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GLOBAL EXISTENCE OF SOLUTIONS OF THE FREE BOUNDARY PROBLEM FOR THE EQUATIONS OF MAGNETOHYDRODYNAMIC COMPRESSIBLE FLUID

PIOTR KACPRZYK

Institute of Mathematics and Cryptology, Cybernetics Faculty
Military University of Technology
S. Kaliskiego 2, 00-908 Warszawa, Poland
E-mail: p.kacprzyk@imbo.wat.edu.pl

Abstract. Global existence of solutions for equations describing a motion of magnetohydrodynamic compresible fluid in a domain bounded by a free surface is proved. In the exterior domain we have an electromagnetic field which is generated by some currents located on a fixed boundary. We have proved that the domain occupied by the fluid remains close to the initial domain for all time.

1. Introduction. In this paper we prove the existence of global solution to equations describing a motion of magnetohydrodynamic compresible fluid in a domain $\Omega_t \subset \mathbb{R}^3$ bounded by a free surface S_t. In a domain $D_t \subset \mathbb{R}^3$ which is exterior to Ω_t we have a gas under a constant pressure p_0. Moreover in the domain D_t we have an electromagnetic field generated by some currents located on a fixed boundary B of D_t.

In the domain Ω_t the motion is described by the following problem

$$
\begin{aligned}
&\varrho(v_t + v\cdot\nabla v) - \operatorname{div}\mathbb{T}(v,p) - \mu_1 \overset{1}{H}\cdot\nabla \overset{1}{H} + \mu_1\nabla\frac{\overset{1}{H}{}^2}{2} = f && \text{in } \tilde\Omega^T,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \tilde\Omega^T,\\
&\mu_1 \overset{1}{H}_t = -\operatorname{rot}\overset{1}{E} && \text{in } \tilde\Omega^T,\\
&\operatorname{rot}\overset{1}{H} = \sigma_1(\overset{1}{E} + \mu_1 v\times \overset{1}{H}) && \text{in } \tilde\Omega^T,\\
&\operatorname{div}(\mu_1 \overset{1}{H}) = 0 && \text{in } \tilde\Omega^T,
\end{aligned}
\tag{1.1}
$$

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where $\tilde{\Omega}^T = \bigcup_{0\le t\le T}\Omega_t \times \{t\}$, $v = v(x,t)$ is the velocity of the fluid, $p = p(\varrho)$ the pressure, $\varrho = \varrho(x,t)$ the density, $\overset{1}{H} = \overset{1}{H}(x,t)$ is the magnetic field, $f = f(x,t)$ the external force field per unit mass, μ_1 the constant magnetic permeability, σ_1 the constant electric conductivity, $\overset{1}{E} = \overset{1}{E}(x,t)$ the electric field,

$$\mathbb{T}(v,p) = \mathbb{D}(v) - pI \tag{1.2}$$

is the stress tensor, where by

$$\mathbb{D}(v) = \{\mu(\partial_{x_i}v_j + \partial_{x_j}v_i) + (\nu-\mu)\delta_{ij}\mathrm{div}v\}_{i,j=1,2,3} \tag{1.3}$$

we denote the dilatation tensor, where ν,μ are the viscosity coefficients of the fluid and I the unit matrix.

In the domain D_t occupied by dielectric (gas) we assume that there is no fluid motion inside ($v=0$). Therefore we have only the electromagnetic field described by the following system

$$\begin{aligned} &\mu_2 \overset{2}{H}_t = -\mathrm{rot}\,\overset{2}{E} && \text{in } \tilde{D}^T,\\ &\mathrm{rot}\,\overset{2}{H} = \sigma_2 \overset{2}{E} && \text{in } \tilde{D}^T,\\ &\mathrm{div}(\mu_2 \overset{2}{H}) = 0 && \text{in } \tilde{D}^T, \end{aligned} \tag{1.4}$$

where $\tilde{D}^T = \bigcup_{0\le t\le T} D_t \times \{t\}$.

On $S_t = \partial\Omega_t \cap \partial D_t$ we assume the following transmission and boundary conditions

$$\begin{aligned} &\mathbb{T}(v,p)n = \Big(-p_0I - \mu_1 \overset{1}{H}\otimes\overset{1}{H} + \mu_1\frac{\overset{1}{H}{}^2}{2}I\Big)n && \text{on } \tilde{S}^T\\ &\frac{1}{\sigma_1}\overset{1}{H} = \frac{1}{\sigma_2}\overset{2}{H} && \text{on } \tilde{S}^T,\\ &\overset{1}{E}\cdot\tau_\alpha = \overset{2}{E}\cdot\tau_\alpha, \quad \alpha=1,2 && \text{on } \tilde{S}^T,\\ &v\cdot n = -\frac{\phi_t}{|\nabla\phi|} && \text{on } \tilde{S}^T, \end{aligned} \tag{1.5}$$

where $\tilde{S}^T = \bigcup_{0\le t\le T} S_t \times \{t\}$, n is the unit outward vector to Ω_t and normal to S_t, τ_α, $\alpha = 1,2$ is the tangent vector to S_t, $\phi(x,t) = 0$ describes S_t at least locally.

Next we assume the boundary conditions on B

$$\begin{aligned} &\overset{2}{H} = H_* && \text{on } B,\\ &\overset{2}{E} = E_* && \text{on } B, \end{aligned} \tag{1.6}$$

where H_* and E_* are connected by conditions

$$\sigma_2 E_{*n} = \frac{1}{A_{\tau_1}A_{\tau_2}}(\partial_{\tau_1}(H_{*\tau_2}A_{\tau_2}) - \partial_{\tau_2}(H_{*\tau_1}A_{\tau_1})),$$

$$\mu_2\partial_t H_{*n} = \frac{1}{A_{\tau_1}A_{\tau_2}}(\partial_{\tau_2}(E_{*\tau_1}A_{\tau_1}) - \partial_{\tau_1}(E_{*\tau_2}A_{\tau_2})),$$

$$-\partial_t\partial_{\tau_1}(H_{*\tau_1}A_{\tau_2}A_n) - \partial_t\partial_{\tau_2}(H_{*\tau_2}A_{\tau_1}A_n) = \partial_{\tau_1}\partial_{\tau_2}(E_{*n}A_n)$$
$$-\mu_2\partial_{\tau_1}(A_{\tau_2}A_{\tau_3}\partial_t H_{*\tau_1}) - \mu_2\partial_{\tau_2}(A_{\tau_1}A_{\tau_3}\partial_t H_{*\tau_2}) - \partial_{\tau_2}\partial_{\tau_1}(E_{*n}A_n),$$

where (τ_1, τ_2, n) are curvilinear coordinates and $A_{\tau_1}, A_{\tau_2}, A_n$ are the Lamé coefficients of transformation $(\tau_1, \tau_2, n) \to (x_1, x_2, x_3)$.

Finally we assume the initial conditions

$$\Omega_{t|\,t=0} = \Omega, \qquad S_{t_{|t=0}} = S, \qquad D_{t_{|t=0}} = D, \tag{1.7}$$

$$\varrho_{|t=0} = \varrho_0, \qquad v_{|t=0} = v_0, \qquad \overset{1}{H}_{|t=0} = \overset{1}{H}_0, \qquad \text{in } \Omega,$$
$$\overset{2}{H}_{|t=0} = \overset{2}{H}_0, \qquad \text{in } D.$$

The aim of this paper is to prove the existence of a global-in-time solution of problem (1.1)–(1.7) which remains for all time close to a constant state. Consider the equation

$$p(\varrho) = p_0, \tag{1.8}$$

where $\varrho \in \mathbb{R}_+$, $p \in C^3(\mathbb{R}_+)$, $p'(\varrho) > 0$.

Then, we introduce the following definition of the constant state.

Definition 1.1. Let $f = 0$. Then by a *constant* (*equilibrium*) *state* we mean a solution (v, ϱ) of problem (1.1)–(1.7) such that $v = 0$, $\varrho = \varrho_e$, $\Omega_t = \Omega_e$ for $t \geq 0$, where ϱ_e is a solution of (1.8) and $|\Omega_e| = \frac{M}{\varrho_e}(|\Omega_e| = vol\Omega_e)$, where $M = \int_{\Omega_t}\varrho(x,t)dx = \int_\Omega \varrho_0(\xi)d\xi$.

The first global existence theorems for equations describing the motion of compressible fluids were proved by V. A. Solonnikov and A. Tani [10] and independently by W. Zajączkowski [20, 21]. Both [10] and [20, 21] are concerned with the barotropic case, but in [20, 21] it is assumed that the pressure of the fluid has the form $p = a\varrho^\gamma$, where $a > 0$ and $\gamma > 1$ are constants. A global existence result for the more general form of pressure, i.e. $p = p(\varrho)$, has been obtained in [16]. Moreover, global existence theorems for viscous compressible heat-conducting fluids can be found in [13, 14, 16].

To prove existence of solutions to the above problem we introduce the Lagrangian coordinates $\xi \in \Omega$. The Lagrangian coordinates connected with the velocity v are the initial data for the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \quad x_{|t=0} = \xi \in \Omega. \tag{1.9}$$

Therefore $x_v(\xi, t) = \xi + \int_0^t \bar{v}(\xi,\tau)d\tau$, where

$$\bar{v}(\xi,t) = v(x_v(\xi,t),t).$$

To introduce the Lagrangian coordinates in D_t we extend v on D_t. Let us denote the extended function by v'. Then we define $\xi \in D$, by the Cauchy data to the problem

$$\frac{dx}{dt} = v'(x,t), \qquad x_{|t=0} = \xi \in D. \tag{1.10}$$

Therefore $x_{v'}(\xi,t) = \xi + \int_0^t \bar{v}'(\xi,\tau)d\tau$, where $\bar{v}'(\xi,t) = v'(x_{v'}(\xi,t),t)$. Then by (1.5)

$$\Omega_t = \{x \in \mathbb{R}^3 : x = x_v(\xi,t),\ \xi \in \Omega\},$$
$$S_t = \{x \in \mathbb{R}^3 : x = x_v(\xi,t),\ \xi \in S\}.$$

Since S_t is determined at least locally by $\phi(x,t)=0$, S is described by $\phi(x_v(\xi,t),t)_{|t=0} = 0$. Moreover, we have

$$\bar n_v = n(x_v(\xi,t),t) = \frac{\nabla_x\phi(x,t)}{|\nabla_x\phi(x,t)|}\Big|_{x=x_v(\xi,t)}.$$

Introduce the following notation:

$$\begin{aligned}
&\|u\|_{l,Q} = \|u\|_{H^l(Q)}, && Q\in\{\Omega,S,D,\Pi,B\},\quad 0\le l\in\mathbb{Z},\\
&\|u\|_{k,p,q,Q^T} = \|u\|_{L_q(0,T,W_p^k(Q))}, && Q\in\{\Omega,S,D,\Pi,B\},\\
&p,q\in[1,+\infty],\quad 0\le k\in\mathbb{Z},
\end{aligned}$$

where $Q^t = Q\times(0,t)$,

$$|u|_{p,Q} = \|u\|_{L_p(Q)},\quad Q\in\{\Omega,S,D,\Pi,B\},\quad p\in[1,+\infty].$$

2. Weak solution. Weak solutions to problem (1.1)–(1.7) we formulate in Lagrangian coordinates.

DEFINITION 2.1. By *weak solutions* for problem (1.1)–(1.7) we mean functions $\bar v, \bar H$ which satisfy the integral identities

$$\begin{aligned}
&\int_0^T\int_\Omega(\bar\varrho\bar v_t\bar\varphi+\mathbb{D}_v(\bar v)\mathbb{D}_v(\bar\varphi))I_v d\xi dt-\int_0^T\int_\Omega(\mu_1\overset{1}{\bar H}\nabla_v\overset{1}{\bar H}\bar\varphi-\mu_1\nabla_v\overset{1}{\bar H}{}^2\bar\varphi)I_v d\xi dt\\
&=\int_0^T\int_\Omega \bar f\bar\varphi I_v d\xi dt+\int_0^T\int_S\Big(-\mu_1\overset{1}{\bar H}\otimes\overset{1}{\bar H}+\mu_1\frac{\overset{1}{\bar H}{}^2}{2}I\Big)\bar n_v\bar\varphi I_v d\xi_S dt I_v d\xi_s dt\\
&\quad+\int_0^T\int_\Omega\nabla_v\bar pI_v d\xi dt-\int_0^T\int_S(\bar p-p_0)\bar n_v\bar\varphi I_v d\xi_s dt,
\end{aligned}\tag{2.1}$$

$$\begin{aligned}
&\int_0^T\int_\Pi\Big(-\mu\bar H_t\bar\psi-\mu\bar v\nabla_v\bar H\bar\psi+\frac{1}{\sigma}\mathrm{rot}_v\bar H\mathrm{rot}_v\bar\psi\Big)I_v d\xi dt\\
&\qquad-\int_0^T\int_\Omega\mu_1(\bar v\times\overset{1}{\bar H})\mathrm{rot}_v\bar\psi I_v d\xi dt=\frac{1}{\sigma_2}\int_0^T\int_B(\bar n_v\times\bar E_*)\bar\psi I_v d\xi_B dt,
\end{aligned}\tag{2.2}$$

where φ, ψ are sufficiently regular, $\bar n_v$ is the unit outward vector normal to S or B. In (2.1), (2.2) we use the notation $\bar A(\xi,t) = A(x_v(\xi,t),t)$,

$$\begin{aligned}
&\bar H_{|\Omega} = \overset{1}{\bar H}, \bar H_{|D} = \overset{2}{\bar H}, \qquad \sigma_{|\Omega}=\sigma_1, \qquad \sigma_{|D}=\sigma_2,\\
&\Pi = \Omega\cup D, \qquad \mu_{|\Omega}=\mu_1, \qquad \mu_{|D}=\mu_2,
\end{aligned}$$

in (2.2) v is the extension on Π,

$$\begin{aligned}
&\mathbb{D}_v(\bar v) = \{\mu(\partial_{x_i}\xi_k\nabla_{\xi_k}\bar v_j+\partial_{x_j}\xi_k\nabla_{\xi_k}\bar v_i)+(\nu-\mu)\delta_{ij}\mathrm{div}_v\bar v\}_{i,j=1,2,3},\\
&\mathrm{rot}_v\bar v = \nabla_v\times\bar v,\\
&\qquad\nabla_v = \partial_x\xi_i\nabla_{\xi_i},\quad \mathrm{div}_v\bar v = \nabla_v\cdot\bar v = \partial_{x_i}\xi_k\nabla_{\xi_k}\bar v_i,\quad \partial_{\xi_i} = \nabla_{\xi_i}.
\end{aligned}$$

Let A be the Jacobi matrix of the transformation $x = x_v(\xi,t)$, then

$$\det A = \exp\Big(\int_0^t \mathrm{div}_v\bar v d\tau\Big) = I_v \quad\text{and if}\quad \sup_{\xi\in\Omega}\sup_{t\in[0,T]}|\nabla_\xi\bar v| < \mu$$

then $0 < c_1(1-\mu t)^3 \le \det\{\partial_\xi x\} \le c_2(1+\mu t)^3$, $t \in [0,T]$, where c_1, c_2 are constants and T is sufficiently small. Moreover $x_{i\xi_j} = \delta_{ij} + \int_0^t \partial_{\xi_j}\bar v_i(\xi,\tau)d\tau$ and $\xi_x = x_\xi^{-1}$. Then we get

$$\sup_{\xi\in\Omega}|x_\xi| \le 1 + \sup_{\xi\in\Omega}\int_0^t |\bar v(\xi,t)|dt$$
$$\le 1 + c\int_0^t \|\bar v\|_{3,\Omega}d\tau \le 1 + c\sqrt{t}\sqrt{\int_0^t \|\bar v\|_{3,\Omega}^2 d\tau} \le 1 + c\sqrt{t}\|\bar v\|_{3,2,2,\Omega^t}.$$

Then $\sup_{x\in\Omega_t}|\xi_x| \le \varphi(a)$, where $a = \sqrt{t}\|\bar v\|_{3,2,2,\Omega^t}$ and φ is an increasing positive function.

To prove the existence of a solution to the above problem we introduce Lagrangian coordinates connected with given divergence-free function u. Moreover we linearize the nonlinear terms with v in (1.1) writing them in the form $u\nabla v$ and $u \times \overset{1}{H}$. Then from (2.1), (2.2) we get

$$\int_0^T\int_\Omega(\bar\varrho\bar v_t\bar\varphi + \mathbb{D}_u(\bar v)\mathbb{D}_u(\bar\varphi))I_u d\xi dt - \int_0^T\int_\Omega(\mu_1 \overset{1}{\bar H}{}' \nabla_u \overset{1}{\bar H}{}'\bar\varphi - \mu_1\nabla_u \overset{1}{H}{}'^2\,\bar\varphi)I_u d\xi dt$$
$$= \int_0^T\int_\Omega \bar f\bar\varphi I_u d\xi dt + \int_0^T\int_S\Big(-\mu_1 \overset{1}{\bar H} \otimes \overset{1}{\bar H} + \mu_1\frac{\overset{1}{\bar H}{}^2}{2}I\Big)\bar n_u\bar\varphi I_u d\xi_S dt d\xi_s dt$$
$$+ \int_0^T\int_\Omega \nabla_u\bar p I_u d\xi dt - \int_0^T\int_S(\bar p - p_0)\bar n_u\bar\varphi I_u d\xi_S dt, \tag{2.3}$$

$$\int_0^T\int_\Pi(-\mu\bar H_t\bar\psi - \mu\bar u\nabla_u\bar H\bar\psi + \frac{1}{\sigma}\mathrm{rot}_u\bar H\mathrm{rot}_u\bar\psi)I_u d\xi dt +$$
$$-\int_0^T\int_\Omega \mu_1(\bar u \times \overset{1}{\bar H})\mathrm{rot}_u\bar\psi I_u d\xi dt = \frac{1}{\sigma_2}\int_0^T\int_B(\bar n_u \times \bar E_*)\bar\psi I_u d\xi_B dt, \tag{2.4}$$

where $\bar u$, $\overset{1}{\bar H}{}'$ are given functions and moreover $\bar\varrho$ is such that

$$0 < \varphi_* \le \bar\varrho \le \varphi^* < \infty. \tag{2.5}$$

Similarly as in [6] we prove

THEOREM 2.1. *Assume that* $\bar v_0 \in H^2(\Omega)$; $\bar v_t(0) \in H^1(\Omega)$; $\bar v_{tt}(0) \in L_2(\Omega)$; $\bar f_t \in L_2(0,T,H^1(\Omega))$; $\bar f_{tt} \in L_2(0,T,L_2(\Omega))$; $\bar f \in L_2(0,T,H^2(\Omega))$; $\bar H_0 \in H^2(\Pi)$; $\bar H_t(0) \in H^1(\Pi)$; $\bar H_{tt}(0) \in L_2(\Pi)$; $\bar E_* \in L_\infty(0,T,H^1(B))$; $\bar E_{*t}, \bar H_{*tt} \in L_2(0,T,L_2(B))$; $\bar H_{*t} \in L_2(0,T,H^2(B))$; $\bar H_* \in L_2(0,T,H^3(B))$, $S, B \in H^{5/2}$.

Then there exists $T^* > 0$ *such that for* $T \le T^*$ *there exists a solution to problem (1.1)–(1.7) such that* $\bar v \in L_2(0,T,H^3(\Omega)) \cap L_\infty(0,T,H^1(\Omega))$; $\bar v_t \in L_\infty(0,T,H^1(\Omega)) \cap L_2(0,T,H^2(\Omega))$; $\bar v_{tt} \in L_\infty(0,T,L_2(\Omega)) \cap L_2(0,T,H^1(\Omega))$; $\bar p_\sigma \in L_2(0,T,H^2(\Omega))$; $\bar p_t \in L_2(0,T,H^1(\Omega))$; $\bar p_{tt} \in L_2(0,T,L_2(\Omega))$; $\bar H \in L_2(0,T,H^3(\Pi)) \cap L_\infty(0,T,H^1(\Pi))$; $\bar H_t \in L_\infty(0,T,H^1(\Pi)) \cap L_2(0,T,H^2(\Pi))$; $\bar H_{tt} \in L_\infty(0,T,L_2(\Pi)) \cap L_2(0,T,H^1(\Pi))$. *Similarly as in [5] we can prove that* $(T^*)^\gamma(\varphi(0)+\beta) \le b$, $b > 0$ *sufficiently small,* $\gamma > 0$ *some constant and*

$$\beta = \|\bar{E}_*\|^2_{0,2,2,B^t} + \|\bar{E}_{*t}\|^2_{0,2,2,B^t} + \|\bar{H}_*\|^2_{3,2,2,B^t} + \|\bar{H}_{*t}\|^2_{2,2,2,B^t} + \|\bar{H}_{*tt}\|^2_{0,2,2,B^t} + \|\bar{f}_t\|^2_{0,2,2,\Omega^t} + \|\bar{f}\|^2_{1,2,2,\Omega^t}, \tag{2.6}$$

$$\varphi(0) = \sum_{i+k\le 2} (\|\partial_t^i \bar{v}(0)\|^2_{k,\Omega} + \|\partial_t^i \bar{H}(0)\|^2_{k,\Pi}). \tag{2.7}$$

Moreover if $\varphi(0)$, β are sufficiently small then we get

$$\begin{aligned}&\|\bar{v}_t\|^2_{1,2,\infty,\Omega^T} + \|\bar{v}\|^2_{1,2,\infty,\Omega^T} + \|\bar{v}\|^2_{3,2,2,\Omega^T} + \|\bar{v}_t\|^2_{2,2,2,\Omega^T} + \|\bar{v}_{tt}\|^2_{1,2,2,\Omega^T}\\ &+ \|\bar{p}_\sigma\|^2_{2,2,2,\Omega^T} + \|\bar{p}_{tt}\|^2_{0,2,2,\Omega^T} + \|\bar{p}_t\|^2_{1,2,2,\Omega^T} + \|\bar{H}_t\|^2_{1,2,\infty,\Pi^T} + \|\bar{H}\|^2_{1,2,\infty,\Pi^T}\\ &+ \|\bar{H}\|^2_{3,2,2,\Pi^T} + \|\bar{v}_{tt}\|^2_{0,2,\infty,\Omega^T} + \|\bar{H}_t\|^2_{2,2,2,\Pi^T} + \|\bar{H}_{tt}\|^2_{1,2,2,\Pi^T} + \|\bar{H}_{tt}\|^2_{0,2,\infty,\Pi^T}\\ &\le c(\varphi(0) + \beta).\end{aligned} \tag{2.8}$$

First, in Section 3, we derive differential inequality (3.62) which makes possible an extension of the local solution of (1.1)–(1.7) step by step from interval $[0,T]$ to $[0,\infty)$. In Section 4 we show the Main Theorem.

MAIN THEOREM. *Assume that $f = \int_\Omega v_0 dx = \int_\Omega \varrho_0 v_0 \cdot \varphi_i dx = 0$, $i = 1,2,3$, where φ_i, $i = 1,2,3$ are defined in Lemma 5.1, $H_* \in H^3(B)$, $H_{*t} \in H^2(B)$, $H_{*tt} \in H^1(B)$, $S_t, B \in H^{\frac{5}{2}}$, $(v(0), \varrho(0), H(0)) \in \mathcal{N}(0)$, $\varphi(0) \le \varepsilon_1$ where ε_1 is sufficiently small. Assume also that $\alpha(t) \le e^{-\mu t}$, where μ is sufficiently large and $\alpha(t)$ is defined in Lemma 4.2. Then there exists a global solution of (1.1)–(1.7) such that $(v(t), \varrho_\sigma(t), H(t)) \in \mathcal{M}(t)$, $t \in \mathbb{R}_+$, where $\mathcal{N}(0)$ and $\mathcal{M}(t)$ are defined in Section 4.*

3. Differential inequality. In this section we obtain a special differential inequality which enables us to prove the global existence. In order to show the differential inequality we consider the motion near the constant state. Let

$$\begin{aligned}&p_\sigma = p - p_0, \quad \varrho_\sigma = \varrho - \varrho_e, \quad \overset{2}{H}_{|B} = 0, \quad f = 0,\\ &\int_\Omega \varrho(0) v_0 dx = 0, \quad \int_\Omega \varrho(0) v_0 \cdot \varphi_i dx = 0, \quad i = 1,2,3,\end{aligned} \tag{3.1}$$

where ϱ_e is introduced in Definition 1.1 and φ_i, $i = 1,2,3$ are defined in Lemma 5.1.

REMARK 3.1. Integrating $(1.1)_1$ over Ω_t we get

$$\frac{d}{dt}\int_{\Omega_t} \varrho v dx - \int_{\Omega_t} \operatorname{div}\mathbb{T}(v, p_\sigma) dx + \mu_1 \int_{\Omega_t} \left(-\operatorname{div}(\overset{1}{H} \otimes \overset{1}{H}) + \nabla \frac{\overset{1}{H}{}^2}{2} \right) dx = \int_{\Omega_t} f dx.$$

Then from (3.1) we get

$$\frac{d}{dt}\int_{\Omega_t} \varrho v dx + \int_{S_t} \left(\mu_1 \overset{1}{H} \otimes \overset{1}{H} - \mu_1 \frac{\overset{1}{H}{}^2}{2} I \right) n dx_{S_t} + \mu_1 \int_{\Omega_t} \left(-\operatorname{div}(\overset{1}{H} \otimes \overset{1}{H}) + \nabla \frac{\overset{1}{H}{}^2}{2} \right) dx = 0.$$

Integrating the last equality by parts we get

$$\int_{\Omega_t} \varrho v dx = \int_\Omega \varrho(0) v_0 dx = 0.$$

Remark 3.2. Let φ_i, $i = 1,2,3$ are defined in Lemma 4.1. Multiplying $(1.1)_1$ by φ_i, $i = 1,2,3$ and integrating over Ω_t we get

$$\frac{d}{dt}\int_{\Omega_t}\varrho v\cdot\varphi_i dx - \int_{\Omega_t}\operatorname{div}(\mathbb{T}(v,p_\sigma)\varphi_i)dx + \mu_1\int_{\Omega_t}\Big(-\operatorname{div}(\overset{1}{H}\otimes\overset{1}{H}\varphi_i) + \nabla\frac{\overset{1}{H}{}^2}{2}\varphi_i\Big)dx = \int_{\Omega_t} f dx.$$

Then from (3.1) we get

$$\frac{d}{dt}\int_{\Omega_t}\varrho v\cdot\varphi_i dx + \int_{S_t}\Big(\mu_1\overset{1}{H}\otimes\overset{1}{H} - \mu_1\frac{\overset{1}{H}{}^2}{2}\Big)\varphi_i\cdot n dx_{S_t} + \mu_1\int_{\Omega_t}\Big(-\operatorname{div}(\overset{1}{H}\otimes\overset{1}{H})\cdot\varphi_i + \nabla\frac{\overset{1}{H}{}^2}{2}\cdot\varphi_i\Big)dx = 0.$$

Integrating last equality by parts we get

$$\int_{\Omega_t}\varrho v\cdot\varphi_i dx = \int_\Omega \varrho(0)v_0\cdot\varphi_i dx = 0, \quad i = 1,2,3.$$

Then we get the problem

$$\begin{aligned}
&\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p_\sigma) - \mu_1\overset{1}{H}\cdot\nabla\overset{1}{H} + \mu_1\nabla\frac{\overset{1}{H}{}^2}{2} = 0 && \text{in } \Omega_t,\quad t\in[0,T],\\
&\varrho_{\sigma t} + \operatorname{div}(\varrho v) = 0 && \text{in } \Omega_t,\quad t\in[0,T],\\
&\mathbb{T}(v,p_\sigma)\bar n = \Big(-\mu_1\overset{1}{H}\otimes\overset{1}{H} + \mu_1\frac{\overset{1}{H}{}^2}{2}I\Big)n && \text{on } S_t,\quad t\in[0,T],\\
&\varrho_\sigma\big|_{t=0} = \varrho_{\sigma 0} = \varrho_0 - \varrho_e,\quad v\big|_{t=0} = v_0, && \text{in } \Omega.
\end{aligned}\tag{3.2}$$

In the sequel we shall use the following Taylor formula for p_σ

$$p_\sigma = (\varrho - \varrho_e)\int_0^1 p'(\varrho_e + s(\varrho - \varrho_e))ds = p_1\varrho_\sigma,\tag{3.3}$$

where the function p_1 is positive.

Lemma 3.1. *Let v, ϱ_σ be a sufficiently smooth solution of (3.2). Then*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_T}\Big(\varrho v^2 + \frac{p_1}{\varrho}\varrho_\sigma^2\Big)dx + c_0\|v\|^2_{1,\Omega_t} \le cX_1^2(1+X_1),\tag{3.4}$$

where $X_1 = \|v\|^2_{2,\Omega_t} + \|\varrho_\sigma\|^2_{2,\Omega_t} + \|\overset{1}{H}\|^2_{1,\Omega_t}$.

Proof. Multiplying $(3.2)_1$ by v, integrating over Ω_t and using continuity equation $(3.2)_2$, boundary condition $(3.2)_3$ and (3.3) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\varrho v^2 dx + \frac{\mu}{2}\mathbb{E}_{\Omega_t}(v) + (\nu-\mu)\|\operatorname{div}v\|^2_{0,\Omega_t} - \int_{\Omega_t}p_1\varrho_\sigma\operatorname{div}v dx\\
&-\mu_1\int_{\Omega_t}\overset{1}{H}\cdot\nabla\overset{1}{H}v dx + \mu_1\int_{\Omega_t}\nabla\frac{\overset{1}{H}{}^2}{2}v dx - \mu_1\int_{S_t}\Big(-\overset{1}{H}\otimes\overset{1}{H} + \frac{\overset{1}{H}{}^2}{2}I\Big)dx_S = 0,
\end{aligned}\tag{3.5}$$

where $\mathbb{E}_{\Omega_t}(v) = \int_{\Omega_t}\sum_{i,j=1}^3(v_{ix_j} + v_{jx_i})^2 dx$.

In [21] it is proved that

$$\frac{\mu}{2}\mathbb{E}_{\Omega_t}(v)+(\nu-\mu)\|\mathrm{div}v\|^2_{0,\Omega_t}\geq\left(\nu-\frac{1}{3}\mu\right)\mathbb{E}_{\Omega_t}(v)\quad\text{for}\quad\nu\geq\frac{1}{3}\mu. \tag{$*$}$$

Next, by the continuity equation $(3.2)_2$ we have

$$-\int_{\Omega_t}p_1\varrho_\sigma\mathrm{div}v dx=\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\frac{p_1\varrho_\sigma^2}{\varrho}dx+J, \tag{3.6}$$

where

$$|J|\leq\varepsilon(\|\varrho_{\sigma t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t})+cX_1^2(1+X_1). \tag{3.7}$$

By continuity equation $(3.2)_2$

$$\|\varrho_{\sigma t}\|^2_{0,\Omega_t}\leq c\|v\|^2_{1,\Omega_t}+c\|v\|^2_{1,\Omega_t}\|\varrho_\sigma\|^2_{2,\Omega_t}, \tag{3.8}$$

and taking into account (3.5)–(3.8) we get estimate (3.4). ■

Next, we have

Lemma 3.2. *Let v, ϱ_σ be a sufficiently smooth solution of (3.2). Then*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho v_t^2+\frac{p_{\varrho\sigma}}{\varrho}\varrho_{\sigma t}^2\right)dx+c_0(\|v_t\|^2_{1,\Omega_t}+\|\varrho_{\sigma t}\|^2_{0,\Omega_t})\leq c\|v\|^2_{1,\Omega_t}+cY_1^2(1+X_2), \tag{3.9}$$

where

$$X_2=\sum_{i+k\leq 2}(\|\partial_t^i v\|^2_{k,\Omega_t}+\|\partial_t^i\varrho_\sigma\|^2_{k,\Omega_t})+\int_0^t\|v\|^2_{3,\Omega_\tau}d\tau+\|\overset{1}{H}_t\|^2_{1,\Omega_t}+\|\overset{1}{H}\|^2_{1,\Omega_t}+\|v\|^2_{1,\Omega_t}, \tag{3.10}$$

$$Y_1=X_2-\int_0^t\|v\|^2_{3,\Omega_\tau}d\tau. \tag{3.11}$$

Proof. Differentiating $(3.2)_1$ with respect to t, multiplying by v_t and integrating over Ω_t yields

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\varrho v_t^2dx+\frac{\mu}{2}E_{\Omega_t}(v_t)+(\nu-\mu)\|\mathrm{div}v_t\|^2_{0,\Omega_t}-\int_{\Omega_t}p_{\sigma p}\varrho_{\sigma t}\mathrm{div}v_tdx\leq cY_1^2(1+X_2), \tag{3.12}$$

where we have used boundary condition $(3.2)_3$.

By Lemma 5.2 and $(*)$,

$$\|v_t\|^2_{1,\Omega_t}\leq c[\mathbb{E}_{\Omega_t}(v_t)+Y_1^2(1+Y_1)]. \tag{3.13}$$

Finally, using continuity equation $(3.2)_2$ we get

$$-\int_{\Omega_t}p_{\sigma\varrho}\varrho_{\sigma t}\mathrm{div}v_tdx=\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma t}^2dx+J, \tag{3.14}$$

where

$$|J|\leq\varepsilon(\|v_t\|^2_{1,\Omega_t}+\|\varrho_{\sigma t}\|^2_{0,\Omega_t})+cY_1^2(1+Y_1). \tag{3.15}$$

In view of inequalities (3.12)–(3.15) and (3.8) we obtain (3.9). ■

Lemmas 3.1 and 3.2 yield

LEMMA 3.3. *Let v, ϱ_σ be a sufficiently smooth solution of (3.2). Then*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\bigg[\varrho(v^2+v_t^2)+\frac{p_1}{\varrho}\varrho_\sigma^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma t}^2\bigg]dx+c_0(\|v\|_{1,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2+\|\varrho_{\sigma t}\|_{0,\Omega_t}^2)$$
$$\le cY_1^2(1+X_2), \tag{3.16}$$

where X_2 and Y_1 are given by (3.10) and (3.11), respectively.

Next, we obtain

LEMMA 3.4. *Let v, ϱ_σ be a sufficiently smooth solution of (3.2). Then*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\bigg(\varrho v_{tt}^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma tt}^2\bigg)dx+c_0(\|v_{tt}\|_{1,\Omega_t}^2+\|\varrho_{\sigma tt}\|_{0,\Omega_t}^2)$$
$$\le c(\|v\|_{1,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2)+cX_2Y_2(1+X_2^2), \tag{3.17}$$

where X_2 is given by (3.10) and

$$Y_2=\sum_{\substack{i+k\le 3\\ i\le 2}}\|\partial_t^i v\|_{k,\Omega_t}^2+\|\varrho_\sigma\|_{2,\Omega_t}^2+\|\varrho_{\sigma t}\|_{2,\Omega_t}^2+\|\varrho_{\sigma tt}\|_{1,\Omega_t}^2+\|\overset{1}{H}_{tt}\|_{1,\Omega_t}^2+\|\overset{1}{H}\|_{2,\Omega_t}^2$$
$$+\|\overset{1}{H}_t\|_{1,\Omega_t}^2. \tag{3.18}$$

The above lemma can be proved in the same way as Lemmas 3.1 and 3.2. To estimate $\mathbb{E}_{\Omega_t}(v_{tt})$ we use here Lemma 5.3. ■

In order to obtain estimates for derivatives with respect to x we rewrite problem (3.2) in the Lagrangian coordinates. We have

$$\begin{aligned}
&\bar\varrho\bar v_t-\nabla_v\mathbb{T}_v(\bar v,p_\sigma)-\mu_1\overset{1}{\bar H}\cdot\nabla_v\overset{1}{\bar H}+\mu_1\nabla_v\frac{\overset{1}{\bar H}{}^2}{2}=0 && \text{in } \Omega^T\equiv\Omega\times(0,T),\\
&\bar\varrho+\bar\varrho\nabla_v\cdot\bar v=0 && \text{in } \Omega^T,\\
&\mathbb{T}_v(\bar v,p_\sigma)\bar n_v=\bigg(-\mu_1\,\overset{1}{\bar H}\otimes\overset{1}{\bar H}+\mu_1\frac{\overset{1}{\bar H}{}^2}{2}\bigg)\bar n_v && \text{on } S^T=S\times(0,T),\\
&\bar v\big|_{t=0}=v_0,\quad \bar\varrho\big|_{t=0}=\varrho_{\sigma 0} && \text{in } \Omega.
\end{aligned} \tag{3.19}$$

Now, introduce a partition of unity $(\{\bar{\Omega}_i\},\{\zeta_i\})$, $\Omega=\bigcup_i\tilde\Omega_i$. Let $\tilde\Omega$ be one of the $\tilde\Omega_i$, s and $\zeta(\xi)=\zeta_i(\xi)$ be the corresponding function. If $\tilde\Omega$ is an interior subdomain then let $\tilde\omega$ be a set such that $\tilde\omega\subset\tilde\Omega$ and $\zeta(\xi)=1$ for $\xi\in\tilde\omega$. Otherwise, we assume that $\bar{\tilde\Omega}\cap S\neq\emptyset$, $\bar{\tilde\omega}\cap S\neq\emptyset$, $\bar{\tilde\omega}\subset\bar{\tilde\Omega}$. Take any $\beta\in\bar{\tilde\omega}\cap S=\bar{\tilde S}$ and introduce local coordinates $\{y\}$ associated with $\{\xi\}$ by

$$y_k=\alpha_{kl}(\xi_l-\beta_l),\quad \alpha_{3k}=n_k(\beta),\quad k=1,2,3, \tag{3.20}$$

where $\{\alpha_{kl}\}$ is a constant orthogonal matrix such that $\tilde S$ is determined by the equation $y_3=F(y_1,y_2)$, $F\in H^{\frac{5}{2}}$ and

$$\tilde\Omega=\{y:|y_i|<d,\ i=1,2,\ F(y')<y_3<F(y')+d,\ y'=(y_1,y_2)\}.$$

Next, we introduce $\bar v'$, $\bar\varrho'$, $\bar\varrho_\sigma$ by

$$\bar v_i'(y) = \alpha_{ij}\bar v_j(\xi)\big|_{\xi=\xi(y)}, \quad i=1,2,3,$$
$$\bar\varrho'(y) = \bar\varrho(\xi)\big|_{\xi=\xi(y)}, \qquad \bar\varrho_\sigma'(y) = \bar\varrho'(y) - \varrho_e,$$

where $\xi = \xi(y)$ is the inverse transformation to (3.20).

Next, we introduce new variables by

$$z_i = y_i, \quad i=1,2, \quad z_3 = y_3 - \tilde F(y), \quad y \in \tilde\Omega,$$

which will be denoted by $z = \Phi(y)$ (where $\tilde F \in H^3$ is an extension of F). Let

$$\hat\Omega = \Phi(\tilde\Omega) = \{z : |z_i| < d,\ i=1,2,\ 0 < z_3 < d\} \quad \text{and} \quad \hat S = \Phi(\tilde S). \tag{3.21}$$

Define

$$\hat v(z) = \bar v'(y)\big|_{y=\Phi^{-1}(z)}, \quad \hat\varrho(z) = \bar\varrho'(y)\big|_{y=\Phi^{-1}(z)}, \quad \hat\varrho_\sigma(z) = \hat\varrho(z) - \varrho_e.$$

Set $\hat\nabla_k = \xi_{lx_k} z_{i\xi_l}\nabla_{z_i}\big|_{\xi=\chi^{-1}(z)}$, where $\chi(\xi) = \Phi(\psi(\xi))$ and $y = \psi(\xi)$ is described by (3.20). We also introduce the following notation:

$$\tilde v(\xi) = \bar v(\xi)\zeta(\xi), \quad \tilde\varrho(\xi) = \bar\varrho(\xi)\zeta(\xi), \quad \tilde\varrho_\sigma(\xi) = \bar\varrho_\sigma(\xi)\zeta(\xi),$$

for $\xi \in \tilde\Omega, \tilde\Omega \cap S = \emptyset$ and

$$\tilde v(z) = \hat v(z)\hat\zeta(z), \quad \tilde\varrho(z) = \hat\varrho(z)\hat\zeta(z), \quad \hat\varrho_\sigma(z) = \hat\varrho_\sigma(z)\hat\zeta(z),$$

for $z \in \hat\Omega = \Phi(\tilde\Omega)$, $\bar{\tilde\Omega} \cap S \neq \emptyset$, where $\hat\zeta(z) = \zeta(\xi)\big|_{\xi=\chi^{-1}(z)}$.

Using the above notation we rewrite problem (3.19) in the following form in an interior subdomain

$$\begin{aligned}
&\bar\varrho\tilde v_{it} - \nabla_{v_j}T_{vij}(\tilde v,\tilde p_\sigma) = -\nabla_{v_j}B_{vij}(\bar v,\zeta) - T_{vij}(\bar v,p_\sigma)\nabla_{v_j}\zeta + \mu_1(\overset{1}{\tilde{\bar H}}\nabla_v\overset{1}{\bar H}_i - (\nabla_{v_i}\overset{1}{\bar H})\overset{1}{\tilde H})\\
&\qquad \equiv k_1, \quad i=1,2,3,\\
&\tilde\varrho_{\sigma t} + \bar\varrho\nabla_v\cdot\tilde v = \bar\varrho\bar v\cdot\nabla_v\zeta \equiv k_2,
\end{aligned}$$

where $\tilde p_\sigma = p_\sigma\zeta$, $\mathbb{B}_v(\bar v,\zeta) = \{B_v^{ij}(\bar v,\zeta)\}_{i,j=1,2,3} = \{\mu(\bar v_i\nabla_v\zeta + \bar v_j\nabla_v\zeta) + (\nu-\mu)\delta_{ij}\bar v\nabla_v\zeta\}_{i,j=1,2,3}$, $\mathbb{T}_v(\bar v,p_\sigma) = \mathbb{D}_v(\bar v) - I\ p_\sigma = \{T_{vij}(\bar v,p_\sigma)\}_{i,j=1,2,3}$ and $\nabla_{v_j} = \xi_{kx_j}\partial_{\xi_k}$.

In boundary subdomains we have

$$\begin{aligned}
&\hat\varrho\tilde v_{it} - \hat\nabla_j\hat T_{ij}(\tilde v,\tilde p_\sigma) = \hat\nabla_j\hat B_{ij}(\hat v,\hat\zeta) - T_{vij}(\hat v,\hat p_\sigma)\hat\nabla_j\hat\zeta + \mu_1(\overset{1}{\tilde H}\hat\nabla\overset{1}{\hat H}_i - (\hat\nabla_i\overset{1}{\hat H})\overset{1}{\tilde H})\\
&\qquad \equiv k_{3i}, \qquad i=1,2,3,,\\
&\tilde\varrho_{\sigma t} + \hat\varrho\hat\nabla\cdot\tilde v = \hat\eta\hat v\cdot\hat\nabla\hat\zeta \equiv k_4,\\
&\hat{\mathbb{T}}(\tilde v,\tilde p_\sigma)\hat n = k_5,
\end{aligned} \tag{3.22}$$

where $k_5 = \hat B_{ij}(\hat v,\hat\zeta)\hat n + \bigg(-\mu_1\overset{1}{\hat H}\otimes\overset{1}{\hat H} + \mu_1\dfrac{\overset{1}{\hat H}{}^2}{2}I\bigg)\hat n$, $\hat\nabla = (\hat\nabla_j)_{j=1,2,3}$ and $\hat{\mathbb{T}}$ and $\hat{\mathbb{B}}$ indicate that the operator ∇_v is replaced by $\hat\nabla$.

In Lemmas 3.5–3.7 below we denote z_1, z_2 by τ, i.e. $\tau = (z_1,z_2)$ and z_3 by n.

LEMMA 3.5. *Let v, ϱ_σ be a sufficiently smooth solution of (3.2). Then*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho v_x^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma x}^2\right)dx+c_0(\|v\|_{2,\Omega_t}^2+\|\varrho_{\sigma x}\|_{0,\Omega_t}^2)\\ \le c(\|v\|_{1,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2+\|\varrho_{\sigma t}\|_{0,\Omega_t}^2+\|p_\sigma\|_{0,\Omega_t}^2)+cX_2^2(1+X_2),\end{aligned}\tag{3.23}$$

where X_2 is given by (3.10), $v_x^2=\sum_{i,j=1}^3 v_{ix_j}^2$, $\varrho_{\sigma x}^2=\sum_{i=1}^3\varrho_{\sigma x_i}^2$.

Proof. First, we consider the following elliptic problem

$$\begin{aligned}&\mu\nabla_v^2\bar v+\nu\nabla_v\nabla_v\cdot\bar v-p_{\sigma\varrho}\nabla_v\bar\varrho=\bar\varrho\bar v_t+\mu_1\bar{\overset{1}{H}}\cdot\nabla_v\bar{\overset{1}{H}}-\mu_1\nabla_v\frac{\bar{\overset{1}{H}}{}^2}{2} &&\text{in } \Omega,\\ &\mathrm{div}_v\bar v=\mathrm{div}_v\bar v &&\text{in } \Omega,\\ &\mathbb{T}_v(\bar v,p_\sigma)\bar n_v=\left(-\mu_1\bar{\overset{1}{H}}\otimes\bar{\overset{1}{H}}+\mu_1\frac{\bar{\overset{1}{H}}{}^2}{2}I\right)\bar n_v &&\text{on } S.\end{aligned}\tag{3.24}$$

Since the complementary condition to (3.24) is satisfied we can apply to problem (3.24) the Agmon-Douglis-Nirenberg theory (see [1]). Thus, we get

$$\begin{aligned}\|\bar v\|_{2,\Omega}^2+\|\bar\varrho_\sigma\|_{1,\Omega}^2&\le c(\|\bar\varrho\bar v_t\|_{0,\Omega}^2+\|\mathrm{div}_v\bar v\|_{1,\Omega}^2+\|\bar{\overset{1}{H}}\|_{1,\Omega}^2)\\ &\le c(\|\bar v_t\|_{0,\Omega}^2+\|\mathrm{div}\bar v\|_{1,\Omega}^2+cX_2^2(\Omega)(1+X_2(\Omega)))+\|\bar v\|_{0,\Omega}^2,\end{aligned}\tag{3.25}$$

where we have used that $\|\mathrm{div}_u\bar v-\mathrm{div}\bar v\|_{1,\Omega}^2\le\varepsilon\|\bar v\|_{2,\Omega}^2$, ($\varepsilon>0$ is sufficiently small) and

$$X_2(\Omega)=\sum_{i+k\le 2}(\|\partial_t^i\bar v\|_{k,\Omega}^2+\|\partial_t^i\bar\varrho_\sigma\|_{k,\Omega}^2)+\int_0^t\|\bar v\|_{3,\Omega}^2d\tau+\|\bar{\overset{1}{H}}\|_{2,\Omega}^2.\tag{3.26}$$

In view of (3.25) we see that in order to obtain inequality (3.23) it remains to get appropriate estimates for $\|\mathrm{div}\bar v\|_{1,\Omega}^2$ and for $\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho v_x^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma x}^2\right)dx$. To do this we first consider boundary subdomains. Differentiate $(3.24)_1$ with respect to τ, multiply the result by $\tilde v_\tau J$ (J is the Jacobian of the transformation $x=x(z)$) and integrate over $\hat\Omega$. Hence using the Korn inequality and equation $(3.22)_2$ we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\hat\Omega}\hat\varrho\tilde v_\tau^2Jdz+c_0\|\tilde v_\tau\|_{1,\hat\Omega}^2-\int_{\hat S}(\hat{\mathbb{T}}(\tilde v,\tilde p_\sigma)\hat n)_\tau\tilde v_\tau Jdz\\ &-\int_{\hat\Omega}\tilde p_{\sigma\tau}\nabla\cdot\tilde v_\tau Jdz\le\varepsilon(\|\hat\varrho_\sigma\|_{0,\hat\Omega}^2+\|\tilde v_\tau\|_{1,\hat\Omega}^2)\\ &+c(\|\hat v\|_{1,\hat\Omega}^2+\|p_\sigma\|_{0,\hat\Omega}^2)+cX_2^2(\hat\Omega)(1+X_2(\hat\Omega)),\end{aligned}\tag{3.27}$$

where

$$\begin{aligned}&X_2(\hat\Omega)=\sum_{i+k\le 2}(\|\partial_t^i\hat v\|_{k,\hat\Omega}^2+\|\partial_t^i\hat\varrho_\sigma\|_{k,\hat\Omega}^2)+\int_0^t\|\hat v\|_{3,\hat\Omega}^2d\tau+\|\hat{\overset{1}{H}}\|_{2,\hat\Omega}^2,\\ &\tilde v_\tau^2=\sum_{i=1}^3\sum_{j=1}^2\tilde v_{iz_j}^2.\end{aligned}\tag{3.28}$$

Using boundary condition $(3.24)_3$ we have

$$\begin{aligned}
-\int_{\hat S}(\hat{\mathbb{T}}(\tilde v,\tilde p_\sigma)\hat n)_\tau \tilde v_\tau J d\tau &= -\int_{\hat S}(\hat B_{ij}(\tilde v,\hat\zeta)\hat n_j)_\tau \tilde v_\tau J d\tau\\
&= \int_{\hat S}\partial_\tau^{1/2}(\hat B_{ij}(\hat v,\hat\zeta)\hat n_j)\partial_\tau^{1/2}(\tilde v_{i\tau}J)d\tau\\
&\le \varepsilon\|\tilde v_\tau\|^2_{1,\hat\Omega}+\|\hat v\|^2_{1,\hat\Omega}+cX_2^2(\hat\Omega),
\end{aligned}\tag{3.29}$$

where to use derivative $\partial_\tau^{1/2}$ we have to apply the Fourier transform.

Next,

$$-\int_{\hat\Omega}\tilde p_{\sigma\tau}\nabla_v\cdot\tilde v_\tau J dz = -\int_{\hat\Omega}p_{\sigma\hat\varrho}\tilde\varrho_{\sigma\tau}\hat\nabla\cdot\tilde v_\tau J dz + J_1,\tag{3.30}$$

where $|J_1|\le\varepsilon\|\tilde v_\tau\|^2_{1,\hat\Omega}+c\|p_\sigma\|^2_{0,\hat\Omega}$ and

$$-\int_{\hat\Omega}p_{\sigma\hat\varrho}\hat\varrho_{\sigma\tau}\hat\nabla\cdot\tilde v_\tau J dz = \frac12\frac{d}{dt}\int_{\hat\Omega}\frac{p_{\sigma\hat\varrho}}{\hat\varrho}\tilde\varrho^2_{\sigma\tau}J dz + J_2,\tag{3.31}$$

where

$$|J_2|\le\varepsilon\|\tilde\varrho_{\sigma\tau}\|^2_{0,\hat\Omega}+c\|\hat v\|^2_{1,\hat\Omega}+cX_2^2(\hat\Omega).\tag{3.32}$$

Taking into account (3.27), (3.29)–(3.32) and assuming that ε is sufficiently small we obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\int_{\hat\Omega}\left(\hat\varrho\tilde v_\tau+\frac{p_{\sigma\hat\varrho}}{\hat\varrho}\tilde\varrho^2_{\sigma\tau}\right)J dz + c_0\|\tilde v_\tau\|^2_{1,\hat\Omega}\\
&\quad\le\varepsilon\|\hat\varrho_{\sigma\tau}\|^2_{0,\hat\Omega}+c(\|\hat v\|^2_{1,\hat\Omega}+\|p_\sigma\|^2_{0,\hat\Omega})+cX_2^2(\hat\Omega)(1+X_2(\hat\Omega)).
\end{aligned}\tag{3.33}$$

Now, applying the operator $(\mu+\nu)\nabla_{z_i}$ to $(3.22)_2$, dividing the result by $\hat\varrho$, adding to $(3.22)_1$ and multiplying both sides of the result by $p_{\sigma\hat\varrho}$ gives

$$\begin{aligned}
&\frac{(\mu+\nu)}{\hat\varrho}p_{\sigma\hat\varrho}\nabla_{z_i}\hat\varrho_{\sigma_t}+p^2_{\sigma\hat\varrho}\nabla_{z_i}\hat\varrho_\sigma = p^2_{\sigma\hat\varrho}\hat\varrho_\sigma\nabla_{z_i}\hat\zeta - p_1 p_{\sigma\hat\varrho}\hat\varrho_\sigma\nabla_{z_i}\hat\zeta\\
&\quad+p_{\sigma\hat\varrho}k_{3i}+\mu p_{\sigma\hat\varrho}(\hat\nabla^2\tilde v_i-\hat\nabla_i\hat\nabla\cdot\tilde v)\\
&\quad+(\mu+\nu)p_{\sigma\hat\eta}(\hat\nabla_i-\nabla_{z_i})\hat\nabla\cdot\tilde v+\frac{(\mu+\nu)}{\hat\varrho}p_{\sigma\hat\varrho}\nabla_{z_i}(\hat\varrho\hat v\cdot\hat\nabla\hat\zeta)\\
&\quad-p_{\sigma\hat\varrho}\hat\varrho\tilde v_{it}-\frac{(\mu-\nu)}{\hat\varrho}p_{\sigma\hat\varrho}\nabla_{z_i}\hat\varrho\hat\nabla\cdot\tilde v,\quad i=1,2,3.
\end{aligned}\tag{3.34}$$

Multiplying the normal component of (3.34) by $\tilde\varrho_{\sigma n}J$, integrating over $\hat\Omega$ we obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\int_{\hat\Omega}\frac{p_{\sigma\hat\varrho}}{\hat\varrho}\tilde\varrho^2_{\sigma n}J dz + c_0\|\tilde\varrho_{\sigma n}\|^2_{0,\hat\Omega}\le(\varepsilon+cd)\|\tilde v_{nn}\|^2_{0,\hat\Omega}\\
&+\varepsilon\|\tilde\varrho_{\sigma n}\|^2_{0,\hat\Omega}+c(\|\tilde v_{z\tau}\|^2_{0,\hat\Omega}+\|\hat v\|^2_{1,\hat\Omega}+\|\tilde v_t\|^2_{0,\hat\Omega}+\|p_\sigma\|^2_{0,\hat\Omega})+cX_2^2(\hat\Omega)(1+X_2(\hat\Omega)),
\end{aligned}\tag{3.35}$$

where d is from formula (3.21).

Now, we write $(3.22)_1$ in the form

$$\hat\varrho\tilde v_{it}-\mu\Delta\tilde v_i-\nu\nabla_{z_i}\nabla\cdot\tilde v=\hat\nabla_i\tilde p_\sigma+k_{3i}-k_{6i},\tag{3.36}$$

where $k_{6i}=(\mu\Delta\tilde v_i+\nu\nabla_{z_i}\nabla\cdot\tilde v)-(\mu\hat\nabla^2\tilde v_i+\nu\hat\nabla_i\hat\nabla\cdot\tilde v)$.

Multiplying the third component of (3.36) by $\tilde{u}_{3nn}J$ and integrating over $\hat{\Omega}$ yields

$$\frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}}\hat{\varrho}\tilde{v}_{3n}Jdz + c_0\|\tilde{v}_{3nn}\|^2_{0,\hat{\Omega}} \le (\varepsilon + cd)\|\tilde{v}_{nn}\|^2_{0,\hat{\Omega}} + c(\|\tilde{v}_{z\tau}\|^2_{0,\hat{\Omega}} + \|\hat{v}\|^2_{1,\hat{\Omega}} + \|\tilde{v}_t\|^2_{1,\hat{\Omega}} + \|\tilde{\varrho}_{\sigma n}\|^2_{0,\hat{\Omega}} + \|p_\sigma\|^2_{0,\hat{\Omega}}) + cX_2^2(\hat{\Omega})(1 + X_2(\hat{\Omega})). \tag{3.37}$$

For an interior subdomain the following estimate is obtained in the same way as (3.33)

$$\frac{1}{2}\frac{d}{dt}\int_{\tilde{\Omega}}\left(\bar{\varrho}\tilde{v}_\xi^2 + \frac{p_{\sigma\bar{\varrho}}}{\bar{\varrho}}\tilde{\varrho}^2_{\sigma\xi}\right)Ad\xi + c_0\|\tilde{v}\|^2_{2,\tilde{\Omega}} \le \varepsilon(\|\hat{\varrho}_{\sigma\xi}\|^2_{0,\bar{\Omega}} + \|\tilde{v}_{\xi\xi}\|^2_{0,\bar{\Omega}}) + c(\|\bar{v}\|^2_{1,\tilde{\Omega}} + \|p_\sigma\|^2_{0,\Omega_t}) + cX_2^2(\tilde{\Omega})(1 + X_2(\tilde{\Omega})), \tag{3.38}$$

where

$$X_2(\tilde{\Omega}) = \sum_{i+k\le 2}(\|\partial_t^i\bar{v}\|_{k,\tilde{\Omega}} + \|\partial_t^i\bar{\varrho}_\sigma\|^2_{k,\tilde{\Omega}}) + \int_0^t\|\bar{v}\|^2_{3,\tilde{\Omega}}d\tau + \|\overset{1}{\bar{H}}\|^2_{2,\tilde{\Omega}} \tag{3.39}$$

and A is the Jacobian of the transformation $x = x(\xi)$.

Finally, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \varrho\bar{v}_\xi^2 Ad\xi \le c(\|\bar{v}\|^2_{1,\Omega} + \|\bar{v}_t\|^2_{1,\Omega}), \tag{3.40}$$

where we have used $(3.19)_1$.

Going back to the old variables ξ in estimates (3.33), (3.35), (3.37) and summing them and (3.38) over all neighbourhoods of the partition of unity, using (3.25) and (3.40), assuming that ε and d are sufficiently small and passing to the variables x we obtain (3.23).

This completes the proof. ∎

LEMMA 3.6. *Let v, ϱ_σ be a sufficiently smooth solution of (3.2). Then*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho v_{xt}^2 + \frac{p_{\sigma\varrho}}{\varrho}\varrho_{xt}^2\right)dx + c_0(\|v_t\|^2_{2,\Omega_t} + \|\varrho_{\sigma t}\|^2_{1,\Omega_t}) \le c(\|v\|^2_{1,\Omega_t} + \|v_t\|^2_{1,\Omega_t} + \|v_{tt}\|^2_{1,\Omega_t} + \|\varrho_{\sigma t}\|^2_{0,\Omega_t} + \|p_\sigma\|^2_{0,\Omega_t}) + cX_2Y_2(1 + X_2^2),$$

where X_2 is given by (3.10) and Y_2 is given by (3.18).

Proof. Differentiating problem (3.24) with respect to t we get the following elliptic problem

$$\begin{aligned}
&\mu\nabla_v^2\bar{v}_t + \nu\nabla_v\nabla_v\cdot\bar{v}_t - p_{\sigma\varrho}\nabla_v\bar{\varrho}_{\sigma t} = \bar{\varrho}\bar{v}_t + \bar{\varrho}\bar{v}_{tt} - \nu(\nabla_v\nabla_v)_t\cdot\bar{v}\\
&\quad + \mu_1\left(\overset{1}{\bar{H}}\cdot\nabla_v\overset{1}{\bar{H}} - \nabla_v\frac{\overset{1}{\bar{H}}{}^2}{2}\right) - \mu(\nabla_v^2)_t\bar{v} + p_{\sigma\varrho\varrho}\varrho_{\sigma t}\nabla_v\varrho_\sigma + p_{\sigma\varrho}(\nabla_v)_t\varrho_\sigma \equiv K_1 \quad \text{in } \Omega,\\
&\mathrm{div}_v\bar{v}_t = \mathrm{div}_v\bar{v}_t \quad \text{in } \Omega,\\
&\mathbb{T}_v(\bar{v}_t, p_{\sigma t})\bar{n}_v = -(\mathbb{T}_v)_t(\bar{v}, p_\sigma)\bar{n}_v - \mathbb{T}_v(\bar{v}, p_\sigma)(\bar{n}_v)_t\\
&\quad + \left[\left(-\mu_1\overset{1}{\bar{H}}\otimes\overset{1}{\bar{H}} + \mu_1\frac{\overset{1}{\bar{H}}{}^2}{2}I\right)\bar{n}_v\right]_t = K_2 \quad \text{on } S.
\end{aligned}$$

By the Agmon-Douglis-Nirenberg theory (see [1]) we have the estimate

$$\|\bar{v}_t\|^2_{2,\Omega} + \|\bar{\varrho}_{\sigma t}\|^2_{1,\Omega} \le c(\|K_1\|^2_{0,\Omega} + \|K_2\|^2_{1/2,S} + \|\mathrm{div}_v \bar{v}_t\|^2_{1,\Omega}),$$

where

$$\|K_1\|^2_{0,\Omega} + \|K_2\|^2_{1/2,S} \le c(\|\bar{\varrho}_{\sigma\xi}\|^2_{0,\Omega} + \|\bar{v}_{tt}\|^2_{0,\Omega} + \|p_\sigma\|^2_{0,\Omega}) + X_2(\Omega)Y_2(\Omega)(1 + X_2^2(\Omega)),$$

$X_2(\Omega)$ is given by (3.26) and

$$Y_2(\Omega) = \sum_{\substack{i+k\le 3\\ i\le 2}} \|\partial_t^i \bar{v}\|^2_{k,\Omega_t} + \|\bar{\varrho}_\sigma\|^2_{2,\Omega} + \|\bar{\varrho}_{\sigma t}\|^2_{2,\Omega} + \|\bar{\varrho}_{\sigma tt}\|^2_{1,\Omega}. \tag{3.41}$$

By the arguments similar to considerations from the proof of Lemma 3.5 we conclude the proof. ∎

LEMMA 3.7. *Let v, ϱ_σ be a sufficiently smooth solution of (3.2). Then*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\Big(\varrho v^2_{xx} + \frac{p_{\sigma\varrho}}{\varrho}\varrho^2_{\sigma xx}\Big)dx + c_0(\|v\|^2_{3,\Omega_t} + \|\varrho_{\sigma x}\|^2_{1,\Omega_t})\\ \le c(\|v\|^2_{2,\Omega_t} + \|v_t\|^2_{1,\Omega_t} + \|\varrho_{\sigma x}\|^2_{0,\Omega_t} + \|p_\sigma\|^2_{0,\Omega_t})\\ + \varepsilon\|v_t\|^2_{2,\Omega_t} + cX_2Y_2(1 + X_2^2),\end{aligned} \tag{3.42}$$

where X_2 and Y_2 are given by (3.10) and (3.18), respectively and

$$v^2_{xx} = \sum_{i,j,k=1}^{3} v^2_{ix_jx_k}, \quad \varrho^2_{\sigma xx} = \sum_{j,k=1}^{3} \varrho^2_{\sigma x_jx_k}.$$

Proof. First, we consider problem (3.24). By the Agmon-Douglis-Nirenberg theory (see [1]) we have

$$\|\bar{v}\|^2_{3,\Omega} + \|\bar{\varrho}_\sigma\|^2_{2,\Omega} \le c(\|\bar{v}_t\|^2_{1,\Omega} + \|\mathrm{div}\bar{v}\|^2_{2,\Omega}) + cX_2(\Omega)Y_2(\Omega)(1 + X_2^2(\Omega)), \tag{3.43}$$

where $X_2(\Omega)$ and $Y_2(\Omega)$ are given by (3.26) and (3.41), respectively. Thus, to obtain (3.42) we have to estimate $\|\mathrm{div}\bar{v}\|^2_{2,\Omega}$ and $\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}(\varrho v^2_{xx} + \frac{p_{\sigma\varrho}}{\varrho}\varrho^2_{\sigma xx})dx$. To do this consider first boundary subdomains. Differentiate $(3.22)_1$ twice with respect to τ, multiply the result by $\tilde{u}_{\tau\tau}J$ and integrate over $\hat{\Omega}$. Using the Korn inequality, continuity equation $(3.22)_2$, and boundary condition $(3.22)_3$ we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}}\Big(\hat{\varrho}\tilde{v}^2_{\tau\tau} + \frac{p_{\sigma\hat{\varrho}}}{\hat{\varrho}}\tilde{\varrho}^2_{\sigma\tau\tau}\Big)Jdz + c_0\|\tilde{v}_{\tau\tau}\|^2_{1,\hat{\Omega}} \le \varepsilon(\|\hat{\varrho}_{\sigma\tau\tau}\|^2_{0,\hat{\Omega}} + \|\tilde{v}_{\tau\tau}\|^2_{1,\hat{\Omega}})\\ + c(\|\hat{v}\|^2_{2,\hat{\Omega}} + \|\hat{\varrho}_{\sigma z}\|^2_{0,\hat{\Omega}}) + cX_2(\hat{\Omega})Y_2(\hat{\Omega})(1 + X_2^2(\hat{\Omega})),\end{aligned} \tag{3.44}$$

where $X_2(\hat{\Omega})$ is given by (3.28) and

$$Y_2(\hat{\Omega}) = \sum_{\substack{i+k\le 3\\ i\le 2}} \|\partial_t^i \bar{v}\|^2_{k,\hat{\Omega}} + \|\hat{\varrho}_\sigma\|^2_{2,\hat{\Omega}} + \|\hat{\varrho}_{\sigma t}\|^2_{2,\hat{\Omega}} + \|\hat{\varrho}_{\sigma tt}\|^2_{1,\hat{\Omega}}.$$

In the same way way we obtain the following estimate in an interior subdomain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}}\Big(\bar{\varrho}\tilde{v}^2_{\xi\xi} + \frac{p_{\sigma\eta}}{\bar{\varrho}}\tilde{\varrho}^2_{\sigma\xi\xi}\Big)Ad\xi + c_0\|\tilde{v}\|^2_{3,\tilde{\Omega}} \le \varepsilon(\|\tilde{\varrho}_{\sigma\xi\xi}\|^2_{0,\tilde{\Omega}} + \|\hat{v}_{\xi\xi\xi}\|^2_{0,\tilde{\Omega}})\\ + c(\|\bar{v}\|^2_{2,\tilde{\Omega}} + \|\bar{\varrho}_{\sigma\xi}\|^2_{0,\tilde{\Omega}} + \|p_\sigma\|^2_{0,\tilde{\Omega}}) + cX_2(\tilde{\Omega})Y_2(\tilde{\Omega})(1 + X_2^2(\tilde{\Omega})),\end{aligned} \tag{3.45}$$

where $X_2(\tilde\Omega)$ is given by (3.39) and

$$Y_2(\tilde\Omega)=\sum_{\substack{i+k\le 3\\ i\le 2}}\|\partial_t^i\bar v\|^2_{k,\tilde\Omega}+\|\bar\varrho_\sigma\|^2_{2,\tilde\Omega}+\|\bar\varrho_{\sigma t}\|^2_{2,\tilde\Omega}+\|\bar\varrho_{\sigma tt}\|^2_{1,\tilde\Omega}.$$

Now, we differentiate the third component of (3.34) by τ, multiply the result by $\tilde\varrho_{\sigma n\tau}J$ and integrate over $\hat\Omega$. We get

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\hat\Omega}\frac{p_{\sigma\hat\varrho}}{\hat\varrho}\tilde\varrho^2_{\sigma n\tau}Jdz+\int_{\hat\Omega}p^2_{\sigma\hat\varrho}\tilde\varrho^2_{\sigma n\tau}Jdz\le\varepsilon\|\tilde\varrho_{\sigma n\tau}\|^2_{0,\hat\Omega}+c(\|\hat v\|^2_{2,\hat\Omega}+\|\hat v_t\|^2_{1,\hat\Omega}\\&+\|\hat\varrho_{\sigma z}\|^2_{1,\hat\Omega}+\|p_\sigma\|^2_{0,\hat\Omega})+cd\|\tilde v\|^2_{3,\hat\Omega}+c\|\tilde v_{z\tau\tau}\|^2_{0,\hat\Omega}\\&+cX_2(\hat\Omega)Y_2(\hat\Omega)(1+X_2^2(\hat\Omega)),\end{aligned}\tag{3.46}$$

where d is from formula (3.21).

In the same way we obtain

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\hat\Omega}\frac{p_{\sigma\hat\varrho}}{\hat\varrho}\tilde\varrho^2_{\sigma nn}Jdz+\int_{\hat\Omega}p^2_{\sigma\hat\varrho}\tilde\varrho^2_{\sigma nn}Jdz\le\varepsilon\|\tilde\varrho_{\sigma nn}\|^2_{0,\hat\Omega}+c(\|\hat v\|^2_{2,\hat\Omega}\\&\quad+\|\hat v_t\|^2_{0,\hat\Omega}+\|\hat\varrho_{\sigma z}\|^2_{1,\hat\Omega}+\|p_\sigma\|^2_{0,\hat\Omega})+cd\|\tilde v\|^2_{3,\hat\Omega}+c\|\tilde v_{zn\tau}\|^2_{0,\hat\Omega}\\&\quad+cX_2(\hat\Omega)Y_3(\hat\Omega)(1+X_2^2(\hat\Omega)).\end{aligned}\tag{3.47}$$

Next, differentiating the third component of (3.36) by τ, multiplying by $\tilde v_{3nn\tau}J$ and integrating over $\hat\Omega$ we have

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\hat\Omega}\hat\varrho\tilde v^2_{3n\tau}Jdz+c_0\|\tilde v_{3nn\tau}\|^2_{0,\hat\Omega}\le\varepsilon\|\tilde v_{3nn\tau}\|^2_{0,\hat\Omega}+c(\|\tilde v\|^2_{2,\hat\Omega}+\|\tilde v_t\|^2_{2,\tilde\Omega}\\&\quad+\|\tilde v_{z\tau\tau}\|^2_{0,\hat\Omega}+\|\hat\varrho_{\sigma n\tau}\|^2_{0,\hat\Omega}+\|\hat\varrho_{\sigma z}\|^2_{0,\hat\Omega}+\|p_\sigma\|^2_{0,\hat\Omega})+cd\|\hat v\|^2_{3,\hat\Omega}+\varepsilon\|\tilde v_t\|^2_{2,\hat\Omega}\\&\quad+cX_2(\hat\Omega)Y_2(\hat\Omega)(1+X_2^2(\hat\Omega)).\end{aligned}\tag{3.48}$$

In order to estimate $\|(\mathrm{div}\tilde v)_{nn}\|^2_{0,\hat\Omega}$ rewrite equation $(3.22)_1$ in the form

$$\begin{aligned}(\nu+\mu)\nabla_{z_i}\mathrm{div}\tilde v&=-\mu(\Delta\tilde v_i-\nabla_{z_i}\mathrm{div}\tilde v)+\hat\varrho\tilde v_{it}-k_{3i}\\&\quad+(\mu\Delta\tilde v_i+\nu\nabla_{z_i}\mathrm{div}\tilde v-\mu\hat\nabla^2\tilde v_i-\nu\hat\nabla_i\hat\nabla\cdot\tilde v)+\mu_1(-\overset{1}{\tilde H}\hat\nabla\overset{1}{\hat H}_i+(\hat\nabla_i\overset{1}{\hat H})\overset{1}{\tilde H})\\&\quad+p_1\hat\varrho_\sigma\hat\nabla_i\hat\zeta+\hat\zeta p_{\sigma\hat\varrho}\hat\nabla_i\hat\varrho_\sigma,\quad i=1,2,3.\end{aligned}\tag{3.49}$$

Differentiating the third component of (3.49) with respect to n gives

$$\begin{aligned}\|(\mathrm{div}\tilde v)_{nn}\|^2_{0,\hat\Omega}&\le cd\|\tilde v_{nnn}\|^2_{0,\hat\Omega}+c(\|\tilde v_\tau\|^2_{2,\hat\Omega}+\|\hat v\|^2_{2,\hat\Omega}+\|\tilde v_t\|^2_{1,\hat\Omega}\\&\quad+\|\hat\varrho_{\sigma z}\|^2_{1,\hat\Omega}+\|p_\sigma\|^2_{0,\hat\Omega})+cX_2(\hat\Omega)Y_2(\hat\Omega).\end{aligned}\tag{3.50}$$

To obtain an estimate for $\|\tilde v_\tau\|^2_{2,\hat\Omega}$ consider the following elliptic problem

$$\begin{aligned}&\mu\hat\nabla^2\tilde v+\nu\hat\nabla\hat\nabla\cdot\tilde v-p_{\sigma\hat\varrho}\hat\varrho_\sigma=\hat\varrho\tilde v_t+(p_1-p_{\sigma\hat\varrho})\hat\varrho_\sigma\hat\nabla\hat\zeta\\&\quad+\mu_1\Big(-\overset{1}{\tilde H}\hat{\cdot}\nabla\overset{1}{\hat H}+\frac12(\hat\nabla(\overset{1}{\tilde H}\overset{1}{\hat H})-(\hat\nabla\hat\zeta)\overset{1}{\hat H}{}^2)\Big)+\hat\nabla\cdot\hat{\mathbb B}(\hat v,\hat\zeta)+\hat{\mathbb T}(\hat v,p_\sigma)\cdot\hat\nabla\hat\zeta,\\&\hat\nabla\cdot\tilde v=\hat\nabla\cdot\tilde v,\\&\hat{\mathbb T}(\tilde v,\tilde p_\sigma)\hat n=k_5,\end{aligned}\tag{3.51}$$

where $\hat{\nabla}\cdot\hat{\mathbb{B}}(\hat{v},\hat{\zeta})=\{\hat{\nabla}_j\hat{\mathbb{B}}_{ij}(\hat{v},\hat{\zeta})\}_{i=1,2,3}$, $\hat{\mathbb{T}}(\hat{v},p_\sigma)\cdot\hat{\nabla}\hat{\zeta}=\{\hat{T}_{vij}(\hat{v},p_\sigma)\hat{\nabla}_j\hat{\zeta}\}_{i=1,2,3}$.

Differentiating (3.51) with respect to τ and next using the Agmon-Douglis-Nirenberg theory we get

$$\begin{aligned}\|\tilde{v}_\tau\|^2_{2,\hat{\Omega}}+\|\hat{\varrho}_{\sigma\tau}\|^2_{1,\hat{\Omega}}\le c(&\|\tilde{v}_{\tau\tau}\|^2_{1,\hat{\Omega}}+\|\hat{v}_{3nn\tau}\|^2_{0,\hat{\Omega}}+\|\hat{v}\|^2_{2,\hat{\Omega}}+\|\hat{v}_t\|^2_{1,\hat{\Omega}}\\&+\|\hat{\varrho}_{\sigma z}\|^2_{0,\hat{\Omega}}+\|p_\sigma\|^2_{0,\hat{\Omega}})+cX_2(\hat{\Omega})Y_2(\hat{\Omega})(1+X_2(\hat{\Omega})).\end{aligned}\tag{3.52}$$

Finally, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\varrho\bar{v}^2_{\xi\xi}Ad\xi\le c\|\bar{v}\|^2_{2,\Omega}+c\|\bar{v}_t\|^2_{1,\Omega}.\tag{3.53}$$

Going back to the old variables ξ in estimates (3.44), (3.46), (3.49), (3.50), (3.52) and summing them and (3.45) over all neighbourhoods of the partition of unity, using (3.43) and (3.53), assuming that ε and d are sufficiently small and passing to the variables x we obtain (3.44). This concludes the proof. ■

In [7] we proved the following lemmas for problem (1.1)–(1.7):

LEMMA 3.8. *For a sufficiently smooth solution (v,ϱ_σ,H) of (1.1)–(1.7), we have*

$$\frac{d}{dt}\|H\|^2_{0,\Pi_t}+\|H\|^2_{1,\Pi_t}\le c\|H\|^2_{1,\Pi_t}\|v\|^2_{1,\Pi_t}.\tag{3.54}$$

LEMMA 3.9. *For a sufficiently smooth solution (v,ϱ_σ,H) of (1.1)–(1.7), we have*

$$\frac{d}{dt}\|H_t\|^2_{0,\Pi_t}+\|H_t\|^2_{1,\Pi_t}\le c(\|H_t\|^2_{1,\Pi_t}\|v\|^2_{1,\Pi_t}+\|v_t\|^2_{1,\Pi_t}\|H\|^2_{1,\Pi_t}).\tag{3.55}$$

LEMMA 3.10. *For a sufficiently smooth solution (v,ϱ_σ,H) of (1.1)–(1.7), we have*

$$\begin{aligned}\frac{d}{dt}\|H_{tt}\|^2_{0,\Pi_t}+\|H_{tt}\|^2_{1,\Pi_t}\le c(&\|H_{tt}\|^2_{1,\Pi_t}\|v\|^2_{1,\Pi_t}+\|v_{tt}\|^2_{1,\Pi_t}\|H\|^2_{1,\Pi_t}\\&+\|v_t\|^2_{1,\Pi_t}\|H_t\|^2_{1,\Pi_t}).\end{aligned}\tag{3.56}$$

LEMMA 3.11. *For a sufficiently smooth solution (v,ϱ_σ,H) of (1.1)–(1.7), we have*

$$\begin{aligned}\frac{d}{dt}\|H_t\|^2_{1,\Pi_t}+\|H_t\|^2_{2,\Pi_t}\le c\varphi(a)[&\|H_{tt}\|^2_{1,\Pi_t}+\|H_t\|^2_{2,\Pi_t}\|v\|^2_{1,\Pi_t}+\|H\|^2_{2,\Pi_t}(\|v\|^2_{3,\Pi_t}\\&+\|v_t\|^2_{2,\Pi_t}+\|v\|^4_{2,\Pi_t})+\|v\|^2_{3,\Pi_t}(\|H_t\|^2_{1,\Pi_t}+\|H\|^2_{2,\Pi_t}\|v\|^2_{2,\Pi_t}+\|v\|^2_{2,\Pi_t}+\|v_t\|^2_{2,\Pi_t})\\&+\|v\|^2_{2,\Pi_t}+a^2\|H\|^2_{3,\Pi_t}].\end{aligned}\tag{3.57}$$

LEMMA 3.12. *For a sufficiently smooth solution (v,ϱ_σ,H) of (1.1)–(1.7), we have*

$$\begin{aligned}\frac{d}{dt}\|H\|^2_{2,\Pi_t}+\|H\|^2_{3,\Pi_t}\le c\varphi(a)[&\|H\|^2_{2,\Pi_t}\|v\|^2_{2,\Pi_t}+\|v\|^2_{2,\Pi_t}+\|H\|^2_{2,\Pi_t}\|v\|^2_{3,\Pi_t}+\\&+\|H_t\|^2_{2,\Pi_t}+\|H\|^2_{3,\Pi_t}\|v\|^2_{2,\Pi_t}].\end{aligned}\tag{3.58}$$

LEMMA 3.13. *For sufficiently smooth solution (v,ϱ_σ,H) of (1.1)–(1.7), we have*

$$\frac{d}{dt}\|H\|^2_{1,\Pi_t}\le c(\|H\|^2_{1,\Pi_t}+\varepsilon\|H_t\|^2_{1,\Pi_t}+\|H\|^2_{2,\Pi_t}\|v\|^2_{0,\Pi_t}).\tag{3.59}$$

Now let $\overset{2}{H}=H_*$ on B, then from Lemmas 3.1–3.13 and inequalities

$$\begin{aligned}&\|\varrho_{\sigma tt}\|^2_{1,\Omega_t}\le c\|v_t\|^2_{2,\Omega_t}+c(\|\varrho_{\sigma t}\|^2_{2,\Omega_t}\|v\|^2_{2,\Omega_t}+\|\varrho_\sigma\|^2_{2,\Omega_t}\|v_t\|^2_{2,\Omega_t}),\\&\|\varrho_{\sigma t}\|^2_{2,\Omega_t}\le c\|v\|^2_{3,\Omega_t}+cX_2Y_2(1+X_2),\end{aligned}$$

(which follow from equations $(3.2)_2$ and $(3.19)_2$, respectively) we get

THEOREM 3.1. *Let $\nu > \frac{1}{3}\mu > 0$ and let Remarks 3.1 and 3.2 be satisfied. Then for a sufficiently smooth solution v, ϱ_σ, H of problem (1.1)–(1.7) we have*

$$\frac{d\bar\phi}{dt}+c_0\Phi \le c_1\bigg(\phi+\int_0^t \|v\|^2_{3,\Omega_\tau}d\tau\bigg)\bigg[1+\bigg(\phi+\int_0^t \|v\|^2_{3,\Omega_\tau}d\tau\bigg)^2\bigg]\Phi+c_2\Psi \quad \textit{for } t\le T, \tag{3.60}$$

where

$$\begin{aligned}\bar\phi(t) &= \int_{\Omega_t}\varrho\sum_{0\le|\alpha|+i\le 2}|D_x^\alpha\partial_t^i v|^2dx+\int_{\Omega_t}\frac{p_1}{\varrho}\varrho_\sigma^2dx\\ &\quad+\int_{\Omega_t}\frac{p_{\sigma\varrho}}{\varrho}\sum_{1\le|\alpha|+i\le 2}|D_x^\alpha\partial_t^i p_\sigma|^2dx+\sum_{i+k\le 2}\|\partial_t^iH\|^2_{k,\Pi_t},\\ \phi(t) &= \sum_{i+k\le 2}(\|\partial_t^iv\|^2_{k,\Omega_t}+\|\partial_t^iH\|^2_{k,\Pi_t}+\|\partial_t^i\varrho_\sigma\|^2_{k,\Omega_t}),\\ \Phi(t) &= \|\varrho_\sigma\|^2_{2,\Omega_t}+\|\varrho_{\sigma t}\|^2_{2,\Omega_t}+\|\varrho_{\sigma tt}\|^2_{1,\Omega_t}+\sum_{\substack{i+k\le 3\\ i\le 2}}(\|\partial_t^iv\|^2_{k,\Omega_t}+\|\partial_t^iH\|^2_{k,\Pi_t})\\ \Psi(t) &= \|p_\sigma\|^2_{0,\Omega_t}+\|H_*\|^2_{3,B}+\|H_*\|^4_{1,B}+\|H_{*t}\|^2_{2,B}(1+\|H_{*t}\|^2_{2,B})+\|H_{*tt}\|^2_{1,B},\end{aligned} \tag{3.61}$$

c_i $(i=1,2)$ are positive constants depending on φ_*, φ^*, μ, ν, $\int_0^t\|v\|^2_{3,\Omega_\tau}d\tau$, $\|S\|_{\frac{5}{2}}$, T and constants of the imbedding theorems and the Korn inequalities; $c_0<1$ is a positive constant depending on μ and ν; ϱ_σ and p_σ are given by (3.1).

4. Global existence. Now, let $\bar\phi(t)$, $\phi(t)$ and $\Phi(t)$ be defined by (3.61). Introduce the spaces

$$\mathcal{N}(t)=\{(v,\varrho_\sigma,H):\phi(t)<\infty\}$$

and

$$\mathcal{M}(t)=\bigg\{(v,\varrho_\sigma,H):\phi(t)+\int_0^t\Phi(\tau)d\tau<\infty\bigg\}.$$

Notice that $(v,\varrho_\sigma,H)\in\mathcal{N}(t)$ iff $\bar\phi(t)<\infty$ and $(v,\varrho_\sigma,H)\in\mathcal{M}(t)$ iff $\bar\phi(t)+\int_0^t\phi(\tau)d\tau<\infty$. Moreover,

$$c'\phi(t)\le\bar\phi(t)\le c''\phi(t), \tag{4.1}$$

where $c', c''>0$ are constants depending on φ_*, φ^* given by (2.5).

From Theorem 2.1 and (4.1) we get

LEMMA 4.1. *Let $(v(0),\varrho_\sigma(0),H(0))\in\mathcal{N}(0)$ and $\varphi(0)<\varepsilon_1$. Then $(v(t),\varrho_\sigma(t),H(t))\in\mathcal{M}(t)$, $t<T$ where T is the time of local existence and*

$$\begin{aligned}\bar\phi(t)+\int_0^t\Phi(\tau)d\tau &\le c\bigg(\varepsilon_1+\int_0^t(\|E_*\|^2_{0,B}+\|E_{*t}\|^2_{0,B}+\|H_*\|^2_{3,B}+\|H_{*t}\|^2_{2,B}+\\ &\quad+\|H_{*tt}\|^2_{0,B})d\tau\bigg)\equiv c(\varepsilon_1+\beta).\end{aligned} \tag{4.2}$$

Proof. From inequality

$$\|\bar H\|^2_{2,\Pi^t}\le c(\varepsilon\|\bar H_t\|^2_{2,2,2,\Pi^t}+c(\varepsilon)\|\bar H\|^2_{2,2,2,\Pi^t}+\|\bar H(0)\|^2_{2,\Pi})$$

and Theorem 2.1 we get (4.2). Next we prove

LEMMA 4.2. *Assume that there exists a local solution of (1.1)–(1.7) in $\mathcal{M}(t)$, $0 \le t \le T$ with initial data in $\mathcal{N}(0)$ sufficiently small and*

$$\alpha(t) = \|p_\sigma\|^2_{0,\Omega_t} + \|H_*\|^2_{3,B} + \|H_*\|^4_{1,B} + \|H_{*t}\|^2_{2,B}(1 + \|H_{*t}\|^2_{2,B}) + \|H_{*tt}\|^2_{1,B} \le e^{-\mu t}$$

$0 \le t \le T$, where $\mu > \frac{1}{2}$. Then there exist constants $\mu_1 > 1$, $\mu_2 > 0$ such that

$$\bar\phi(t) \le \mu_1 e^{-\mu_2 t}\left(\bar\phi(0) + \frac{c_2}{\mu - \mu_2}\right), \tag{4.3}$$

if $\mu > \mu_2$.

Proof. Consider inequality (3.63) and assume that $\varepsilon_1 + \beta$ from (4.2) (see Lemma 4.1) is so small that

$$c_1\left(\phi + \int_0^t \|v\|^2_{3,\Omega_\tau} d\tau\right)\left[1 + \left(\phi + \int_0^t \|v\|^2_{3,\Omega_\tau} d\tau\right)^2\right] < \frac{c_0}{4}. \tag{4.4}$$

Then inequality (3.61) implies

$$\frac{d\bar\phi}{dt} + \frac{3}{4}c_0\Phi$$
$$< c_2(\|p_\sigma\|^2_{0,\Omega_t} + \|H_*\|^2_{3,B} + \|H_*\|^4_{1,B} + \|H_{*t}\|^2_{2,B}(1 + \|H_{*t}\|^2_{2,B}) + \|H_{*tt}\|^2_{1,B}). \tag{4.5}$$

Applying the same argument as in the proof of Lemma 6.2 of [19] yields

$$\|p_\sigma\|^2_{0,\Omega_t} \le \varepsilon(\|p_{\sigma x}\|^2_{0,\Omega_t} + \|v_{xx}\|^2_{0,\Omega_t}) + c(\varepsilon)(\|v\|^2_{0,\Omega_t} + \|v_t\|^2_{0,\Omega_t}). \tag{4.6}$$

Since

$$\|p_{\sigma x}\|^2_{0,\Omega_t} \le c_4\|\varrho_{\sigma x}\|^2_{0,\Omega_t}$$

inequalities (4.5) and (4.6) imply for sufficiently small ε

$$\frac{d\bar\phi}{dt} + \frac{3}{4}c_0\Phi < c_5(\|v\|^2_{0,\Omega_t} + \|v_t\|^2_{0,\Omega_t}) + c_2(\|H_*\|^2_{3,B} + \|H_*\|^4_{1,B}$$
$$+ \|H_{*t}\|^2_{2,B}(1 + \|H_{*t}\|^2_{2,B}) + \|H_{*tt}\|^2_{1,B}). \tag{4.7}$$

Now, multiplying (3.16) by a constant c_6 so large that $c_0c_6 - c_5 > 0$ and $c_6 > 1$, adding to (4.7) and using Lemma 4.1 we obtain

$$\frac{d}{dt}(\bar\phi + c_6J) + \frac{3}{4}c_0\Phi + (c_0c_6 - c_5)(\|v\|^2_{1,\Omega_t} + \|v_t\|^2_{1,\Omega_t} + \|\varrho_{\sigma t}\|^2_{0,\Omega_t}) < c_7(\beta + c\varepsilon_1)\phi$$
$$+ c_2(\|H_*\|^2_{3,B} + \|H_*\|^4_{3,B} + \|H_{*t}\|^2_{2,B}(1 + \|H_{*t}\|^2_{2,B}) + \|H_{*tt}\|^2_{1,B}), \tag{4.8}$$

where $J = \frac{1}{2}\int_{\Omega_t}\left[\varrho(v^2 + v_t^2) + \frac{p_1}{\varrho}\varrho_\sigma^2 + \frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma t}^2\right]dx$. Since $\frac{\bar\phi}{c''} \le \phi \le \Phi$ and $\bar\phi \ge J$ for sufficiently small $\beta + c\varepsilon_1$ ($\beta + c\varepsilon_1$ so small that $c_7(\beta + c\varepsilon_1) < \frac{1}{4}c_0$) inequality (4.8) implies

$$\frac{d}{dt}(\bar\phi + c_6J) + c_8(\bar\phi + c_6J) \le c_2e^{-\mu t}, \tag{4.9}$$

where $c_8 = \frac{c_0}{4c''c_6}$ ($c'' > 0$ is a constant from (4.1)).

Inequality (4.9) yields (4.3) with $\mu_1 = c_6 + 1$ and $\mu_2 = c_8$. This completes the proof of Lemma 4.2. ∎

LEMMA 4.3. *Let the assumptions of Lemma 4.2 be satisfied and $\bar\phi(0) < \varepsilon_1$. Then $\bar\phi(T) \le \varepsilon_1$, where $T > 0$ is the time of local existence.*

Proof. If $\mu > 0$ and T are sufficiently large then from Lemma 4.2

$$\bar{\varphi}(T) \le \mu_1 e^{-\mu_2 T}\left(\bar{\varphi}(0) + \frac{c_2}{\mu - \mu_2}\right) \le \bar{\varphi}(0).$$

Proof of Main Theorem. The theorem is proved step by step using local existence in a fixed time interval. Under the assumptions that

$$(v(0), \varrho_\sigma(0), H(0)) \in \mathcal{N}(0). \tag{4.10}$$

Theorem 2.1 and Lemma 4.1 yield local existence of solutions of (1.1)–(1.7).

By (4.8), Lemma 4.1 implies that the local solution belongs to $\mathcal{M}(t)$, $t \le T$. For small ε_1 and β the existence time T is correspondingly large, so we can assume it is a fixed positive number. The constants in those theorems depend on Ω_t and shape of S_t and $\int_0^t \|v\|^2_{3,\Omega_\tau} d\tau$, so generally they are functions of t.

But in view of (4.1) with sufficiently small ε_1, β we obtain

$$\left|\int_0^t v d\tau\right| \le c\beta \qquad t \in [0, T]. \tag{4.11}$$

Hence from the relation

$$x = \xi + \int_0^t v(x(\xi,\tau),\tau) d\tau, \quad \xi \in S, \quad t \le T, \tag{4.12}$$

for sufficiently small ε_1, β and fixed T, the shape of Ω_t, $t \le T$ does not change too much, so the constants from the immbeding theorems can be chosen independent of time. Now we wish to extend the solution to the interval $[T, 2T]$. Using Lemma 4.3 we can prove the existence of local solution in $\mathcal{M}(t)$, $T \le t \le 2T$. To prove

$$\bar{\phi}(2T) \le \varepsilon_1, \tag{4.13}$$

we need inequality (3.61) where the constans depend on the constants from the imbedding theorems and Korn inequalities for $t \in [T, 2T]$. Therefore we have to show that the shape of S_t and $\int_0^t \|v\|^2_{3,\Omega_\tau} d\tau$, $t \le 2T$, do not change more than for $t \le T$. Assume that there exists a local solution in the interval $[0, kT]$. Then in view of Lemma 4.2, we have for $t \in [0, kT]$

$$\begin{aligned}
&\left|\int_0^t v dx\right| \le \left|\int_0^t \|v\|_{2,\Omega_\tau} d\tau\right| \\
&\le c_1 \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \|v\|_{2,\Omega_t} dt \le c_1 T^{\frac{1}{2}} \sum_{i=0}^{k-1} \left(\int_{iT}^{(i+1)T} \|v\|^2_{2,\Omega_t} dt\right)^{\frac{1}{2}} \\
&\le c_1 T^{\frac{1}{2}} \sum_{i=0}^{k-1} \left(\int_{iT}^{(i+1)T} \bar{\phi}(t) dt\right)^{\frac{1}{2}} \\
&\le c_1 \left(\frac{T}{\mu_2}\right)^{\frac{1}{2}} \sum_{i=0}^{k-1} \left[\frac{c_2}{\mu - \mu_2} e^{-iT\mu} + \bar{\phi}(iT)\right]^{\frac{1}{2}} \\
&\le c\left(\frac{T}{\mu_2}\right)^{\frac{1}{2}} \frac{1}{1 - e^{-\mu_2 T/2}} \left[\frac{1}{(\mu - \mu_2)^{\frac{1}{2}}}\left(1 + \frac{1}{1 - e^{-\mu_2 T/2}}\right) + (\bar{\phi}(0)\mu_1)^{\frac{1}{2}}\right]
\end{aligned} \tag{4.14}$$

and

$$\int_0^t \|v\|^2_{3,\Omega_\tau} d\tau \le \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \|v\|^2_{3,\Omega_\tau} d\tau \le \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \Phi dt$$
$$\le \frac{1}{\mu_2} \sum_{i=0}^{k-1} \left(\frac{c_2}{\mu} e^{-\mu iT} + \mu_2 \bar{\phi}(iT) \right)$$
$$\le \frac{c}{\mu_2(1-e^{-\mu_2 T})} \left(\frac{1}{\mu} + \mu_1 \bar{\phi}(0) + \frac{1}{(\mu-\mu_2)(1-e^{-\mu_2 T})} \right).$$

since from (4.9)

$$A((i+1)T) \le \frac{c_2}{\mu-\mu_2} e^{-\mu_2(i+1)T} + A(iT) e^{-\mu_2 T},$$

where $A = \bar{\phi} + c_6 I$, then we have

$$\sum_{i=0}^{k-1} \bar{\phi}(iT) \le \sum_{i=0}^{k-1} A(iT) \le \frac{\mu_1 \bar{\phi}(0)}{1-e^{-\mu_2 T}} + \frac{c_2}{\mu-\mu_2} \frac{e^{-\mu_2 T}}{1-e^{-\mu_2 T}} + \frac{c_2}{\mu-\mu_2} \frac{e^{-2\mu_2 T}}{1-e^{-\mu_2 T}}$$
$$+ \cdots + \frac{c_2}{\mu-\mu_2} \frac{e^{-n\mu_2 T}}{1-e^{-\mu_2 T}} + \cdots \le \frac{1}{1-e^{\mu_2 T}} \left(\mu_1 \bar{\phi}(0) + \frac{c_2 e^{-\mu_2 T}}{(\mu-\mu_2)(1-e^{-\mu_2 T})} \right),$$

also

$$\int_{iT}^{(i+1)T} A(t) dt \le \frac{c_2}{(\mu-\mu_2)\mu_2} e^{-\mu iT} + \frac{1}{\mu_2} A(iT)$$

and

$$A((i+1)T) + \mu_2 \int_{iT}^{(i+1)T} A(t) dt \le \frac{c_2}{\mu} e^{-\mu iT} + A(iT), \quad i = 0, 1, \ldots, k-1.$$

Taking $k = 2$, ε_1 sufficiently small and μ sufficiently large we see that $\int_0^t v(x(\xi,\tau),\tau)d\tau$ is small for any $t \in [0, 2T]$, so (4.14) implies that the shape of S_t and $\int_0^t \|v\|^2_{3,\Omega_\tau} d\tau$ change no more than in $[0, T]$, and then the differential inequality (3.62) can also be shown for this interval with the same constants. Hence in view of Lemma 4.1 the solutions of (1.1)–(1.7) belongs to $\mathcal{M}(t)$, $t \in [T, 2T]$. Next Lemmas 4.1–4.3 imply (4.14).

Repeating the above considerations for the intervals $[kT, (k+1)T]$, $k \ge 2$, we prove the existence for all $t \in \mathbb{R}_+$.

5. Korn inequality

LEMMA 5.1. *Let $\Omega_t \subset \mathbb{R}^3$ be a bounded domain. Let (v, ϱ_σ) be a solution of* $(1.1)_1$, $(1.1)_2$, $(1.5)_1$ *and* $f = \int_\Omega v_0 dx = \int_\Omega \varrho_0 v_0 \cdot \varphi_i dx$, $i = 1, 2, 3$ *and*

$$\mathbb{E}_{\Omega_t}(v) = \int_{\Omega_t} (\partial_{x_i} v_j + \partial_{x_j} v_i)^2 dx < \infty. \tag{5.1}$$

Then there exists a constant $c > 0$ such that

$$\|v\|^2_{1,\Omega_t} \le c \left(\mathbb{E}_{\Omega_t}(v) + \left(\int_{\Omega_t} |(\varrho - \varrho_e)||v| dx \right)^2 \right) \tag{5.2}$$

Proof. Introduce a function u by

$$u = \sum_{i=1}^{3} b_i\varphi_i(x) + v, \tag{5.3}$$

where $\varphi_i = (x - \bar{x}) \times e_i$, $\bar{x} = \frac{1}{|\Omega_t|}\big(\int_{\Omega_t} x_1dx, \int_{\Omega_t} x_2dx, \int_{\Omega_t} x_3dx\big)$, $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$, $i = 1, 2, 3$.

Define $b = (b_1, b_2, b_3)$ by

$$b = \frac{1}{2|\Omega_t|}\int_{\Omega_t} \mathrm{rot} v dx. \tag{5.4}$$

Since rot $\varphi_i = 2e_i$, $i = 1, 2, 3$, equations (5.3) and (5.4) imply

$$\int_{\Omega_t} \mathrm{rot} u dx = 0. \tag{5.5}$$

From (5.4) we have $\int_{\Omega_t} \varphi_i dx = 0$, $i = 1, 2, 3$ so

$$\int_{\Omega_t} udx = \int_{\Omega_t} vdx \quad \text{and} \quad \mathbb{E}_{\Omega_t}(\varphi_i) = 0, \quad i = 1, 2, 3, \tag{5.7}$$

so

$$\mathbb{E}_{\Omega_t}(u) = \mathbb{E}_{\Omega_t}(v). \tag{5.8}$$

By Theorem 1 of [9] we have

$$\partial_{x_j} w_i = \varepsilon_{ikl}\partial_{x_k} S_{jl}, \quad i = 1, 2, 3, \quad w = \mathrm{rot} u, \tag{5.9}$$

$S_{ij} = \partial_{x_i} u_j + \partial_{x_j} u_i$, so by (5.6) and Lemma 2.4 of [8] it follows that

$$\|\mathrm{rot} u\|_{0,\Omega_t}^2 \le c \sum_{i,j=1}^{3} \|S_{ij}\|_{0,\Omega_t}^2 = c\mathbb{E}_{\Omega_t}(u) = c\mathbb{E}_{\Omega_t}(v). \tag{5.10}$$

Employing the identity

$$\partial_{x_j} u_i = \frac{1}{2}(\partial_{x_j} u_i + \partial_{x_i} u_j) + \frac{1}{2}(\partial_{x_j} u_i - \partial_{x_i} u_j)$$

and (5.10) we have

$$\|\nabla u\|_{0,\Omega_t}^2 \le c(\mathbb{E}_{\Omega_t}(u) + \|\mathrm{rot} u\|_{0,\Omega_t}^2) \le c\mathbb{E}_{\Omega_t}(u) = c\mathbb{E}_{\Omega_t}(v). \tag{5.11}$$

Using 5.3 we obtain

$$\|\nabla v\|_{0,\Omega_t}^2 \le c(\mathbb{E}_{\Omega_t}(v) + |b|^2). \tag{5.12}$$

From Remark 3.2 using (5.3) we get systems of equations

$$\sum_{i=1}^{3} b_i \int_{\Omega_t} \varphi_i \cdot \varphi_j dx = \int_{\Omega_t} u\varphi_j dx + \frac{1}{\varrho_e}\int_{\Omega_t} (\varrho - \varrho_e)v \cdot \varphi_j dx. \tag{5.13}$$

Since det $\Gamma \ne 0$, where $\Gamma = \{\Gamma_{ij}\}$, $\Gamma_{ij} = \int_{\Omega_t} \varphi_i\varphi_j dx$ we can calculate b from (5.14), so

$$|b|^2 \le c\bigg(\|u\|_{0,\Omega_t}^2 + \bigg(\int_{\Omega_t} |\varrho - \varrho_e|\,|v|dx\bigg)^2\bigg). \tag{5.14}$$

Now by Poincaré inequality and (5.7), (5.8), and Remark 3.1, we obtain

$$\|u\|_{0,\Omega_t}^2 \le 2\left\|u - \frac{1}{|\Omega_t|}\int_{\Omega_t} udx\right\|_{0,\Omega_t}^2 + 2\left\|\frac{1}{|\Omega_t|}\int_{\Omega_t} udx\right\|_{0,\Omega_t}^2 \le$$
$$\le c\left(\|\nabla u\|_{0,\Omega_t}^2 + \left\|\frac{1}{|\Omega_t|}\int_{\Omega_t} vdx\right\|_{0,\Omega_t}^2\right) \le c\left(\mathbb{E}_{\Omega_t}(u) + \left(\int_{\Omega_t} |\varrho - \varrho_e|\,|v|dx\right)^2\right). \tag{5.15}$$

From (5.3) and (5.14) we get

$$\|v\|_{0,\Omega_t}^2 \le c\left(\|u\|_{0,\Omega_t}^2 + \left(\int_{\Omega_t} |\varrho - \varrho_e||v|dx\right)^2\right). \tag{5.16}$$

Then from (5.8), (5.12), (5.15) and (5.16) we get (5.2)

LEMMA 5.2. *Let $\Omega_t \subset \mathbb{R}^3$ be a bounded domain. Let (v, ϱ_σ) be a solution of* $(1.1)_1$, $(1.1)_2$, $(1.5)_1$ *and $f = \int_\Omega v_0 dx = \int_\Omega \varrho_0 v_0 \cdot \varphi_i dx$, $i = 1, 2, 3$ and*

$$\mathbb{E}_{\Omega_t}(v_t) = \int_{\Omega_t} (\partial_{x_i} v_{jt} + \partial_{x_j} v_{it})^2 dx < \infty. \tag{5.17}$$

Then there exists a constant $c > 0$ such that

$$\|v_t\|_{1,\Omega_t}^2 \le c\left(\mathbb{E}_{\Omega_t}(v_t) + \|v\|_{2,\Omega_t}^4 + \left(\int_{\Omega_t} |\varrho - \varrho_e|\,|v_t|dx\right)^2\right). \tag{5.18}$$

Proof. Introduce a function u by

$$u = \sum_{i=1}^{3} b_i \varphi_i(x) + v_t, \tag{5.19}$$

where $\varphi_i = (x - \bar{x}) \times e_i$, $\bar{x} = \frac{1}{|\Omega_t|}\left(\int_{\Omega_t} x_1 dx, \int_{\Omega_t} x_2 dx, \int_{\Omega_t} x_3 dx\right)$, $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$, $i = 1, 2, 3$.

Define $b = (b_1, b_2, b_3)$ by

$$b = \frac{1}{2|\Omega_t|}\int_{\Omega_t} \mathrm{rot} v_t dx. \tag{5.20}$$

Since rot $\varphi_i = 2e_i$, $i = 1, 2, 3$, equations (5.19) and (5.20) imply

$$\int_{\Omega_t} \mathrm{rot} u dx = 0. \tag{5.21}$$

From (5.20) we have $\int_{\Omega_t} \varphi_i dx = 0$, $i = 1, 2, 3$ so

$$\int_{\Omega_t} udx = \int_{\Omega_t} v_t dx \quad \text{and} \quad \mathbb{E}_{\Omega_t}(\varphi_i) = 0, \quad i = 1, 2, 3, \tag{5.22}$$

so

$$\mathbb{E}_{\Omega_t}(u) = \mathbb{E}_{\Omega_t}(v_t). \tag{5.23}$$

By Theorem 1 of [9] we have

$$\partial_{x_j} w_i = \varepsilon_{ikl}\partial_{x_k} S_{jl}, \quad i = 1, 2, 3, \quad w = \mathrm{rot} u, \tag{5.24}$$

$S_{ij} = \partial_{x_i} u_j + \partial_{x_j} u_i$, so by (5.21) and Lemma 2.4 of [8] it follows that

$$\|\mathrm{rot} u\|_{0,\Omega_t}^2 \leq c \sum_{i,j=1}^{3} \|S_{ij}\|_{0,\Omega_t}^2 = c\mathbb{E}_{\Omega_t}(u) = c\mathbb{E}_{\Omega_t}(v_t). \tag{5.25}$$

Employing the identity

$$\partial_{x_j} u_i = \frac{1}{2}(\partial_{x_j} u_i + \partial_{x_i} u_j) + \frac{1}{2}(\partial_{x_j} u_i - \partial_{x_i} u_j)$$

and (5.25) we have

$$\|\nabla u\|_{0,\Omega_t}^2 \leq c(\mathbb{E}_{\Omega_t}(u) + \|\mathrm{rot} u\|_{0,\Omega_t}^2) \leq c\mathbb{E}_{\Omega_t}(u) = c\mathbb{E}_{\Omega_t}(v_t). \tag{5.26}$$

Using (5.19) we obtain

$$\|\nabla v_t\|_{0,\Omega_t}^2 \leq c(\mathbb{E}_{\Omega_t}(v_t) + |b|^2). \tag{5.27}$$

Integrating $(3.2)_1$ over Ω_t we get

$$\int_{\Omega_t} v_t \varrho_e dx = -\int_{\Omega_t} \varrho v \cdot \nabla v dx - \int_{\Omega_t} (\varrho - \varrho_e) v_t dx \tag{5.28}$$

and multiplying $(3.2)_1$ by φ_i, $i = 1, 2, 3$ and integrating over Ω_t using (5.20) we get systems of equations

$$\begin{aligned}\sum_{j=1}^{3} b_j \int_{\Omega_t} \varrho_e \varphi_i \cdot \varphi_j dx = \int_{\Omega_t} \varrho_e u \cdot \varphi_i dx + \int_{\Omega_t} \varrho v \cdot \nabla v \cdot \varphi_i dx \\ + \int_{\Omega_t} (\varrho - \varrho_e) v_t \cdot \varphi_i dx.\end{aligned} \tag{5.29}$$

Since $\det \Gamma \neq 0$, where $\Gamma = \{\Gamma_{ij}\}$, $\Gamma_{ij} = \int_{\Omega_t} \varphi_i \varphi_j dx$, we can calculate b from (5.29), so

$$|b|^2 \leq c\bigg(\|u\|_{0,\Omega_t}^2 + \|v\|_{2,\Omega_t}^4 + \bigg(\int_{\Omega_t} |\varrho - \varrho_e||v_t| dx\bigg)^2\bigg). \tag{5.30}$$

Now by Poincaré inequality and (5.22), (5.23), (5.28) we obtain

$$\begin{aligned}\|u\|_{0,\Omega_t}^2 &\leq 2\bigg\|u - \frac{1}{|\Omega_t|}\int_{\Omega_t} u dx\bigg\|_{0,\Omega_t}^2 + 2\bigg\|\frac{1}{|\Omega_t|}\int u dx\bigg\|_{0,\Omega_t}^2 \\ &\leq \bigg(\|\nabla u\|_{0,\Omega_t}^2 + \bigg\|\frac{1}{|\Omega_t|}\int_{\Omega_t} v_t dx\bigg\|_{0,\Omega_t}^2\bigg) \\ &\leq c\bigg(\mathbb{E}_{\Omega_t}(u) + \|v\|_{2,\Omega_t}^4 + \bigg(\int_{\Omega_t} |\varrho - \varrho_e||v_t| dx\bigg)^2\bigg).\end{aligned} \tag{5.31}$$

From (5.19) we get

$$\|v_t\|_{0,\Omega_t}^2 \leq c(\|u\|_{0,\Omega_t}^2 + |b|^2). \tag{5.32}$$

Then from (5.27), (5.30), (5.31) and (5.32) we get (5.18). ■

Similarly as Lemma 5.2 we prove

LEMMA 5.3. *Let* $\Omega_t \subset \mathbb{R}^3$ *be a bounded domain. Let* (v, ϱ_σ) *be a solution of* $(1.1)_1$, $(1.1)_2$, $(1.5)_1$ *and* $f = \int_\Omega v_0 dx = \int_\Omega \varrho_0 v_o \cdot \varphi_i dx$, $i = 1,2,3$ *and*

$$\mathbb{E}_{\Omega_t}(v_{tt}) = \int_{\Omega_t} (\partial_{x_i} v_{jtt} + \partial_{x_j} v_{itt})^2 dx < \infty. \tag{5.33}$$

Then there exists constant $c > 0$ *such that*

$$\|v_{tt}\|^2_{1,\Omega_t} \le c\bigg(\mathbb{E}_{\Omega_t}(v_{tt}) + \|\varrho_{\sigma t}\|^2_{1,\Omega_t} \|v\|^4_{1,\Omega_t} + \|v\|^4_{1,\Omega_t} + \|v\|^2_{1,\Omega_t}\|v_t\|^2_{1,\Omega_t} + \|v_t\|^2_{0,\Omega_t}\|\varrho_{\sigma t}\|^2_{0,\Omega_t} + \bigg(\int_{\Omega_t} |\varrho - \varrho_e|\, |v_{tt}| dx \bigg)^2 \bigg). \tag{5.34}$$

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