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A REVIEW ON THE IMPROVED REGULARITY FOR THE PRIMITIVE EQUATIONS

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Abstract. In this work we will study some types of regularity properties of solutions for the geophysical model of hydrostatic Navier-Stokes equations, the so-called Primitive Equations (PE). Also, we will present some results about uniqueness and asymptotic behavior in time.

1. Introduction. The knowledge of seas and oceans has always been a human interest. We cannot forget that at least two thirds of the Earth surface are covered by oceans, and it is surrounded by the atmosphere. From the beginning of the XIXth century, some scientists such as Pierre Simon de Laplace thought that the physical laws that govern atmosphere and ocean could serve to predict the future weather and climate. Nevertheless, it was not until the XXth century that people started to treat this prediction by solving differential problems in mathematical physics.

The dynamics of geophysical fluids is a subject born in the fifties, relating to Oceanography and Meteorology, and studies large scale fluids (in space and, sometimes, in time). What Meteorology tries to describe are the weather changes, the coast winds, the influence of topography in the local or regional weather, the general circulation, the climate variation,... On the other hand, Oceanography studies "upwelling" phenomena (circulation of deep water), oceanic streams (as the Mexico Gulf Stream) and large scale general circulation (meso-scale and climate scale).

According to J. L. Lions, R. Temam and S. Wang [17], in order to understand the turbulent behavior of both the atmosphere and the ocean, and to predict the climate, the following requirements are needed:

(a) to establish the equations and mathematical models that govern the movement and the atmosphere and ocean states, and the interactions appearing among them;

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(b) to know the mathematical basis of these equations and models;
(c) to design and compute numerical approximations to these equations.

The atmosphere is a compressible fluid described mathematically by the Hydrodynamic and Thermodynamic equations, where centripetal and Coriolis forces are also acting. Such equations describe the big scale movements, and small scales are considered as "noises" in the numerical treatment. However, because of the vertical scale is much smaller than the horizontal scales, we can use the hydrostatic approximations in order to obtain the Primitive Equations of the atmosphere and the ocean.

The interaction between atmosphere and ocean can be observed when the wind force moves the ocean or when the ocean influences the behavior of the atmosphere. From a physical point of view, the water dynamics, the distribution of temperature and salinity, and the chemical and biological components of water are interesting in Oceanography. In Meteorology, air dynamics, temperature, humidity and pressure are interesting.

Although it seems that the incompressible (or slightly incompressible) Navier-Stokes equations, with variable density and free surface, is one of the most realistic models to simulate the hydrodynamic behavior in Oceanography, the high complexity of this model and the high dimensions of the domain of study motivate some simplifications (see [19]). We describe here one of them.

This work is organized as follows. In section 2, we present the physical derivation of the model and some mathematical simplifications. In section 3 the (PE) problem is considered: fist, we present some functional spaces and definitions; secondly, we show the main steps in the proof of the strong regularity for the solution of (PE), global in time for small data and local in time for any data; and then, we deal with the asymptotic behavior in time. Section 4 is devoted to the uniqueness of solution, proved for weak solutions when some additional hypotheses on the derivative with respect to the z-variable are made. In the fifth section, we show the existence of a very weak solution for the linear problem that will help us to weaken the regularity hypothesis on the data in order to prove strong regularity for (PE). Finally, in section 6 we prove anisotropic global regularity and uniqueness of solution global in time for a 2D (PE) model provided with friction boundary conditions on the bottom.

2. Derivation of the Primitive Equations of the ocean model. The ocean can be considered as a slightly compressible fluid, with the influence of centripetal and Coriolis forces. The set of equations that form the so called "large scale ocean model" are: the momentum equation, the continuity equation, the thermodynamic equation (with temperature θ), the diffusion equation for the salinity S and the state equation for the density:

$$(1) \qquad \left\{ \begin{aligned} \rho\frac{D\mathbf{U}}{Dt} + 2\rho\mathbf{W}\times\mathbf{U} + \rho\mathbf{W}\times(\mathbf{W}\times\mathbf{r}) + \nabla P + \rho\mathbf{g} &= D, \\ \frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{V} &= 0, \\ \frac{D\theta}{Dt} &= Q_\theta, \\ \frac{DS}{Dt} &= Q_S, \\ \rho &= \rho(\theta, S), \end{aligned} \right.$$

where $\mathbf{V}$ is the 3D velocity field, P is the pressure, $\mathbf{g} = (0, 0, g)$ is the gravity, $2\rho\mathbf{W} \times \mathbf{V}$ is the Coriolis term and $\rho\mathbf{W} \times (\mathbf{W} \times \mathbf{r})$ the centripetal forces ($\mathbf{W} = f(0, \cos\lambda, \sin\lambda)$ is the Earth rotation vector, f its module, $\lambda = \lambda(y)$ is the latitude and $\mathbf{r}$ is the Earth ratio). On the other hand, D is the molecular dissipation, Q_θ and Q_S are the temperature and salinity diffusions, respectively.

We will use the following operators: $\nabla = (\partial_x, \partial_y, \partial_z)$ the 3D gradient, with $\nabla\cdot$ the divergence operator and $\frac{D}{Dt}$ the material derivative, i.e.

$$\frac{D}{Dt} = \partial_t + \mathbf{U} \cdot \nabla.$$

In what follows, we will do a β-plane approximation, that means to suppose that the earth surface can be approached locally by the tangent plane at a central point of this neighborhood, where β is the deformation angle from the sphere over the plane. In this case, the domain of ocean Ω, can be described in cartesian coordinates as:

$$\Omega = \{(x, y, z) = (\mathbf{x}, z) \in \mathbb{R}^3,\, \mathbf{x} \in S,\, -H(\mathbf{x}) < z < 0\}.$$

Its boundary is $\partial\Omega = \overline{\Gamma_b} \cup \Gamma_l \cup \Gamma_s$ where the bottom Γ_b, the sidewalls Γ_l and the surface Γ_s are defined by:

$$\Gamma_b = \{(\mathbf{x}, z) \in \mathbb{R}^3 : \mathbf{x} \in S,\, z = -H(\mathbf{x})\},$$
$$\Gamma_l = \{(\mathbf{x}, z) \in \mathbb{R}^3 : \mathbf{x} \in \partial S,\, -H(\mathbf{x}) < z < 0\},$$
$$\Gamma_s = \{(\mathbf{x}, 0) : \mathbf{x} \in S\},$$

where the horizontal section S is an open set in $\mathbb{R}^2$ and the depth H is a non-negative continuous function over S.

In order to avoid theoretical and computational difficulties, two main simplifications are considered in (1):

a) *Boussinesq approximation*, that neglects the differences of density in all the equations of the system except the gravity term and the state equation. In this way, once a medium density ρ_0 is fixed, then $\rho = \rho_0 + \rho'$ with $\rho' << \rho_0$. The continuity equation is then the incompressibility equation for the velocity $\mathbf{U}$. The inclusion of the centripetal forces in the gradient of a potential function p (along with the pressure), they allow to consider the following model of Navier-Stokes with anisotropic viscosities:

$$(BEs)\begin{cases} \dfrac{D}{Dt}\mathbf{U} - \nabla \cdot (D_\nu(\mathbf{U})) + 2\mathbf{W} \times \mathbf{U} + \nabla p = -\dfrac{\rho'}{\rho} g\mathbf{e}_3, \\ \rho = \rho(\theta, S), \quad \nabla \cdot \mathbf{U} = 0, \\ \dfrac{D}{Dt}\theta - \nabla \cdot (D_{\nu_\theta}(\theta)) = 0, \quad \dfrac{D}{Dt}S - \nabla \cdot (D_{\nu_S}(S)) = 0. \end{cases}$$

Here, $\frac{D}{Dt} = \partial_t + \mathbf{U} \cdot \nabla$ is the material derivative, $\nu, \nu_\theta, \nu_S > 0$ are anisotropic (eddy) diffusion coefficients (with different order in horizontal and vertical) of $(\mathbf{U}, \theta, S)$ respectively, where $D_\nu(\mathbf{U}) = \nabla_\nu \mathbf{U} + \nabla_\nu \mathbf{U}^t$ and $\nabla_\nu = (\nu_{\mathbf{x}} \nabla_{\mathbf{x}}, \nu_z \partial_z)^t$, with $\nabla_{\mathbf{x}} = (\partial_x, \partial_y)^t$ the horizontal gradient operator.

b) *Hydrostatic approximation.* An analysis of spatial scales says that the aspect quotient (ratio between the vertical Z and horizontal L characteristic lengths) is small, namely:

$$\delta = \frac{Z}{L} \approx 10^{-3}.$$

It is also possible to observe that the vertical water velocity is much smaller than the horizontal ones, which is modelled approximating the third momentum equation by the so-called hydrostatic equation:

$$\frac{\partial p}{\partial z} = -\rho\, g,$$

which relates the ocean pressure and density with the gravity, and that has become a fundamental equation in Oceanography. This analysis also shows that for the viscosities in each direction to be of the same order (respect to δ), we have to suppose:

$$\nu_z = \delta^2 \nu_v, \quad \nu_{\mathbf{x}} = \nu_h, \quad \text{with } \nu_v = O(1) \text{ and } \nu_h = O(1). \tag{2}$$

By simplicity, we only treat the (nonlinear) system for velocity $\mathbf{U} = (\mathbf{u}, v)$ (where $\mathbf{u} = (u_1, u_2)$ and v are the horizontal and vertical velocities respectively) and pressure p, because of the system coupled with temperature and salinity (of convection-diffusion type) do not introduce any new mathematical difficulties. This system is called Hydrostatic Navier-Stokes equations, which can be described as follows:

$$(HNS) \begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} + v\partial_z \mathbf{u} - \nu_h \Delta_{\mathbf{x}}\mathbf{u} - \nu_v \partial^2_{zz}\mathbf{u} + \alpha \mathbf{u}^{\perp} + \dfrac{1}{\rho_0}\nabla_{\mathbf{x}} p = \mathbf{0}, \\ \rho = \rho_0 + \rho'(\theta, S), \qquad \partial_z p = -\rho g, \qquad \nabla_{\mathbf{x}}\cdot \mathbf{u} + \partial_z v = 0, \end{cases}$$

where $\alpha = 2f\sin(\lambda)$. The surface Γ_s is the same as before, where the new Ω, Γ_l and Γ_b (with $h = \frac{H}{Z}$) are described as follows:

$$\begin{aligned} \Omega &= \{(\mathbf{x}, z) \in \mathbb{R}^3,\ \mathbf{x} \in S,\ -h(\mathbf{x}) < z < 0\}, \\ \Gamma_l &= \{(\mathbf{x}, z) \in \mathbb{R}^3 :\ \mathbf{x} \in \partial S,\ -h(\mathbf{x}) < z < 0\}, \\ \Gamma_b &= \{(\mathbf{x}, z) \in \mathbb{R}^3 : \mathbf{x} \in S,\ z = -h(\mathbf{x})\}. \end{aligned} \tag{3}$$

A derivation of (HNS) of the ocean from the hydrostatic approximation hypothesis is obtained in the works of J. L. Lions, R. Temam and S. Wang, [15, 16]. Such hypothesis can be justified as the limit of the weak solution of the Navier-Stokes equations or (BEs) when $\delta \to 0$ imposing (2) (see the work of O. Besson and M. R. Laydi, [3], for the stationary case, and the work of P. Azérad and F. Guillén-González, [2], for the evolutionary case).

2.1. *The boundary conditions.* The exchange between atmosphere and ocean is determined by the interface conditions, called surface boundary conditions when the isolated model of the ocean is considered. A simplifying hypothesis is the "rigid lid" hypothesis; namely, the interface atmosphere-ocean is assumed flat, thanks to two facts:

(a) the water density is much greater than the air density; $\rho^a/\rho \approx 10^{-3}$, where ρ^a and ρ are the air and oceanic water density respectively. Then, the atmosphere-ocean interface is very stable considering great spatial scales, due to the intensity of the gravitational force.

(b) in the oceanic scale, the vertical displacement of the tides waves usually is neglected in most of Global Circulation models.

Denoting with the upper-index ^{a} the variables of the atmosphere, the surface boundary conditions are:

$$v|_{\Gamma_s} = 0, \quad \mathbf{u}|_{\Gamma_s} = \mathbf{u}^a|_{\Gamma_s}.$$

Nevertheless, due to the difference of density between both states, a thin boundary layer appears in the atmosphere (of 1 km of thickness) and very fine in the ocean (between 10 and 100 m). A possible modelling of this boundary layer is given by:

$$v = 0 \qquad -\rho_0\,\nu_v\,\partial_z \mathbf{u} = \rho^a\,C_D^a\,(\mathbf{u}^a - \mathbf{u})|\mathbf{u}^a - \mathbf{u}|^\alpha \quad \text{on } \Gamma_s,$$

where C_D^a is a momentum transfer coefficient. Following the references [16, 15], we consider the simplification:

$$v = 0, \quad \nu_v\,\partial_z \mathbf{u} = \boldsymbol{\Upsilon} \quad \text{on } \Gamma_s,$$

where $\boldsymbol{\Upsilon}$ is the wind stress tensor on the surface of the ocean, which is given as a datum or as a linear function of $\mathbf{u}$: for instance $\boldsymbol{\Upsilon} = -C|\mathbf{u}^a|^\alpha(\mathbf{u}^a - \mathbf{u})$.

With respect to the bottom and sidewalls, we will always impose the slip condition $(\mathbf{u}, v)\cdot\mathbf{n} = 0$ on $\Gamma_l \cup \Gamma_b$. On Γ_l this condition yields $\mathbf{u}_{|\Gamma_l} = 0$ (allowing vertical sliding on the sidewalls). On the bottom, two additional conditions should be imposed, that could be of adherence or friction type:

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_b \qquad (\text{or } (\nabla_\nu \mathbf{u})\,\mathbf{n} + \beta\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_b)$$

where $\beta = \beta(\mathbf{x}) > 0$ is a coefficient depending of the bottom roughness. From a physical point of view, the election of homogeneous Dirichlet boundary conditions for the velocity on the bottom is only justify when the molecular viscosity of the fluid is important. Nevertheless, in many geophysical models eddy viscosity is considered, neglecting the molecular viscosity. On the other hand, one knows that the use of the friction boundary condition in the Navier-Stokes equations prevents the appearance of boundary layers.

2.2. *The reduced model.* The unknowns $(\mathbf{u}, v, p)$ of the (HNS) system have different roles: the horizontal velocity $\mathbf{u}$ satisfies an evolution problem and therefore needs initial data (*prognostic variable*). The vertical velocity v can be determined from $\mathbf{u}$ (*diagnostic variable*). Indeed, integrating the incompressibility equation in $(z, 0)$ and using the rigid lid hypothesis $v_{|\Gamma_s} = 0$, one has:

$$v(t; \mathbf{x}, z) = \int_z^0 \nabla_{\mathbf{x}} \cdot \mathbf{u}(t; \mathbf{x}, s)ds. \tag{4}$$

With regard to the pressure, integrating the hydrostatic equation in $(z, 0)$, one has:

$$p(t; \mathbf{x}, z) = p_s(t; \mathbf{x}) + \int_z^0 (\rho\,g)(t; \mathbf{x}, s)ds = p_s(t; \mathbf{x}) - \rho_0 g z + g\int_z^0 \rho'(\theta, S)(t; \mathbf{x}, s)ds,$$

where $p_s(t; \mathbf{x}) = p(t; \mathbf{x}, 0)$ is a potential function only defined in the surface of the ocean, namely the atmospheric pressure plus the surface lid pressure (this latter is the pressure exerted by undulations of a free surface), and $-\rho_0 g z + g\int_z^0 \rho'(\theta, S)(t; \mathbf{x}, s)ds$ is the baroclinic pressure (where $-\rho_0 g z$ is an average of the pressure exerted by the water column between z and 0). Then, horizontal gradient of pressure is rewritten as:

$$\frac{1}{\rho_0}\nabla_{\mathbf{x}} p = \frac{1}{\rho_0}\nabla_{\mathbf{x}} p_s + \mathbf{F}(\theta, S).$$

On the other hand, using in (4) the slip boundary condition on the bottom $(\mathbf{u}, v)\cdot\mathbf{n}_{|\Gamma_b} = 0$, one arrives to the constraint (see [16, 14])

$$\nabla_{\mathbf{x}} \cdot \langle \mathbf{u} \rangle = 0 \text{ in } (0,T) \times S, \quad \text{where } \langle \mathbf{u} \rangle(t;\mathbf{x}) = \int_{-h(\mathbf{x})}^{0} \mathbf{u}(t;\mathbf{x},z)dz. \tag{5}$$

Then, we arrive at the following reduced system, that will be called *Primitive Equations* :

$$(PE)\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} + v\partial_z \mathbf{u} - \nu_h \Delta_{\mathbf{x}} \mathbf{u} - \nu_v \partial^2_{zz}\mathbf{u} + \alpha \mathbf{u}^{\perp} \\ \qquad\qquad\qquad + \dfrac{1}{\rho_0}\nabla_{\mathbf{x}} p_s = \mathbf{F} \text{ in } (0,T)\times\Omega, \\ \qquad\qquad\qquad \nabla_{\mathbf{x}}\cdot\langle\mathbf{u}\rangle = 0 \text{ in } (0,T)\times S, \\ \qquad\qquad\qquad \mathbf{u}|_{t=0} = \mathbf{u}_0 \ \text{ in } \Omega, \\ \nu_v \partial_z \mathbf{u}|_{\Gamma_s} = \mathbf{\Upsilon}, \quad \mathbf{u}|_{\Gamma_b\cup\Gamma_l} = \mathbf{0} \ \text{ in } (0,T), \end{cases}$$

where v depends on $\mathbf{u}$ as in (4).

This system displays some advantages, from the computational point of view, eliminating the unknown v and reducing the unknown pressure to a surface function. Nevertheless, due to the dependency of v with respect to $\nabla_{\mathbf{x}}\cdot\mathbf{u}$, an anisotropy in the regularity of the derivatives of v is produced. For example, in the weak solution framework, $\mathbf{u}\in\mathbf{H}^1(\Omega)$, hence using the incompressibility equation, $\partial_z v = -\nabla_{\mathbf{x}}\cdot\mathbf{u} \in L^2(\Omega)$ but $\nabla_{\mathbf{x}} v \notin L^2(\Omega)$ in general. This anisotropy implies that the nonlinear terms of the momentum equations are less regular than in the Navier-Stokes case. Another fact to consider is that whereas (BEs) is a differential model, (PE) it is a integral-differential one.

Remark 1. When in the Navier-Stokes model, free surface is considered (as a new unknown), the rigid lid condition ($v = 0$ on Γ_s) must be changed by the free surface equation, arriving at the so-called $3D$ Shallow Water model.

3. Regularity for the primitive equations model (PE)

3.1. *Functional spaces and definitions.* Before making a mathematical study of problem (PE), we describe the functional spaces and the definitions of very weak, weak and strong solution:

$$C^\infty_{b,l}(\Omega) = \{\varphi \in C^\infty(\Omega)^2;\ supp(\varphi) \text{ is a compact set } \subseteq \overline{\Omega}\backslash(\Gamma_b\cup\Gamma_l)\},$$

$$\mathbf{H}^1_{b,l}(\Omega) = \overline{C^\infty_{b,l}(\Omega)}^{H^1} = \{\mathbf{v}\in H^1(\Omega)^2;\ \mathbf{v} = 0 \text{ on } \Gamma_b\cup\Gamma_l\},$$

$$\mathbf{H}^{-1}_{b,l}(\Omega) = \text{ dual space of } H^1_{b,l}(\Omega),$$

$$\mathcal{V} = \{\varphi\in C^\infty_{b,l}(\Omega)^2;\ \nabla_{\mathbf{x}}\cdot\langle\varphi\rangle = 0 \ \text{ in } S\},$$

$$\mathbf{H} = \overline{\mathcal{V}}^{L^2} = \{\mathbf{v}\in\mathbf{L}^2(\Omega);\ \nabla_{\mathbf{x}}\cdot\langle\mathbf{v}\rangle = 0 \ \text{ in } S,\ \langle\mathbf{v}\rangle\cdot\mathbf{n}_{|\partial S} = 0\},$$

$$\mathbf{V} = \overline{\mathcal{V}}^{H^1} = \{\mathbf{v}\in\mathbf{H}^1(\Omega);\ \nabla_{\mathbf{x}}\cdot\langle\mathbf{v}\rangle = 0 \ \text{ in } S,\ \mathbf{v}_{|\Gamma_b\cup\Gamma_l} = \mathbf{0}\}.$$

Taking regular test functions in (PE) and integrating by parts, we obtain:

DEFINITION 1. Let $\mathbf{u}_0 \in \mathbf{H}$, $\mathbf{F} \in L^2(0,T;\mathbf{H}^{-1}_{b,l}(\Omega))$ and $\boldsymbol{\Upsilon} \in L^2(0,T;\mathbf{H}^{-1/2}(\Gamma_s))$ be given functions. We say $\mathbf{u} : (0,T) \times \Omega \to \mathbb{R}^2$ is a *weak solution* of (PE) in $(0,T)$ if

$$\mathbf{u} \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}),$$

satisfying the variational formulation: $\forall\, \varphi \in C^1([0,T];\mathcal{V})$ such that $\varphi(T) = \mathbf{0}$,

$$\begin{aligned}&\int_0^T \int_\Omega \big(-\mathbf{u} \cdot \big(\partial_t \varphi + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\varphi + u_3 \partial_z \varphi\big) + \alpha \mathbf{u}^\perp \cdot \varphi\big)\, d\Omega\, dt \\ &+ \int_0^T \int_\Omega (\nu_h \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} \varphi + \nu_v \partial_z \mathbf{u} \cdot \partial_z \varphi)\, d\Omega dt \\ &= \int_\Omega \mathbf{u}_0 \cdot \varphi(0)\, d\Omega + \int_0^T \langle \mathbf{F}, \varphi \rangle_\Omega dt + \int_0^T \langle \boldsymbol{\Upsilon}, \varphi \rangle_{\Gamma_s} dt,\end{aligned}$$

and, moreover, $\mathbf{u}$ satisfying the energy inequality:

$$\begin{aligned}&\frac{1}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \Big(\nu_h \|\nabla_{\mathbf{x}} \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \nu_v \|\partial_z \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}\Big)\, ds \\ &\le \frac{1}{2}\|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \langle \mathbf{F}, \mathbf{u} \rangle_\Omega ds + \int_0^t \langle \boldsymbol{\Upsilon}, \mathbf{u} \rangle_{\Gamma_s} ds \quad c.p.d.\, t \in (0,T).\end{aligned} \tag{6}$$

In the case $T = +\infty$, we say that $\mathbf{u}$ is a weak solution of (PE) in $(0,+\infty)$ if $\mathbf{u}$ is a weak solution of (PE) in $(0,T)$, $\forall T < +\infty$.

Observe that $\langle \cdot, \cdot \rangle_\Omega$ denotes the duality between $\mathbf{H}^{-1}_{b,l}(\Omega)$ and $\mathbf{H}^1_{b,l}(\Omega)$, and $\langle \cdot, \cdot \rangle_{\Gamma_s}$ denotes the duality between $\mathbf{H}^{-1/2}(\Gamma_s)$ and $\mathbf{H}^{1/2}(\Gamma_s)$. In this section, u_3 will denote the vertical velocity associated to $\mathbf{u}$.

Finally, we denote the V-norm by $\|\varphi\|^2_V = \nu_h \|\nabla_{\mathbf{x}} \varphi\|^2_{\mathbf{L}^2(\Omega)} + \nu_v \|\partial_z \varphi\|^2_{\mathbf{L}^2(\Omega)}$, and the $H^1_{b,l}(\Omega)$-norm by $\|\varphi\|^2_{H^1(\Omega)} = \|\nabla_{\mathbf{x}} \varphi\|^2_{L^2(\Omega)} + \|\partial_z \varphi\|^2_{L^2(\Omega)}$.

DEFINITION 2. Let $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{F} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $\boldsymbol{\Upsilon} \in L^2(0,T;\mathbf{H}^{1/2}(\Gamma_s))$ and $\partial_t \boldsymbol{\Upsilon} \in L^2(0,T;\mathbf{H}^{-1/2}(\Gamma_s))$ be given functions. If $\mathbf{u}$ is a weak solution of (PE) in $(0,T)$, we say that $\mathbf{u}$ is a *strong solution* if it satisfies the following additional regularity:

$$\mathbf{u} \in L^\infty(0,T;\mathbf{V}) \cap L^2(0,T;\mathbf{H}^2(\Omega) \cap \mathbf{V}), \quad \partial_t \mathbf{u} \in L^2(0,T;\mathbf{H}).$$

The existence of weak solution of (PE) is well-known from the works of Lewandowski [14] and Lions-Temam-Wang [16] in domain whose depth is strictly bounded from below (i.e., $h \ge h_{min} > 0$ in $\overline{S}$). They use a Galerkin method in order to obtain the velocity $\mathbf{u}$ in a space with the restriction $\nabla \cdot \langle \mathbf{u} \rangle = 0$. The pressure will be recovered later thanks to a De Rham Lemma, specific for this kind of spaces. In domain without this restriction the existence of weak solution is obtained as a consequence of a limit process applied to the Navier-Stokes equations with anisotropic viscosity, when the aspect quotient tends to zero (see Besson-Laydi [3] for the stationary case and Azerad-Guillén [2] for the evolutionary case). Other proofs by internal approximations can be seen in [6] for the stationary case and [9] for the evolutionary case.

The novelty of the results of the authors is the proof of existence of strong solution for the nonlinear system (PE) and the uniqueness. The linear stationary case has been studied by M. Ziane, [21]. One of the main difficulties for this study is the treatment of the boundary conditions: Neumann non homogeneous on the surface and Dirichlet homogeneous on the bottom and sidewalls. Uniqueness of weak solution is still an open problem, but the regularity hypothesis for it has been weakened.

3.2. *Strong regularity for the Primitive Equations.* We start our study by the linear evolutionary system associated to the primitive equations (for simplicity in the exposition, we will omit the Coriolis term):

$$(S)\begin{cases} \partial_t \mathbf{v} - \nu_h \Delta_{\mathbf{x}} \mathbf{v} - \nu_v \partial_{zz}^2 \mathbf{v} + \nabla_{\mathbf{x}} q_s = \mathbf{F} \text{ in } (0,T)\times\Omega, \\ \qquad\qquad\qquad\qquad \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \rangle = 0 \text{ in } (0,T)\times S, \\ \qquad\qquad\qquad\qquad \mathbf{v}|_{t=0} = \mathbf{u}_0 \text{ in } \Omega, \\ \nu_v \partial_z \mathbf{v}|_{\Gamma_s} = \boldsymbol{\Upsilon}, \qquad \mathbf{v}|_{\Gamma_b\cup\Gamma_l} = \mathbf{0} \text{ in } (0,T). \end{cases}$$

The associated stationary problem will be called (S_{st}).

Theorem 1 (Weak solution of (S_{st})). *Let $S \subseteq \mathbb{R}^d$ ($d = 1$ or 2) and $\Omega \subseteq \mathbb{R}^{d+1}$ be Lipstchitz-continuous domains defined by* (3). *If $\mathbf{F} \in \mathbf{H}^{-1}_{b,l}(\Omega)$ and $\boldsymbol{\Upsilon} \in \mathbf{H}^{-1/2}(\Gamma_s)$, then the problem (S_{st}) has a unique solution $\mathbf{v} \in \mathbf{H}^1(\Omega)$. Moreover, there exists a constant $C = C(\Omega) > 0$ such that if $\nu = \min\{\nu_h,\nu_v\}$, we obtain:*

$$\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega)} \le \frac{C}{\nu^2}\{\|\boldsymbol{\Upsilon}\|^2_{\mathbf{H}^{-1/2}(\Gamma_s)} + \|\mathbf{F}\|^2_{\mathbf{H}^{-1}_{b,l}(\Omega)}\}. \tag{7}$$

In [3], [6] and [14], there are different proofs of this result.

Theorem 2 (Strong solution for (S_{st}) [21]). *Let $S \subseteq \mathbb{R}^d$ ($d = 1$ or 2) be a C^3 domain and $h \in C^3(\overline{S})$ the depth satisfying $h \ge h_{\min} > 0$ in $\overline{S}$. If $\mathbf{F} \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{\Upsilon} \in \mathbf{H}_0^{1/2+\varepsilon}(\Gamma_s)$ (for some $\varepsilon > 0$), then there exists a (unique) strong solution $\mathbf{v}$ of (S_{st}) (i.e., $\mathbf{v} \in \mathbf{H}^2(\Omega)\cap V$). Moreover, there exists a constant $C = C(\Omega) > 0$ such that:*

$$\|\mathbf{v}\|^2_{\mathbf{H}^2(\Omega)} \le \frac{C}{\nu^2}\{\|\mathbf{F}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\Upsilon}\|^2_{\mathbf{H}_0^{1/2+\varepsilon}(\Gamma_s)}\}. \tag{8}$$

This result of strong regularity ([10, 20]) must be extended to the linear evolutionary case (S). First of all, we get a lift of the boundary conditions: In this way, we define the operator $B : \mathbf{a} \in \mathbf{H}^{-1/2}(\Gamma_s) \to \mathbf{u} = B\mathbf{a} \in \mathbf{V}$, where $\mathbf{u}$ is the weak solution of the hydrostatic Stokes problem (S_{st}) with $\mathbf{F} = \mathbf{0}$ and $\boldsymbol{\Upsilon} = \mathbf{a}$. Then, we consider $\mathbf{e}(t) = B(\boldsymbol{\Upsilon}(t))$ which has strong regularity, and we prove that $\partial_t\mathbf{e}(t)$ coincides with $B(\partial_t\boldsymbol{\Upsilon}(t))$ which has weak regularity, and therefore $\mathbf{e} \in C^0([0,T];\mathbf{V})$. Secondly, we consider the homogeneous problem satisfied by $\mathbf{y} = \mathbf{v} - \mathbf{e}$. The estimates of energy deduced for $\mathbf{e}$ and $\partial_t\mathbf{e}$ thanks to Theorem 1 and Theorem 2 yield the following result:

Theorem 3 (Strong solution of (S)). *Let $S \subseteq \mathbb{R}^d$ ($d = 1$ or 2) be a C^3-domain and $h \in C^3(\overline{S})$ the depth with $h \ge h_{min} > 0$ in $\overline{S}$. If $\mathbf{F} \in \mathbf{L}^2((0,T)\times\Omega)$, $\mathbf{u}_0 \in V$, $\boldsymbol{\Upsilon} \in L^2(0,T;\mathbf{H}_0^{1/2+\varepsilon}(\Gamma_s))$, for any $\varepsilon > 0$, with $\partial_t\boldsymbol{\Upsilon} \in L^2(0,T;\mathbf{H}^{-1/2}(\Gamma_s))$, then there exists a unique strong solution $\mathbf{v}$ of (S) in $(0,T)$. Moreover, there exists a constant $C > 0$ such that:*

$$\begin{aligned}\|\mathbf{v}\|^2_{L^\infty(\mathbf{V})} + \|\mathbf{v}\|^2_{L^2(\mathbf{H}^2(\Omega))} + \|\partial_t\mathbf{v}\|^2_{L^2(\mathbf{H})} \le C\{\|\mathbf{u}_0\|^2_V + \|\boldsymbol{\Upsilon}(0)\|^2_{\mathbf{H}^{-1/2}(\Gamma_s)} \\ + \|\mathbf{F}\|^2_{L^2(\mathbf{L}^2(\Omega))} + \|\boldsymbol{\Upsilon}\|^2_{L^2(\mathbf{H}_0^{1/2+\varepsilon}(\Gamma_s))} + \|\partial_t\boldsymbol{\Upsilon}\|^2_{L^2(\mathbf{H}^{-1/2}(\Gamma_s))}\}\end{aligned} \tag{9}$$

Once the linear problem has been studied, we deal with the strong regularity for the nonlinear problem (PE). We use the previous theorem to lift the boundary conditions of

(PE) problem and thus to study the homogeneous boundary problem satisfied by $(\mathbf{w}, \pi_s)$, where $\mathbf{w} = \mathbf{u} - \mathbf{v}$, $\pi_s = p_s - q_s$, for $(\mathbf{v}, q_s)$ the solution of (S):

$$(NL) \begin{cases} \partial_t \mathbf{w} - \nu_h \Delta_{\mathbf{x}} \mathbf{w} - \nu_v \partial^2_{zz} \mathbf{w} + (\mathbf{w} + \mathbf{v}) \partial_x (\mathbf{w} + \mathbf{v}) \\ \qquad\qquad + (w_3 + v_3) \partial_z (\mathbf{w} + \mathbf{v}) + \partial_x \pi_s = \mathbf{0} \ \text{ in } (0,T) \times \Omega, \\ \nabla_{\mathbf{x}} \cdot \langle \mathbf{w} \rangle = 0 \ \text{ in } (0,T) \times S, \quad \mathbf{w}|_{t=0} = 0 \ \text{ in } \Omega, \\ \nu_v \partial_z \mathbf{w} = 0 \ \text{ on } (0,T) \times \Gamma_s, \quad \mathbf{w} = 0 \ \text{ on } (0,T) \times (\Gamma_b \cup \Gamma_l), \end{cases}$$

with $w_3 = \int_z^0 \nabla_{\mathbf{x}} \cdot \mathbf{w}\, ds$ and in a similar manner for v_3.

In the spirit of Galerkin method, we approach $\mathbf{w}$ functions by $\mathbf{w}_m$. They are the Galerkin approximates in the m-dimensional space V_m, spanned by an orthogonal and unitary base in V of eigenfunctions of the hydrostatic operator $A : V \to V'$ such that:

$$\langle A\mathbf{u}, \mathbf{v} \rangle_{V',V} = \int_\Omega \big(\nu_h \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} \mathbf{v} + \nu_v \partial_z \mathbf{u} \cdot \partial_z \mathbf{v} \big) d\Omega \quad \forall\, \mathbf{u}, \mathbf{v} \in V, \tag{10}$$

is associated to the homogeneous boundary conditions (Neumann on the surface and Dirichlet on the bottom and sidewalls). In order to obtain estimates in the $\mathbf{H}^2(\Omega)$-norm we take $A\mathbf{w}_m(t) \in V_m$ as test functions, obtaining:

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{w}_m\|_V^2 + \|A\mathbf{w}_m\|^2_{\mathbf{L}^2(\Omega)} = G(\mathbf{w}_m, \mathbf{v}), \tag{11}$$

for a certain function G. Using the estimates in the strong norm for $\mathbf{v}$ (depending on data) and controlling the terms in $\mathbf{w}_m$ with the term appearing on the left side of (11), we try to bound G. The big difficulty appears in the terms:

$$I_1 = - \int_\Omega (\mathbf{w}_m \cdot \nabla_{\mathbf{x}}) \mathbf{w}_m \cdot A\mathbf{w}_m d\Omega \quad \text{and} \quad I_2 = - \int_\Omega (w_m)_3 \partial_z \mathbf{w}_m \cdot A\mathbf{w}_m d\Omega$$

corresponding to the nonlinear term of (PE). Observe that I_2 is less regular than I_1 due to the anisotropy regularity of the vertical velocity. In order to bound the I_2-term, the following lemma ([10]) will be basic:

Lemma 4. *Let $\Omega \subseteq \mathbb{R}^N$ ($N = 2$ or 3) be the domain considered in the (PE) problem. Then, for all function $\mathbf{v} \in W^{1,p}(\Omega)^{N-1}$ ($p > 1$), if we define v_3 as $v_3(\mathbf{x}, z) = - \int_{-h(\mathbf{x})}^z \nabla_{\mathbf{x}} \cdot \mathbf{v}(\mathbf{x}, s) ds$, we have:*

$$\|v_3\|_{L^p(\Omega)} \le h_{max} \|\nabla_{\mathbf{x}} \cdot \mathbf{v}\|_{\mathbf{L}^p(\Omega)}.$$

In the 2D case, using the Gagliardo-Nirenberg's inequality, we obtain:

$$\begin{aligned} I_2 &\le \|(w_m)_3\|_{L^4(\Omega)} \|\partial_z w_m\|_{L^4(\Omega)} \|Aw_m\|_{L^2(\Omega)} \\ &\le C h_{\max} \|\partial_x w_m\|_{L^4(\Omega)} \|Aw_m\|^{3/2}_{L^2(\Omega)} \|\partial_z w_m\|^{1/2}_{L^2(\Omega)} \\ &\le C h_{\max} \|w_m\|_{H^1(\Omega)} \|Aw_m\|^2_{L^2(\Omega)} \\ &\le \frac{C}{\nu^{1/2}} h_{\max} \|w_m\|_V \|Aw_m\|^2_{L^2(\Omega)}. \end{aligned}$$

Similar estimates for the remaining terms lead to:

$$\begin{aligned} &\frac{d}{dt} \|w_m\|_V^2 + \|Aw_m\|^2_{L^2(\Omega)} (1 - C_1 h_{max} \|w_m\|_V) \\ &\qquad \le C_2 \|w_m\|_V^4 + a(t) \|w_m\|_V^2 + b(t), \end{aligned} \tag{12}$$

where $a(t)$, $b(t)$ are certain functions belonging to $L^1(0,T)$ depending on the data. Hence, under smallness hypothesis on the data, allows to apply the Gronwall's Lemma an obtain the following result ([10]):

THEOREM 5 (Global strong solution for small data in the 2D case). *Let $S \subseteq \mathbb{R}$ be an interval and $h \in C^3(\overline{S})$ such that $h \geq h_{min} > 0$ in $\overline{S}$. Suppose that $u_0 \in V$, $F \in L^2(0,T;L^2(\Omega))$ and $\Upsilon \in L^2(0,T;H_0^{1/2+\varepsilon}(\Gamma_s))$, for any $\varepsilon > 0$, with $\partial_t \Upsilon \in L^2(0,T;H^{-1/2}(\Gamma_s))$. If the following* **smallness hypothesis** *is satisfied:* $\forall t \in [0,T]$,

$$(H)_{2D} \begin{cases} exp\Big(-\dfrac{1}{4K_2}t + \displaystyle\int_0^t a(s)ds\Big)\{2(\|u_0\|_V^2 + K_1\|\Upsilon(0)\|^2_{\mathbf{H}^{-1/2}(\Gamma_s)}) \\ \qquad + \displaystyle\int_0^t exp\Big(\dfrac{1}{4K_2}s - \displaystyle\int_0^s a(\sigma)d\sigma\Big)b(s)ds\} < M^2, \end{cases}$$

where M is a positive constant small enough, K_1 and K_2 are constants, and a and b are the functions appearing in (12)*, then there exists a unique strong solution (u,p_s) of (PE) in $(0,T)$ (p_s is unique up to an additive constant depending on t).*

Moreover, in [10], the asymptotic in time behavior when $t \uparrow +\infty$, exponentially decreasing in $H^1(\Omega)$-norm is proved if we impose $(H)_{2D}$ $\forall t \in (0,+\infty)$ and an additional smallness condition on the data Υ and F when $t \uparrow \infty$. Finally, a fixed point argument yields the existence of a strong solution local in time if $h_{\max}$ is small enough.

In the 3D case, applying some interpolation inequalities, we obtain ([10]):

$$I_2 \leq C\frac{h_{\min}}{\nu^{1/4}}\|A\mathbf{w}_m\|^{5/2}_{\mathbf{L}^2(\Omega)}\|\mathbf{w}_m\|_V^{1/2},$$

and therefore the previous argument cannot be applied. In the search of a solution, in [11] we focus our study in the anisotropy of the vertical velocity. Recall that $\partial_z w_3 = -\nabla_{\mathbf{x}} \cdot \mathbf{w} \in L^2(\Omega)$, and by a Poincaré vertical inequality we have $w_3 \in L^2(\Omega)$. However, $\nabla_{\mathbf{x}} w_3 \notin L^2(\Omega)$ in general. Thus, we treat the regularity for the $\mathbf{x}$ and z separately. The novelty is the fact of considering anisotropic spaces and anisotropic estimates (see [11] for the proofs):

DEFINITION 3. Given $p,q \in [1,+\infty]$, we say that a function $\mathbf{u}$ belongs to $L^q_z L^p_{\mathbf{x}}(\Omega)$ if:

$$\mathbf{u}(\cdot,z) \in L^q(S_z) \quad \text{and} \quad \|\mathbf{u}(\cdot,z)\|_{L^q(S_z)} \in L^p(-h_{\max},0).$$

and its norm is given by the expression:

$$\big\|\|\mathbf{u}(\cdot,z)\|_{L^q(S_z)}\big\|_{L^p(-h_{\max},0)}$$

PROPOSITION 6 (Interpolation inequalities). (**a**) *Let $v \in L^2(\Omega)$ be a function such that $\partial_z v \in L^2(\Omega)$ and $(vn_z)|_{\Gamma_b} = 0$. Then, $v \in L^\infty_z L^2_{\mathbf{x}}(\Omega)$ and satisfies the estimate:*

$$\|v\|^2_{L^\infty_z L^2_{\mathbf{x}}} \leq 2\,\|v\|_{L^2(\Omega)}\|\partial_z v\|_{L^2(\Omega)}. \tag{13}$$

More generally, if $v \in H^1(\Omega)$ then $v \in L^\infty_z L^2_{\mathbf{x}}(\Omega)$, and there exists a constant $C = C(\Omega) > 0$ such that:

$$\|v\|^2_{L^\infty_z L^2_{\mathbf{x}}} \leq C(\Omega)\|v\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega). \tag{14}$$

(**b**) *Let $v \in L^2(\Omega)$ be a function such that $\nabla_{\mathbf{x}} v \in L^2(\Omega)^2$ and $(v n_{x_i})|_{\Gamma_b \cup \Gamma_l} = 0$ $(i = 1, 2)$. Then, $v \in L^2_z L^4_{\mathbf{x}}(\Omega)$ and satisfies the estimate:*

$$\|v_i\|^2_{L^2_z L^4_{\mathbf{x}}} \leq 4\, \|v_i\|_{L^2(\Omega)} \|\nabla_{\mathbf{x}} v_i\|_{L^2(\Omega)}. \tag{15}$$

More generally, if $v \in H^1(\Omega)$ then $v \in L^2_z L^4_{\mathbf{x}}$, and there exists a constant $C = C(\Omega) > 0$ such that:

$$\|v\|^2_{L^2_z L^4_{\mathbf{x}}} \leq C(\Omega) \|v\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}. \tag{16}$$

PROPOSITION 7 (New estimates for v_3). *Let $\mathbf{v} \in L^2(\Omega)^2$ be a function such that $\nabla_{\mathbf{x}} \cdot \mathbf{v} \in H^1(\Omega)$. Then, if we consider v_3 defined in function of $\nabla_{\mathbf{x}} \cdot \mathbf{v}$ as in Lemma 4, we obtain that $v_3 \in L^\infty_z L^4_{\mathbf{x}}(\Omega)$ and satisfies the estimate:*

$$\|v_3\|_{L^\infty_z L^4_{\mathbf{x}}} \; \leq \; C\,(\Omega)\, \|\nabla_{\mathbf{x}} \cdot \mathbf{v}\|^{1/2}_{L^2(\Omega)} \|\nabla_{\mathbf{x}} \cdot \mathbf{v}\|^{1/2}_{H^1(\Omega)}. \tag{17}$$

Using this inequality, we bound the I_2-term in the form:

$$\begin{aligned} I_2 &\leq \|(w_3)_m\|_{L^\infty_z L^4_{\mathbf{x}}} \|\partial_z \mathbf{w}_m\|_{L^2_z L^4_{\mathbf{x}}} \|A\mathbf{w}_m\|_{L^2(\Omega)} \\ &\leq \frac{C}{\nu^{3/2}} \|A\mathbf{w}_m\|^2_{\mathbf{L}^2(\Omega)} \|\mathbf{w}_m\|_V \end{aligned}$$

for $C = C(\Omega) > 0$ a constant. Now, following a similar argument to Theorem 5, and writing precisely the influence of the data of type $L^2(0,T)$ and $L^\infty(0,T)$, and the explicit dependence on the viscosity (with constants only depending on the domain), we have [11]:

THEOREM 8 (Strong global in time solution for small data in the 3D case). *Let $S \subset \mathbb{R}^2$ be a C^3 domain and $h \in C^3(\overline{S})$ the depth function such that $h \geq h_{\min} > 0$ in $\overline{S}$. Suppose that $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ with $\mathbf{F}_1 \in L^2(0,T;\mathbf{L}^2(\Omega))$ and $\mathbf{F}_2 \in L^\infty(0,T;\mathbf{L}^2(\Omega))$, $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}_1 + \boldsymbol{\Upsilon}_2$ with $\boldsymbol{\Upsilon}_1 \in L^2(0,T;\mathbf{H}^{1/2+\varepsilon}_0(\Gamma_s))$ and $\boldsymbol{\Upsilon}_2 \in L^\infty(0,T;\mathbf{H}^{1/2+\varepsilon}_0(\Gamma_s))$ for any $\varepsilon > 0$, such that $\partial_t \boldsymbol{\Upsilon}_1 \in L^2(0,T;\mathbf{H}^{-1/2}(\Gamma_s))$ and $\partial_t \boldsymbol{\Upsilon}_2 \in L^\infty(0,T;\mathbf{H}^{-1/2}(\Gamma_s))$. If, moreover, the data satisfy the following* **"smallness conditions":**

$$(H)_{3D} \left\{ \begin{array}{ll} \|\mathbf{F}_1\|_{L^2_T(\mathbf{L}^2)} + \|\boldsymbol{\Upsilon}_1\|_{L^2_T(\mathbf{H}^{1/2+\varepsilon}_0)} < c\,\nu^{3/2}, & \|\partial_t \boldsymbol{\Upsilon}_1\|_{L^2_T(\mathbf{H}^{-1/2})} < c\,\nu^{5/2}, \\ \|\mathbf{F}_2\|_{L^\infty_T(\mathbf{L}^2)} + \|\boldsymbol{\Upsilon}_2\|_{L^\infty_T(\mathbf{H}^{1/2+\varepsilon}_0)} < c\,\nu^2, & \|\partial_t \boldsymbol{\Upsilon}_2\|_{L^\infty_T(\mathbf{H}^{-1/2})} < c\,\nu^3, \\ \|\mathbf{u}_0\|_{\mathbf{H}^1} < c\,\nu \sqrt{\dfrac{\nu}{\bar{\nu}}}, & \|\boldsymbol{\Upsilon}_1(0)\|_{\mathbf{H}^{-1/2}} + \|\boldsymbol{\Upsilon}_2(0)\|_{\mathbf{H}^{-1/2}} < c\,\nu^2 \sqrt{\dfrac{\nu}{\bar{\nu}}}, \end{array} \right.$$

where $\nu = \min\{\nu_h, \nu_v\}$, $\bar{\nu} = \max\{\nu_h, \nu_v\}$ and c is a constant small enough (depending on Ω), then there exists a (unique) strong solution $(\mathbf{u}, p_s)$ of (PE) in $(0,T)$ (p_s is unique up to an additive function depending on t).

REMARK 2. We have denoted $L^q_T(L^p) = L^q(0,T;L^p(\Omega))$, $H^{-1/2} = H^{-1/2}(\Gamma_s)$ and $H^{1/2+\varepsilon}_0 = H^{1/2+\varepsilon}_0(\Gamma_S)$.

On the other hand, if we try to eliminate the smallness hypotheses on the data, we start from the following expression relative to (12) but for the 3D case:

$$\begin{aligned} \frac{d}{dt}\|\mathbf{w}\|^2_V + \|A\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} \leq \frac{C}{\nu^{3/2}} \|A\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} \|\mathbf{w}\|_V + \frac{C}{\nu^{11}} \|\mathbf{w}\|^{10}_V \\ + a(t)\|\mathbf{w}\|^2_V + b(t), \end{aligned} \tag{18}$$

where $a(t)$ and $b(t)$ belong to $L^1(0,T)$, depend on ν and on the data. Unlike the fixed point argument made in [10], which imposed smallness for $h_{\max}$, in [11] we use a new argument that avoids this hypothesis. It is the following: Since $\mathbf{w}_m(0)=\mathbf{0}$ and $\mathbf{w}_m$ is a time continuous function valued in $\mathbf{H}^1(\Omega)$, we can find a time T_m^1 (see [11] for more details) such that:

$$\|\mathbf{w}_m(t)\|_V \le \frac{\nu^{3/2}}{2C}, \quad \forall t\in[0,T_m^1].$$

From this point, bounding from below $T_m^1 \ge T^1 > 0$, the proof of the existence of strong solution in $(0,T^1)$ can be concluded in a standard manner. ∎

3.3. *Time asymptotic behavior.* In [11] the time asymptotic behavior towards a steady solution is studied (generated by the second member $\mathbf{F}_2$ and Neumann boundary condition $\boldsymbol{\Upsilon}_2$, which now are time independent functions). The objective is to obtain a result of convergence in norm $\mathbf{V}$, which in principle forces us to know under what conditions the strong regularity of the stationary problem is obtained:

$$(PE)_{st}\begin{cases} -\nu_h\Delta_{\mathbf{x}}\mathbf{v}-\nu_v\partial^2_{zz}\mathbf{v}+(\mathbf{v}\cdot\nabla_{\mathbf{x}})\mathbf{v}+v\partial_z\mathbf{v}+\alpha\mathbf{v}^\perp+\nabla_{\mathbf{x}}p_s=\mathbf{F}_2 \text{ in } \Omega,\\ \nabla_{\mathbf{x}}\cdot\langle\mathbf{v}\rangle=0 \quad \text{in } S,\\ \nu_v\partial_z\mathbf{v}_{|\Gamma_s}=\boldsymbol{\Upsilon}_2, \quad \mathbf{v}_{|\Gamma_b\cup\Gamma_l}=\mathbf{0}.\end{cases}$$

The following result is obtained in [11]:

THEOREM 9. *If data $(\mathbf{F}_2,\boldsymbol{\Upsilon}_2)$ are small enough in the $\mathbf{L}^2(\Omega)\times\mathbf{H}_0^{1/2+\varepsilon}(\Gamma_s)$-norm, then there exists a unique strong solution $\mathbf{v}$ of $(PE)_{st}$, and there exists $C=C(\Omega)>0$ such that:*

$$\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega)} \le \frac{C}{\nu^2}\{\|\mathbf{F}_2\|^2_{\mathbf{H}^{-1}(\Omega)}+\|\boldsymbol{\Upsilon}_2\|^2_{\mathbf{H}^{-1/2}(\Gamma_s)}\}, \tag{19}$$

$$\|\mathbf{v}\|^2_{\mathbf{H}^2(\Omega)} \le \frac{C}{\nu^2}\{\|\mathbf{F}_2\|^2_{\mathbf{L}^2(\Omega)}+\|\boldsymbol{\Upsilon}_2\|^2_{\mathbf{H}_0^{1/2+\varepsilon}(\Gamma_s)}\}. \tag{20}$$

Finally, the asymptotic behavior obtained in [11] can be written as:

THEOREM 10 (Convergence towards steady solution). *Let $\mathbf{u}$ a strong solution of (PE) in $(0,+\infty)$ with second member $\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2$, where $\mathbf{F}_1\in L^2(0,+\infty;\mathbf{L}^2(\Omega))$ and $\mathbf{F}_2\in\mathbf{L}^2(\Omega)$ (independent on t), and the Neumann condition $\boldsymbol{\Upsilon}=\boldsymbol{\Upsilon}_1+\boldsymbol{\Upsilon}_2$, where $\boldsymbol{\Upsilon}_1\in L^2(0,+\infty;\mathbf{H}_0^{1/2+\varepsilon}(\Gamma_s))$ for some $\varepsilon>0$, such that $\partial_t\boldsymbol{\Upsilon}_1\in L^2(0,+\infty;\mathbf{H}^{-1/2}(\Gamma_s))$, and $\boldsymbol{\Upsilon}_2\in\mathbf{H}_0^{1/2+\varepsilon}(\Gamma_s)$ for some $\varepsilon>0$ (also independent on t). Assuming smallness hypotheses (H) with $T=+\infty$, if $\mathbf{v}$ is the steady strong solution of $(PE)_{st}$ with second member $\mathbf{F}_2$ and Neumann boundary condition $\boldsymbol{\Upsilon}_2$, then $\mathbf{u}(t)\to\mathbf{v}$ in the $\mathbf{H}^1(\Omega)$ norm as $t\to+\infty$.*

4. Uniqueness of weak/strong solution. The smaller regularity of the nonlinear term (of vertical convection) in the PE system causes that greater regularity is needed to demonstrate uniqueness of solution than in the Navier-Stokes case (see, for instance, the book of P. L. Lions [18] for this case). Assuming $(\mathbf{u},v)$ and $(\underline{\mathbf{u}},\underline{v})$ two possible solutions, the main difficulty is to control the terms:

$$J_1=\int_\Omega(\mathbf{u}-\underline{\mathbf{u}})\cdot\nabla_{\mathbf{x}}\underline{\mathbf{u}}\cdot(\mathbf{u}-\underline{\mathbf{u}}) \quad\text{and}\quad J_2=\int_\Omega(v-\underline{v})\partial_z\underline{\mathbf{u}}\cdot(\mathbf{u}-\underline{\mathbf{u}})$$

Using anisotropic estimations of Lemmas 6 and 7, the following inequalities hold:

$$J_1 \le \|\mathbf{u}-\underline{\mathbf{u}}\|^2_{L^2_z L^4_{\mathbf{x}}} \|\nabla_{\mathbf{x}}\underline{\mathbf{u}}\|_{L^\infty_z L^2_{\mathbf{x}}} \le C\|\nabla_{\mathbf{x}}\underline{\mathbf{u}}\|_{L^\infty_z L^2_{\mathbf{x}}} \|\mathbf{u}-\underline{\mathbf{u}}\|_{\mathbf{L}^2}\|\mathbf{u}-\underline{\mathbf{u}}\|_{\mathbf{H}^1},$$

$$J_2 \le \|v-\underline{v}\|_{L^\infty_z L^2_{\mathbf{x}}} \|\partial_z\underline{\mathbf{u}}\|_{L^2_z L^4_{\mathbf{x}}} \|\mathbf{u}-\underline{\mathbf{u}}\|_{L^2_z L^4_{\mathbf{x}}} \le C\|\partial_z\underline{\mathbf{u}}\|_{L^2_z L^4_{\mathbf{x}}} \|\mathbf{u}-\underline{\mathbf{u}}\|^{1/2}_{L^2}\|\mathbf{u}-\underline{\mathbf{u}}\|^{3/2}_{H^1}.$$

Consequently, one arrives at

THEOREM 11 (Weak/strong uniqueness [4]). *Let* $\mathbf{u}$ *a weak solution of* (PE) *in* $(0,T)$. *If there exists* $\underline{\mathbf{u}}$ *a solution of* (PE) *in* $(0,T)$ *such that:*

$$\nabla_{\mathbf{x}}\underline{\mathbf{u}} \in L^2(0,T;L^\infty_z L^2_{\mathbf{x}}) \qquad \text{and} \qquad \partial_z\underline{\mathbf{u}} \in L^4(0,T;L^2_z L^4_{\mathbf{x}}), \tag{21}$$

then both solutions must coincide in $[0,T)$.

In [12] the previous result is improved, eliminating the additional regularity imposed for $\nabla_{\mathbf{x}}\underline{\mathbf{u}}$. For this, the following new anisotropic estimation is used:

LEMMA 12. *Let* $u \in H^1_{b,l}(\Omega)$ *such that* $\partial_z u \in H^1(\Omega)$. *Then* $u \in L^\infty_z L^4_{\mathbf{x}}$ *and there exists a constant* $C = C(\Omega) > 0$ *such that:*

$$\|u\|_{L^\infty_z L^4_{\mathbf{x}}} \le C(\Omega)\|u\|^{1/4}_{L^2(\Omega)}\|u\|^{1/4}_{H^1(\Omega)}\|\partial_z u\|^{1/4}_{L^2(\Omega)}\|\partial_z u\|^{1/4}_{H^1(\Omega)}. \tag{22}$$

Using this inequality in the J_1 term, (previously integrated by parts) one has,

$$J_1 \le C\|\underline{\mathbf{u}}\|^{1/4}_{L^2(\Omega)}\|\underline{\mathbf{u}}\|^{1/4}_{H^1(\Omega)}\|\partial_z\underline{\mathbf{u}}\|^{1/4}_{L^2(\Omega)}\|\partial_z\underline{\mathbf{u}}\|^{1/4}_{H^1(\Omega)}\|\mathbf{u}-\underline{\mathbf{u}}\|^{1/2}_{L^2(\Omega)}\|\mathbf{u}-\underline{\mathbf{u}}\|^{3/2}_{H^1(\Omega)}.$$

Consequently the uniqueness of weak solution is obtained, changing the additional regularity (21) to

$$\partial_z\underline{\mathbf{u}} \in L^\infty(0,T;\mathbf{L}^2) \cap L^2(0,T;\mathbf{H}^1). \tag{23}$$

This uniqueness result also holds, when Robin boundary conditions at bottom are imposed, but only in domains with sidewalls [12].

REMARK 3. In $2D$ domains, an additional hypothesis that implies uniqueness is $\partial_z\underline{u} \in L^4(0,T;L^2(\Omega))$. In any case ($2D$ or $3D$), the additional regularity is not assured in general for a weak solution, hence uniqueness of weak solution is an open problem. We will see in Section 6 that in $2D$ domains this open problem is solved obtaining the additional regularity (23) for $\partial_z u$ (supposing L^2 regularity for $\partial_z u_0$ and $\partial_z f$).

Finally, in [12] it is also proved that (23) is a sufficient condition to deduce strong regularity:

THEOREM 13. *Let* $S \subseteq \mathbb{R}^2$ *be a* C^3 *domain and* $h \in C^3(\overline{S})$ *with* $h \ge h_{min} > 0$ *in* $\overline{S}$. *Let* $\mathbf{F} \in L^2(0,T;\mathbf{L}^2)$, $\mathbf{u}_0 \in \mathbf{V}$, $\boldsymbol{\Upsilon} \in L^2(0,T;\mathbf{H}_0^{1/2+\varepsilon}(\Gamma_s))$, *for some* $\varepsilon > 0$, *such that* $\partial_t\boldsymbol{\Upsilon} \in L^2(0,T;\mathbf{H}^{-3/2}(\Gamma_s))$ *with* $\boldsymbol{\Upsilon}(0) \in \mathbf{H}^{-1/2}(\Gamma_s)$. *Assuming* $\mathbf{u}$ *a weak solution of* (PE) *in* $(0,T)$ *such that* $\partial_z\mathbf{u} \in L^\infty(0,T;\mathbf{L}^2)\cap L^2(0,T;\mathbf{H}^1)$, *then* $\mathbf{u}$ *is the unique strong solution of* (PE) *in* $(0,T)$.

For the proof of this result, the method is standard but it is necessary to prove some new anisotropic estimates that appear in the following lemma (note that the hypothesis $h \ge h_{\min} > 0$ is necessary):

LEMMA 14. *a) Let $v \in L^2(\Omega)$ be a function such that $\partial_z v \in L^2(\Omega)$. Then, $v \in L^\infty_z L^2_{\mathbf{x}}(\Omega)$ and*

$$h_{\min}\|v\|^2_{L^\infty_z L^2_{\mathbf{x}}} \le \|v\|^2_{L^2(\Omega)} + 2\|v\|_{L^2(\Omega)}\|\partial_z v\|_{L^2(\Omega)}. \tag{24}$$

b) Let $v \in H^1(\Omega)$ be a function such that $\partial_z v \in H^1(\Omega)$. Then, $v \in L^\infty_z L^4_{\mathbf{x}}(\Omega)$ and

$$h^{1/2}_{\min}\|v\|_{L^\infty_z L^4_{\mathbf{x}}} \le C\|v\|^{1/4}_{L^2(\Omega)}\|v\|^{1/4}_{H^1(\Omega)}(\|v\|^{1/4}_{L^2(\Omega)}\|v\|^{1/4}_{H^1(\Omega)} + \|\partial_z v\|^{1/4}_{L^2(\Omega)}\|\partial_z v\|^{1/4}_{H^1(\Omega)}). \tag{25}$$

The difference between (13) and (24), and between (22) and (25) is that in the inequalities of Lemma 14 there are no homogeneous boundary conditions for the functions.

5. Non-regular data for Primitive Equations. The analysis of the regularity for the data imposed in order to obtain strong solution for Primitive Equations does not seem to be optimal. We can observe that if $\mathbf{v} \in \mathbf{H}^1(\Omega)$ then $\nu_v \partial_z \mathbf{v}|_{\Gamma_s} = \boldsymbol{\Upsilon} \in \mathbf{H}^{1/2+\varepsilon}_0(\Gamma_s)$, but in the "classical works" it is usual to impose that $\partial_t \boldsymbol{\Upsilon} \in \mathbf{H}^{-1/2}(\Gamma_s)$ to obtain $\partial_t \mathbf{v} \in \mathbf{L}^2(\Omega)$. Here, we explain the reason why we replaced this hypothesis in Theorems 3 and 8 by $\partial_t \boldsymbol{\Upsilon} \in L^2(0,T;\mathbf{H}^{-3/2}(\Gamma_s))$.

The result is a generalization of that one of C. Conca for the stationary Stokes problem ([7]) to the hydrostatic Stokes problem (i.e., the linear stationary Primitive Equations problem). In [7], the *very weak solution* is defined for the Stokes problem, and corresponds to the regularity that can be obtained for this system in the case that the Dirichlet boundary data only belong to $L^2(\partial\Omega)$ (usually the data belong to $H^{1/2}(\partial\Omega)$).

As we said before, we will use this very weak solution to weaken the regularity demanded for the data $\partial_t \boldsymbol{\Upsilon}$ in order to obtain strong solution for the Primitive Equations, global in time for small data and local in time for any data.

The difficulties that the linear Primitive Equations model present versus the Stokes problem are: the hydrostatic pressure, the new free divergence condition and the mixed boundary data (nonhomogeneous Neumann on the surface and homogeneous Dirichlet in other case).

The existence of very weak solution will be proved for the linear stationary hydrostatic (Stokes) problem, and then generalized for the evolutionary case.

In order to fix ideas, we write the following problem: knowing the external forces $\mathbf{F} \in \mathbf{L}^2(\Omega)$ and the wind stress tensor on the surface $\boldsymbol{\Upsilon} \in \mathbf{H}^{-3/2}(\Gamma_s)$, we want to find the horizontal velocity $\mathbf{u}$ and the surface pressure p:

$$\begin{cases} -\nu\Delta\mathbf{u} - \nu_3\partial^2_{zz}\mathbf{u} + \nabla p = \mathbf{F} & \text{in } \Omega, \\ \nabla\cdot\langle\mathbf{u}\rangle = 0 & \text{in } S, \\ \nu_3\partial_z\mathbf{u} = \boldsymbol{\Upsilon} & \text{on } \Gamma_s, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_b \cup \Gamma_l. \end{cases} \tag{26}$$

5.1. *The dual problem.* The dual problem associated to (26) is the following:

$$\begin{cases} -\nu\Delta\phi - \nu_3\partial^2_{zz}\phi + \nabla\pi = \mathbf{g} & \text{in } \Omega, \\ \nabla\cdot\langle\phi\rangle = -\varphi & \text{in } S, \\ \nu_3\partial_z\phi = \mathbf{0} & \text{on } \Gamma_s, \\ \phi = \mathbf{0} & \text{on } \Gamma_b \cup \Gamma_l, \end{cases} \tag{27}$$

where $\mathbf{g} \in \mathbf{L}^2(\Omega)$, $\varphi \in \mathcal{H}$, and

$$\mathcal{H} = \left\{ \varphi / \, \varphi \in H^1(S), \, \int_S \varphi d\mathbf{x} = 0 \right\}.$$

Using the mixed formulation of the problem ([8]) and generalizing Ziane's results of H^2-regularity for (27) (only proved for $\varphi \equiv \mathbf{0}$), we prove that:

THEOREM 15. *Let $h \in C^3(S)$ the depth function and $\partial S \in C^3$. If $\mathbf{g} \in \mathbf{L}^2(\Omega)$ and $\varphi \in \mathcal{H}$, then there exists a unique solution of* (27) *with $\phi \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{b,l}(\Omega)$, $\pi \in H^1(S)$, satisfying moreover that:*

$$\|\phi\|^2_{H^2(\Omega)} + \|\pi\|^2_{H^1(S)} \leq C\{\|\mathbf{g}\|^2_{L^2(\Omega)} + \|\varphi\|^2_{\mathcal{H}}\}. \tag{28}$$

5.2. *The very weak regularity*

DEFINITION 4. A pair $(\mathbf{u}, p)$ is called a *very weak solution of* (26) iff the following conditions are satisfied:

$$\left\{ \begin{array}{c} \mathbf{u} \in \mathbf{L}^2(\Omega), \qquad p \in (H^1(S))'/\mathbb{R}, \\ \displaystyle\int_\Omega \mathbf{u} \cdot \mathbf{g} \, d\Omega + \langle p, \varphi \rangle_S = l(\mathbf{g}, \varphi), \\ \forall \mathbf{g} \in \mathbf{L}^2(\Omega), \, \forall \varphi \in H^1(S) \text{ such that } \displaystyle\int_S \varphi \, d\mathbf{x} = 0, \end{array} \right. \tag{29}$$

where $\langle \cdot, \cdot \rangle_S$ denotes the duality between $(H^1(S))'$ and $H^1(S)$, where $l : \mathbf{L}^2(\Omega) \times \mathcal{H} \to \mathbb{R}$ is defined by:

$$\left\{ \begin{array}{ll} l(\mathbf{g}, \varphi) = \displaystyle\int_\Omega \mathbf{F} \cdot \phi d\Omega + \langle \mathbf{\Upsilon}, \phi \rangle_{\Gamma_s} & \text{si } \mathbf{F} \in \mathbf{L}^2(\Omega), \\ l(\mathbf{g}, \varphi) = \langle \mathbf{F} \, \phi \rangle_\Omega + \langle \mathbf{\Upsilon}, \phi \rangle_{\Gamma_s} & \text{si } \mathbf{F} \in (\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{b,l}(\Omega))', \end{array} \right.$$

where (ϕ, π) is the solution of the dual problem (27) and $\langle \cdot, \cdot \rangle_{\Gamma_s}$ the duality between $H^{-3/2}(\Gamma_s)$ and $H^{3/2}_0(\Gamma_s)$ (and $\langle \cdot, \cdot \rangle_\Omega$ the duality between $(\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{b,l}(\Omega))'$ and $\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{b,l}(\Omega)$). It is easy to see that l is a continuous linear operator from $\mathbf{L}^2(\Omega) \times \mathcal{H}$ into $\mathbb{R}$.

Using (27), we rewrite the previous definition as:

$$\left\{ \begin{array}{c} \mathbf{u} \in \mathbf{L}^2(\Omega), \qquad p \in \mathcal{H}', \\ \displaystyle\int_\Omega \mathbf{u} \cdot \left(-\nu \Delta \phi - \nu_3 \partial^2_{zz} \phi + \nabla \pi \right) d\Omega - \langle p, \nabla \cdot \langle \phi \rangle \rangle_S = \langle \mathbf{F}, \phi \rangle_\Omega + \langle \mathbf{\Upsilon}, \phi \rangle_{\Gamma_s}, \\ \forall \phi \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{b,l}(\Omega), \, \nabla \cdot \langle \phi \rangle \in H^1(S) \quad \text{and} \quad \partial_z \phi|_{\Gamma_s} = 0, \quad \forall \pi \in H^1(S). \end{array} \right. \tag{30}$$

Therefore, we give the following result:

THEOREM 16. *Given $\mathbf{F} \in (\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{b,l}(\Omega))'$ and $\mathbf{\Upsilon} \in \mathbf{H}^{-3/2}(\Gamma_s)$ there exists a unique very weak solution $(\mathbf{u}, p)$ of* (29) *in $\mathbf{L}^2(\Omega) \times (H^1(S))'/\mathbb{R}$ ($p \in (H^1(S))'$ unique up to additive constant). Moreover,*

$$\|\mathbf{u}\|_{L^2(\Omega)} + \|p\|_{(H^1(S))'/\mathbb{R}} \leq C\{\|\mathbf{F}\|_{(H^2(\Omega) \cap H^1_{b,l}(\Omega)^2)'} + \|\mathbf{\Upsilon}\|_{H^{-3/2}(\Gamma_s)}\}. \tag{31}$$

As in [7], the proof of Theorem 16 needs the result:

PROPOSITION 17. *The space $(H^1(S))'/\mathbb{R}$ is isomorphic to $\mathcal{H}'$, the dual space of $\mathcal{H}$.*

Scheme of the proof of Theorem 16. Since $l : \mathbf{L}^2(\Omega) \times \mathcal{H} \to \mathbb{R}$ is a linear continuous operator, there exists a unique pair $(\mathbf{u}, \widetilde{p}) \in \mathbf{L}^2(\Omega) \times \mathcal{H}'$ ($\mathcal{H}'$ the dual space of $\mathcal{H}$) such that:

$$\int_\Omega \mathbf{u} \cdot \mathbf{g} \, d\Omega + \langle \widetilde{p}, \varphi \rangle_{\mathcal{H}', \mathcal{H}} = l(\mathbf{g}, \varphi) \quad \forall \mathbf{g} \in \mathbf{L}^2(\Omega), \forall \varphi \in \mathcal{H}.$$

From Proposition 17 we can identify $\widetilde{p}$ with a distribution p in $(H^1(S))'/\mathbb{R}$ such that $\langle \widetilde{p}, \varphi \rangle_{\mathcal{H}', \mathcal{H}} = \langle p, \varphi \rangle_{(H^1(S))', H^1(S)}$, $\forall \varphi \in \mathcal{H}$. Therefore, we conclude that $(\mathbf{u}, p)$ is a solution of (29), and this proves the existence of solution. The uniqueness follows from the method used in the construction of the solution. For the continuous dependence of the solution with respect to the data the estimate (28) is used. ■

Once the regularity of problem (26) is obtained, we get:

PROPOSITION 18. *Let* $(\mathbf{u}, p) \in \mathbf{L}^2(\Omega) \times \mathcal{H}'$ *the unique solution of* (29). *Then,* $(\mathbf{u}, p)$ *satisfy* $(26)_{1-2}$ *in the sense of distributions in* Ω *and* S *respectively.*

Finally, it is possible to give a meaning to the boundary conditions in certain dual spaces, defining what we call "generalized traces" and which coincides with the standard trace operator for regular functions (see [5] for the details).

As we explained before, the final version of the regularity result for Primitive Equations (S) is:

THEOREM 19. *Let* $S \subseteq \mathbb{R}^d$ *(*$d = 1$ *or* 2*) a* C^3*-domain and* $h \in C^3(\bar{S})$ *the depth function satisfying* $h \geq h_{min} > 0$ *in* $\bar{S}$*. If* $\mathbf{F} \in \mathbf{L}^2(0,T) \times \Omega)$*,* $\mathbf{u}_0 \in \mathbf{V}$*,* $\mathbf{\Upsilon} \in L^2(0,T; \mathbf{H}_0^{1/2+\varepsilon}(\Gamma_s)) \cap L^\infty(0,T; \mathbf{H}^{-1/2}(\Gamma_s))$*, for any* $\varepsilon > 0$ *with* $\partial_t \mathbf{\Upsilon} \in L^2(0,T; \mathbf{H}^{-3/2}(\Gamma_s))$ *and* $\mathbf{\Upsilon}(0) \in \mathbf{H}^{-1/2}(\Gamma_s)$*, then there exists a unique strong solution* $\mathbf{v}$ *of* (S) *in* $(0,T)$*. Moreover, there exists a constant* $C > 0$ *such that:*

$$\begin{aligned} (32) \quad & \|\mathbf{v}\|^2_{L^\infty(V)} + \|\mathbf{v}\|^2_{L^2(\mathbf{H}^2(\Omega))} + \|\partial_t \mathbf{v}\|^2_{L^2(H)} \leq C\{\|\mathbf{u}_0\|^2_V + \|\mathbf{F}\|^2_{L^2(\mathbf{L}^2(\Omega))} \\ & + \|\mathbf{\Upsilon}(0)\|^2_{\mathbf{H}^{-1/2}(\Gamma_s)} + \|\mathbf{\Upsilon}\|^2_{L^\infty(\mathbf{H}^{-1/2}(\Gamma_s))} + \|\mathbf{\Upsilon}\|^2_{L^2(\mathbf{H}_0^{1/2+\varepsilon}(\Gamma_s))} + \|\partial_t \mathbf{\Upsilon}\|^2_{L^2(\mathbf{H}^{-3/2}(\Gamma_s))}\}. \end{aligned}$$

REMARK 4. In the case of $S \subseteq \mathbb{R}^2$ of C^∞-class, the hypotheses $\mathbf{\Upsilon} \in L^\infty(0,T; \mathbf{H}^{-1/2}(\Gamma_s))$ and $\mathbf{\Upsilon}(0) \in \mathbf{H}^{-1/2}(\Gamma_s)$ are not needed. Indeed, from $\mathbf{\Upsilon} \in L^2(0,T; \mathbf{H}_0^{1/2+\varepsilon}(\Gamma_s))$ and $\partial_t \mathbf{\Upsilon} \in L^2(0,T; \mathbf{H}^{-3/2}(\Gamma_s))$ we can obtain $\mathbf{\Upsilon} \in C([0,T]; \mathbf{H}^{-1/2}(\Gamma_s))$ with continuous dependence (see [13]).

REMARK 5 (Application to the nonlinear evolutionary Primitive Equations). The extension of Theorem 19 to the nonlinear case is identical to the extension obtained in [10, 11], replacing Theorem 3 by Theorem 19.

6. Regularity and uniqueness for the 2D model. The main object is to obtain existence of weak solution u with additional weak regularity for $\partial_z u$ for the case of friction on the bottom $\partial_z u_{|\Gamma_b} = \beta u_{|\Gamma_b}$. This model was obtained, in the 2D case, from (BEs) with friction boundary condition on the bottom as the aspect quotient δ tends to zero ([5]) (note that the "usual" model is obtained in the same way when homogeneous Dirichlet boundary conditions on the bottom are considered). In particular, this solution is unique.

The problem is: Find velocity (u, v) and pressure p such that:

$$(PE)_{2D}\begin{cases} \partial_t u + u\partial_x u + v\partial_z u - \nu_h \partial_x^2 u - \nu_v \partial_z^2 u + \partial_x p_s = f \text{ in } (0,T)\times\Omega, \\ v(t;x,z) = \int_z^0 \partial_x u(t;x,s)ds \quad \text{in } (0,T)\times\Omega, \quad \langle u\rangle = 0 \text{ in } (0,T)\times S, \\ \nu_v \partial_z u|_{\Gamma_s} = \alpha|u^a|(u^a - u), \quad u_{|\Gamma_l} = 0, \quad \nu_v\partial_z u|_{\Gamma_b} = \beta(x)u \text{ in } (0,T), \\ u|_{t=0} = u_0 \text{ in } \Omega. \end{cases}$$

REMARK 6. In $2D$ domains, the constraint derive to $\langle u\rangle \equiv 0$ that is deduced from $\partial_x\langle u\rangle = 0$ in $(0,T)\times S$ and $\langle u\rangle = 0$ on $(0,T)\times\partial S$.

REMARK 7. To assure that the model is dissipative, one must impose $\gamma(x)\geq 0$, with

$$\gamma(x) = \left\{\beta(x)\left(1+\frac{\nu_h}{\nu_v}|D'(x)|^2\right) - \frac{\nu_h}{2}D''(x)\right\},$$

which is derived from the limit of dissipative hypothesis of $2D$ (BEs) as $\delta\to 0$.

DEFINITION 5. We say that u is a *weak-vorticity solution* of (PE) in $(0,T)$ if it is a weak solution ($u\in L^\infty(0,T;H)\cap L^2(0,T;V)$), that satisfies the additional regularity:

$$\partial_z u \in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)).$$

REMARK 8. The function $\partial_z u$ can be called the vorticity of $(PE)_{2D}$, because it is the limit of the 2D (BEs) vorticity as $\delta\to 0$ ([4]).

We state the main result:

THEOREM 20. *Let $h\in H^2(S)$ with $|h'|>0$ on ∂S, $\beta\in H_0^1(S)$, $f\in L^2(0,T;L^2(\Omega))$, $\partial_z f\in L^2(0,T;H^{-1}(\Omega))$, $u^a\in L^\infty(0,T;H_0^1(S))$, $\partial_t u^a \in L^2(0,T;L^1(S))$, $u_0\in H$, $\partial_z u_0\in L^2(\Omega)$. Assuming the dissipative hypothesis $\gamma(x)\geq 0$ in S and that the depth function h satisfies $|h'(x)|/D(x)\leq c/dist(x,\partial S)$ in S, there exists a unique weak-vorticity solution of $(PE)_{2D}$ in $(0,T)$.*

In the proof of this result, a problem satisfied by $\partial_z u$ is used. Indeed, differentiating (PE) with respect to z, one has that $w = \partial_z u$ satisfies the initial-boundary problem:

$$\begin{cases} \partial_t w + u\partial_x w + v\partial_z w - \nu_h\partial_x^2 w - \nu_v \partial_z^2 w = \partial_z f, \\ w|_{\Gamma_s} = \alpha|u^a|(u^a-u)/\nu_v, \quad w|_{\Gamma_l} = 0, \quad w|_{\Gamma_b} = \beta(x)u/\nu_v, \\ w|_{t=0} = \partial_z u_0 \end{cases}$$

that we will call the vorticity problem. Notice that, given u and v, this problem is linear and parabolic, and in addition the pressure "has disappeared". Therefore, we can expect weak regularity for $\partial_z u$. But, boundary conditions for $\partial_z u$ on Γ_s and Γ_b depend on u and u and p are coupled by the (PE) problem. Consequently, the problem satisfied by $\partial_z u$ depends also on the pressure, because a lifting of the nonhomogeneous boundary conditions must be done.

The main problems to solve are two: First, we need to improve the regularity of the pressure, in order to obtain a weak solution w of the vorticity problem. Then, we need to identify w with $\partial_z u$, where the difficulty is that $\partial_z u$ in principle only has $L^2(0,T;L^2(\Omega))$ regularity.

The following lemma guarantees a certain weight regularity for the pressure that a posteriori will be sufficient to obtain a weak solution of the vorticity problem.

LEMMA 21. *Under hypothesis of Theorem* 20, *if* (u, v, p) *is a weak solution of* $(PE)_{2D}$, *then one has:*

$$\sqrt{h}\,\partial_x p_s \in L^2(0,T;H^{-1}(S)).$$

In the context of weak solution of Navier-Stokes, the regularity of the pressure is obtained from the regularity of the rest of the terms of the momentum equations. In particular, the term $\partial_t u$ implies time regularity of H^{-1} dual type. In this case, using that $\langle u\rangle = 0$, in particular $\partial_t\langle u\rangle = 0$, we can improve the time regularity for the pressure, integrating previously the equation in vertical.

We have then to identify the solution vorticity w with $\partial_z u$. As we already said, the main difficulty is that $\partial_z u$ only belongs to $L^2(0,T;L^2(\Omega))$, which causes that the well-known results of uniqueness cannot be applied. Then, an alternative is to compare u with a suitable function $\widetilde{u}$ such that $\langle\widetilde{u}\rangle = 0$ on S and $\partial_z\widetilde{u} = w$ (that can be directly obtained).

One has that $\widetilde{u}$ (jointly with a potential function $\widetilde{p}_s$ defined in S) satisfies the problem:

$$\partial_t\widetilde{u} + u\,\partial_x\widetilde{u} + v\,\partial_z\widetilde{u} - \nu_h\partial^2_{xx}\widetilde{u} - \nu_v\partial^2_{zz}\widetilde{u} + \partial_x\widetilde{p}_s = G, \tag{33}$$

with the same initial and boundary conditions as u, where

$$G = u\,\partial_x\widetilde{u} + \int_z^0 \partial_x\left(u\partial_z\widetilde{u}\right)(x,s)ds + f.$$

It is important to notice that $G = f$ whereas $\widetilde{u} = u$. Making a uniqueness argument for problems satisfied by u and $\tilde{u}$, and using the additional regularity for $\partial_z\widetilde{u}$ (since $\widetilde{u}$ is a weak-vorticity solution), one can conclude that $u = \widetilde{u}$ (see [5]).

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