega\wedge x)\cdot\nabla u = |\omega|\partial_\theta u$ and let us perform a simple scaling analysis. Define the non-dimensional quantities $\tilde u = |\omega| u/A$, where $A \in \mathbb{R}$ is a characteristic acceleration of the flow, and $\tilde x = x\sqrt{|\omega|/\nu}$. Then, dividing (2) by A and omitting $\tilde{}$, (2) simplifies to the non-dimensional equation

$$-\Delta u + \frac{k}{\sqrt{\nu|\omega|}}\,\partial_3 u - \partial_\theta u = f \quad \text{in } \mathbb{R}^3.$$

Note that $\frac{k}{\sqrt{\nu|\omega|}}$ is a new non-dimensional characteristic number of the flow. For fixed $r = |x'|$ let us interpret $\frac{k}{\sqrt{\nu|\omega|}}\partial_3 u - \partial_\theta u$ as a directional derivative defined by the unit vector

$$d_\omega(x') = \frac{1}{\sqrt{r^2 + k^2/\nu\omega}} \left(\frac{k}{\sqrt{\nu|\omega|}} e_3 - (-x_2, x_1, 0)^T \right) \in \mathbb{R}^3$$

which is tangential to the cylinder $\mathcal{C}_r = \{x \in \mathbb{R}^3 : \, |x'| = r\}$. Hence, defining the curve

$$\gamma_\omega(s) = \Big(- r\cos s, -r\sin s, \frac{k}{\sqrt{\nu|\omega|}}\, s \Big)^T$$

on $\mathcal{C}_r$ with tangential vector $\sqrt{r^2 + k^2/\nu\omega}\; d_\omega$, we get that

$$\frac{d}{ds} u(\gamma_\omega(s)) = \big(\sqrt{r^2 + k^2/\nu\omega}\; d_\omega \cdot \nabla u\big)(\gamma_\omega(s)).$$

Obviously d_ω converges to the third unit vector e_3, whereas the curve $\gamma_\omega(s)$ has no reasonable limit on the cylinder $\mathcal{C}_r$. In this sense, the information on the directional derivative $d_\omega \cdot \nabla u$ on $\mathcal{C}_r$ is lost in the limit $\omega = 0$. This discrepancy vanishes for $r = 0$, but gets larger as $r \to \infty$. Therefore, the weight $\frac{1}{1+|x'|}$ has to occur in Theorem 1.4.

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