ON THE STOKES AND NAVIER-STOKES FLOWS IN A PERTURBED HALF-SPACE

TAKAYUKI KUBO and YOSHIHIRO SHIBATA

Department of Mathematical Sciences, School of Science and Engineering, Waseda University
3-4-1, Ohkubo Shinjuku-ku, Tokyo 169-8555, Japan
E-mail: kubo@aoni.waseda.jp, yshibata@waseda.jp

Abstract. We give the $L_p$-$L_q$ estimate for the Stokes semigroup in a perturbed half-space and some global in time existence theorems for small solutions to the Navier-Stokes equation.

1. Background. The non-stationary Stokes system is given by the equations:

(S) \begin{align*}
\vec{u}_t - \Delta \vec{u} + \nabla \pi &= \vec{f}, \quad \text{div} \vec{u} = 0, \quad \text{in} \ (0, T) \times \Omega, \\
\vec{u}|_{\partial \Omega} &= 0, \quad \vec{u}|_{t=0} = \vec{a}
\end{align*}

with unknown velocity $\vec{u} = T(u_1, \ldots, u_n)$ and pressure $\pi$ (scalar function) in some domain $\Omega \subset \mathbb{R}^n (n \geq 2)$, whose boundary is denoted by $\partial \Omega$ and assumed to be a $C^{1,1}$ hypersurface at least. Here and hereafter, $TM$ means the transposed $M$ and $n$-vectors of functions are denoted by letters with arrow. If we define the spaces $J_p(\Omega)$ and $G_p(\Omega)$ by the relations:

\begin{align*}
J_p(\Omega) &= \text{the closure of} \ \{ \vec{u} \in C_0^\infty(\Omega)^n \ | \ \text{div} \vec{u} = 0 \ \text{in} \ \Omega \} \ \text{in} \ L_p(\Omega)^n, \\
G_p(\Omega) &= \{ \nabla \pi \in L_p(\Omega)^n \ | \ \pi \in L_{p, \text{loc}}(\Omega) \},
\end{align*}

we know the unique decomposition (so called Helmholtz decomposition)

(HD) \[ L_p(\Omega)^n = J_p(\Omega) \oplus G_p(\Omega) \]

with a linear continuous projection $P : L_p(\Omega)^n \to J_p(\Omega)$ for many types of domains (cf. Fujiwara and Morimoto [18], Farwig and Sohr [16], [17], Galdi [19], Miyakawa [32], Simader and Sohr [37] and references therein). Then, we can define the Stokes operator

2000 Mathematics Subject Classification: Primary 76D07; Secondary 35B40.

Key words and phrases: Stokes equation, perturbed half space, $L_p$-$L_q$ estimate, Navier-Stokes equation.

Research of the second author supported by Grant-in-Aid for Scientific Research (B)-12440055, Ministry of Education, Sciences, Sports and Culture, Japan.

The paper is in final form and no version of it will be published elsewhere.
by

\[(SO) \quad A = P(-\Delta)\]

with definition domain:

\[(SD) \quad D_p(A) = \{ \vec{u} \in J_p(\Omega) \cap W_p^2(\Omega)^n \mid \vec{u} \mid_{\partial \Omega} = 0 \} \].

Having the Stokes operator \(A\) in hand, the non-stationary Stokes equation \((S)\) can be formulated as an ordinary differential equation in the Banach space \(J_p(\Omega)\):

\[(O) \quad \vec{u}'(t) + A\vec{u}(t) = P\vec{f}(t), \quad \vec{u}(0) = \vec{a}.\]

Hence, the question is whether \(A\) generates an analytic semigroup. Through the Laplace transform, this question is related to the resolvent estimate:

\[(R) \quad |\lambda||| (\lambda + A)^{-1} \vec{f}||_{L_p(\Omega)} + ||(\lambda + A)^{-1} \vec{f}||_{W_p^2(\Omega)} \leq C_{\epsilon,p} ||\vec{f}||_{L_p(\Omega)}\]

for \(\lambda \in \Sigma_\epsilon = \{ z \in \mathbb{C} \setminus \{0\} \mid |\arg z| \leq \pi - \epsilon \}\) with some \(\epsilon \in (0, \pi/2)\), where \(1 < p < \infty\). In fact, once obtaining \((R)\), we have the representation formula:

\[(Rp) \quad T(t)\vec{f} = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}(\lambda + A)^{-1}\vec{f} d\lambda, \quad \vec{f} \in J_p(\Omega)\]

where \(\Gamma = \{ \lambda = e^{i\theta} s \mid s \geq \epsilon \} \cup \{ \lambda = e^{-i\theta} s \mid s \geq \epsilon \} \cup \{ \lambda = e^{i\theta} s \mid -\theta \leq s \leq \theta \}\) with some \(\theta \in (\pi/2, \pi)\) and \(\epsilon > 0\), which combined with \((R)\) implies not only the generation of the analytic semigroup \(\{T(t)\}_{t \geq 0}\) by \(A\) but also the semigroup estimates:

\[(SE) \quad ||T(t)\vec{a}||_{L_p(\Omega)} \leq C_p e^{\epsilon t} ||\vec{a}||_{L_p(\Omega)}\]

\[(SE) \quad ||T(t)\vec{a}||_{W_p^2(\Omega)} \leq C_p e^{\epsilon t} \||\vec{a}||_{L_p(\Omega)}\]

for any \(t > 0\) (cf. Pazy [33]).

Concerning the references for \((R)\), when \(\Omega = \mathbb{R}^n\), since the Helmholtz projection commutes with the Laplacian, the resolvent estimate \((R)\) is reduced to that for the Laplacian. The case of the half-space \(\Omega = \mathbb{R}^n_+\) was settled by McCracken [30], where

\[\mathbb{R}^n_+ = \{ x = (x_1, \ldots, x_n) \in \mathbb{R} \mid x_n > 0 \},\]

and the case of bounded domains by Giga [20] and Solonnikov [38]. The case of exterior domains was treated by Borchers and Sohr [7], Farwig and Sohr [16], Borchers and Varnhorn [9] and Varnhorn [42]. When \(\Omega\) is a perturbed half-space which is a domain such that \(\Omega \cap B^R = \mathbb{R}^n_+ \cap B^R\) for some \(R > 0\) where \(B^R = \{ x \in \mathbb{R}^n \mid |x| > R \}\), \((R)\) was proved by Farwig and Sohr [16]. The case of cones in \(\mathbb{R}^3\) was settled by Deuring [15]. The case of aperture domains was settled by Farwig and Sohr [17]. The case of infinite layers like \(\mathbb{R}^{n-1} \times (-1,1)\) was settled by Wiegner [5] and Abe and Shibata [1] and [2]. The case of the asymptotically flat layer was settled by Abels [4].

To obtain \(L_p - L_q\) estimates:

\[(1) \quad ||T(t)\vec{a}||_{L_q(\Omega)} \leq C_{p,q} e^{\epsilon t} t^{-\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} ||\vec{a}||_{L_p(\Omega)}\]

\[(2) \quad ||\nabla T(t)\vec{a}||_{L_q(\Omega)} \leq C_{p,q} e^{\epsilon t} t^{-\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{1}{2}} ||\vec{a}||_{L_p(\Omega)}\]
for \( t > 0 \) and \( 1 \leq p \leq q \leq \infty \) \((p \neq \infty, q \neq 1)\), we combine (SE) with the Sobolev inequality:
\[
\|u\|_{W^q_p(\Omega)} \leq C\|\nabla^m u\|_{L^p(\Omega)}^{\alpha} \|u\|_{L^p(\Omega)}^{1-a} + \|u\|_{L^p(\Omega)}
\]
provided that \( 0 \leq j < m, 1 \leq p < \infty, m-j-n/p \) is not non-negative integer, \( j/m < a \leq 1 \) and \( 1/q = j/n+1/p - am/n \geq 0 \). The estimates (1) and (2) play an important role in the study of Navier-Stokes equation. In fact, by using the Stokes semigroup, we can reduce the Navier-Stokes equation:
\[
(\text{NS}) \quad \begin{cases}
\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = \nabla \pi + \Delta \vec{u}, \quad \text{div} \vec{u} = 0 \quad \text{in} \ (0, T) \times \Omega \\
\vec{u}|_{\partial \Omega} = 0, \quad \vec{u}|_{t=0} = \vec{a}
\end{cases}
\]
to the integral equation:
\[
(1) \quad \vec{u}(t) = T(t)\vec{a} - \int_0^t T(t-s)P((\vec{u}(s) \cdot \nabla)\vec{u}(s)) \, ds,
\]
where we have set
\[
(\vec{v} \cdot \nabla)\vec{w} = T\left(\left(\sum_{j=1}^{n} v_j \partial_j\right)w_1, \ldots, \left(\sum_{j=1}^{n} v_j \partial_j\right)w_n\right), \quad \partial_j = \partial/\partial x_j,
\]
for the vectors of functions \( \vec{v} = T(v_1, \ldots, v_n) \) and \( \vec{w} = T(w_1, \ldots, w_n) \). Employing the argument due to Kato \[24\] and using (1) and (2) we can prove the locally in time existence theorem of (1). More precisely, we see that for any initial data \( \vec{a} \in J_n(\Omega) \) there exists a time \( t_0 > 0 \) such that the integral equation (1) admits a unique solution \( \vec{u}(t) \in C^0(\{0, t_0\}, J_n(\Omega)) \) with \( \nabla \vec{u}(t) \in C^0(\{0, t_0\}, L_n(\Omega)) \) (cf. Giga and Miyakawa \[21\]).

However, in proving a globally in time existence of solutions to (1) at least with small initial data as well as in the study of time-asymptotic behaviour, we have to show (1) and (2) without \( e^{ct} \). To show this, we need more precise analysis of \( (\lambda + A)^{-1} \) near \( \lambda = 0 \). That \( \lambda = 0 \) is in the resolvent set was derived in the bounded domain case by Giga \[20\] and Solonnikov \[38\], and in the infinite layer case by Abe and Shibata \[2\], which implies that (1) and (2) hold, replacing \( e^{ct} \) by \( e^{-ct} \) with some constant \( c > 0 \).

When \( \Omega = \mathbb{R}^n \), applying the Young inequality to the concrete solution formula, we have (1) and (2) without \( e^{ct} \), namely
\[
(4) \quad \|T(t)\vec{a}\|_{L^q_p(\Omega)} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})} \|\vec{a}\|_{L^p(\Omega)}, \quad \forall t > 0,
\]
\[
(5) \quad \|\nabla T(t)\vec{a}\|_{L^q_p(\Omega)} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{1}{2}} \|\vec{a}\|_{L^p(\Omega)}, \quad \forall t > 0,
\]
for \( 1 \leq p \leq q \leq \infty \) \((p \neq \infty, q \neq 1)\). When \( \Omega = \mathbb{R}^n_+ \), applying the Fourier multiplier theorem to the concrete solution formula obtained by Ukai \[41\] and using the Sobolev inequality:
\[
(6) \quad \|\nabla^j u\|_{L^q_p(\mathbb{R}^n_+)} \leq C\|\nabla^m u\|_{L^p(\mathbb{R}^n_+)}^{\alpha} \|u\|_{L^p(\mathbb{R}^n_+)}^{1-a}
\]
provided that \( 0 \leq j < m, 1 \leq p < \infty, m-j-n/p \) is not non-negative integer, \( j/m < a \leq 1 \) and \( 1/q = j/n+1/p - am/n \geq 0 \), we have (4) and (5) for \( 1 \leq p \leq q \leq \infty \) \((p \neq \infty, q \neq 1)\) (cf. Borchers and Miyakawa \[8\] and Desch, Hieber and Prüss \[14\]).
When Ω is an exterior domain, (4) holds for $1 \leq p \leq q < \infty$ ($p \neq \infty$, $q \neq 1$) but (5) holds only for $1 \leq p \leq q \leq n$ ($q \neq 1$). This result was first proved by Iwashita [23] for $1 < p \leq q < \infty$ in (4) and $1 < p \leq q \leq n$ in (5) when $n \geq 3$. The refinement of his result was done by the following authors: Chen [10] ($n = 3, q = \infty$), Shibata [35] ($n = 3, q = \infty$), Borchers and Varnhorn [9] ($n = 2$, (4) for $p = q$), Dan and Shibata [11], [12] ($n = 2$), Dan, Kobayashi and Shibata [13] ($n = 2, 3$), and Maremonti and Solonnikov [31] ($n \geq 2$). Especially, that Iwashita’s restriction: $q \leq n$ in (5) is unavoidable was shown by Maremonti and Solonnikov [31].

When Ω is an aperture domain, Abels [3] proved (4) for $1 < p \leq q < \infty$ and (5) for $1 < p \leq q < n$ when $n \geq 3$; and Hishida [22] proved (4) for $1 \leq p \leq q \leq \infty$ and (5) for $1 \leq p \leq q \leq n$ ($q \neq 1$) and $1 \leq p < n < q < \infty$ when $n \geq 3$.

Moreover, (I) was solved globally in time for small initial data in $J_n(\Omega)$ by using (4) and (5) in the following papers: Kato [24] in the whole space case; Ukai [41] and Borchers and Miyakawa [8] in the half-space case; Iwashita [23] and Dan and Shibata [11] in the exterior domain cases; Abe and Shibata [2] in the infinite layer case; Hishida [22] in the aperture domain case.

In this paper, we report on the results about (4) and (5) and the related topics in the perturbed half-space cases. And also, we report on some results on the Navier-Stokes flow in the perturbed half-space case. The detailed proofs of the results stated below are found in the papers due to Kubo and Shibata [28] and [29].

2. Notation. Before stating our main theorem precisely, we outline our notation used throughout the paper. If $X$ is a subset in the complex number field $\mathbb{C}$ or functional space, then $X^n$ denotes the $n$-th product:

$$X^n = \{(x_1, \ldots, x_n) \mid x_j \in X, j = 1, \ldots, n\}.$$  

If $X$ be a subset of $\mathbb{C}$, then $X \setminus (-\infty, 0]$ is defined by

$$X \setminus (-\infty, 0] = X \setminus \{x + i0 \in \mathbb{C} \mid -\infty < x \leq 0\}.$$  

Given an $n$-vector of functions $\vec{v} = T(v_1, \ldots, v_n)$ and point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we set

$$\vec{v}' = T(v_1, \ldots, v_{n-1}), \quad x' = (x_1, \ldots, x_{n-1}).$$  

The dot $\cdot$ stands for the usual inner products both of $\mathbb{R}^n$ and of $\mathbb{R}^{n-1}$. Given $R > 0$, we set

$$B_R = \{x \in \mathbb{R}^n \mid |x| < R\}, \quad B^R = \{x \in \mathbb{R}^n \mid |x| > R\}, \quad B_R^+ = B_R \cap \mathbb{R}_+^n.$$  

For the differentiation, we use the symbols:

$$\partial_x^\alpha u = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u \text{ for } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n,$$

$$\partial_{x'}^\alpha u = \partial_{x_1'}^{\alpha_1} \cdots \partial_{x_{n-1}'}^{\alpha_{n-1}} u \text{ for } \alpha' = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^{n-1},$$

$$\partial_x^\alpha \vec{u} = T(\partial_x^\alpha u_1, \ldots, \partial_x^\alpha u_n), \quad \partial_{x'}^\alpha \vec{u} = T(\partial_{x'}^\alpha u_1, \ldots, \partial_{x'}^\alpha u_n)$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}$ is the set of all natural numbers, and moreover we set

$$\nabla^j u = (\partial_x^\alpha u \mid |\alpha| = j), \quad \nabla u = \nabla^1 u, \quad \nabla^j \vec{u} = T(\nabla^j u_1, \ldots, \nabla^j u_n).$$
Sobolev spaces of vector-valued functions are used as well as of scalar functions. Thus, given a domain $D$ in $\mathbb{R}^n$, $\| \cdot \|_{L_p(D)}$ denotes the usual $L_p$ norm on $D$ and we set

$$\|u\|_{W^{m,p}(D)} = \sum_{|\alpha| \leq m} \|\partial_x^\alpha u\|_{L_p(D)}, \quad \|\bar{u}\|_{L_p(D)} = \sum_{j=1}^n \|u_j\|_{L_p(D)}, \quad \|\bar{u}\|_{W^{m,p}(D)} = \sum_{j=1}^m \|u_j\|_{W^{m,p}(D)}.$$  

$L_p(D)$ denotes the usual $L_p$ space on $D$ and $C_0^\infty(D)$ the set of all functions in $C^\infty(\mathbb{R}^n)$ whose support is compact and contained in $D$. Therefore, we know that the Stokes operator ($SO$) with domain ($SD$) generates the resolvent estimate. Sohr [16] proved the Helmholtz decomposition (HD) and the resolvent estimate (R) on $\mathbb{R}$.

By $C_{A,B,\ldots...}$ we denote the constants depending on the quantities $A, B, \ldots...$. For two Banach spaces $X$ and $Y$, $\mathscr{L}(X,Y)$ denotes the set of all bounded linear operators from $X$ into $Y$. $\mathcal{C}((U_\epsilon, X)$ denotes the set of all $X$-valued holomorphic functions defined on $U_\epsilon = \{z \in \mathbb{C} \mid |z| < \epsilon\}$. $BC(I; X)$ and $C^k(I; X)$ denote the set of all $X$-valued bounded continuous functions and $C^k$ functions defined on $I$, respectively.

3. Main results about Stokes flow in the perturbed half-space. In this section, we will state our main results concerning the Stokes system ($S$) in the half-space and the perturbed half-space, which is defined as follows.

**Definition 1.** (1) The half-space $\mathbb{R}^n_+$ is defined by

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$  

(2) Let $\Omega$ be a domain in $\mathbb{R}^n$. We call $\Omega$ a perturbed half-space if there exists a number $R > 0$ such that

$$(7) \quad \Omega \cap B^R = \mathbb{R}^n_+ \cap B^R.$$  

As we already stated in section 1, when $\Omega$ is a perturbed half-space, Farwig and Sohr [16] proved the Helmholtz decomposition (HD) and the resolvent estimate (R) on $\Omega$. Therefore, we know that the Stokes operator ($SO$) with domain ($SD$) generates the analytic semigroup $\{T(t)\}_{t \geq 0}$ on $J_p(\Omega)$. Then, we have the following theorem.

**Theorem 1.** Let $\Omega$ be a perturbed half-space in $\mathbb{R}^n (n \geq 2)$ whose boundary $\partial \Omega$ is a $C^{1,1}$ hypersurface. Then, the Stokes semigroup $\{T(t)\}_{t \geq 0}$ satisfies the following two estimates:

$$(8) \quad \|T(t)a\|_{L_p(\Omega)} \leq C_{p,q,t} t^{-\frac{n}{2} \left(\frac{1}{2} - \frac{1}{p}\right)} \|a\|_{L_p(\Omega)},$$  

$$(9) \quad \|\nabla T(t)a\|_{L_p(\Omega)} \leq C_{p,q,t} t^{-\frac{n}{2} \left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}} \|a\|_{L_p(\Omega)},$$  

for any $a \in J_p(\Omega)$, $t > 0$ and $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$).

The main step in our proof of Theorem 1 is to show the following local energy decay estimate.

**Theorem 2.** Let $\Omega$ be a perturbed half-space in $\mathbb{R}^n (n \geq 2)$ whose boundary $\partial \Omega$ is a $C^{1,1}$ hypersurface. Let $1 < p < \infty$ and $R$ be a number such that (7) holds. Then, the Stokes
semigroup \( \{T(t)\}_{t \geq 0} \) satisfies the following estimate:

\[
\|\partial_t^j T(t) P\bar{a}\|_{W^2_p(\Omega \cap B_R^+)} \leq C_{p,R} t^{-\frac{n+1}{2} - j} \|\bar{a}\|_{L_p(\Omega)}
\]

for any \( t \geq 1, \ j \in \mathbb{N}_0 \) and \( \bar{a} \in L_{p,R}(\Omega) \).

If we consider the Stokes system in the half-space:

\[
\bar{v}_t - \Delta \bar{v} + \nabla \pi = 0, \quad \text{div} \bar{v} = 0 \quad \text{in} \ (0, \infty) \times \mathbb{R}^n_+,
\]

then we know by Ukai [41] and Borchers and Miyakawa [8] that the solution \( \bar{v} \) of (11) satisfies the \( L_p - L_q \) estimate:

\[
\|\bar{v}(t)\|_{L_q(\mathbb{R}^n_+)} \leq C_{p,q} t^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \|\bar{b}\|_{L_p(\mathbb{R}^n_+)},
\]

\[
\|\nabla \bar{v}(t)\|_{L_q(\mathbb{R}^n_+)} \leq C_{p,q} t^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{q} \right) - \frac{1}{2}} \|\bar{b}\|_{L_p(\mathbb{R}^n_+)}
\]

for any \( t > 0 \) and \( 1 \leq p \leq q \leq \infty \) (\( p \neq \infty, q \neq 1 \)). Since

\[
\|\bar{v}(t)\|_{L_p(B_R^+)} \leq C_R \|\nabla \bar{v}(t)\|_{L_p(B_R^+)}
\]

as follows from the boundary condition: \( \bar{v}|_{x_n = 0} = 0 \), using (13) and Theorem 2, we have

\[
\|T(t) P\bar{a}\|_{W^2_p(\Omega \cap B_R^+)} \leq C_{p,R} t^{-\frac{n}{2p} - \frac{1}{2}} \|\bar{a}\|_{L_p(\Omega)}
\]

for any \( \bar{a} \in L_p(\Omega) \) and \( t \geq 1 \). Combining (12), (13) and (14) by the cut-off technique and following the argument due to Hishida [22, the proof of Theorem 2.1], we can show Theorem 1.

In order to prove Theorem 2, we need some precise information about solutions to the resolvent problem in \( \mathbb{R}^n_+ \):

\[
(\lambda - \Delta) \bar{w} + \nabla \theta = \bar{f}, \quad \text{div} \bar{w} = 0 \quad \text{in} \ \mathbb{R}^n_+,
\]

\[
\bar{w}|_{x_n = 0} = 0,
\]

which is stated in the following two theorems.

**Theorem 3.** Let \( R(\lambda) \) and \( \Pi(\lambda) \) denote the solution operators of (15) which are defined by

\[
\bar{w} = R(\lambda) \bar{f} = T(R_1(\lambda) \bar{f}, \ldots, R_n(\lambda) \bar{f}) \quad \text{and} \quad \theta = \Pi(\lambda) \bar{f}
\]

for \( \lambda \in \mathbb{C} \setminus (-\infty, 0] \). Let \( R > 0, 1 < p < \infty \) and set

\[
\mathcal{B}^j_{p,R} = \mathcal{L}(L_p(B_{R}^+)^n, W^j_p(B_{R}^+))
\]

for \( j = 1, 2 \). Then there exist operators \( G^j_{\pi}(\lambda) \in \mathcal{A}(U_{1/16}, R^2_{p,R}), k = 1, 2, 3, j = 1, \ldots, n, \) and \( G^k_{\pi}(\lambda) \in \mathcal{B}^1_{p,R}, k = 1, 2, 3 \) such that

\[
R_j(\lambda) \bar{f} = \lambda^{-\frac{n+1}{2}} G^1_j(\lambda) \bar{f} + (\lambda^{\frac{n}{2}} \log \lambda) G^2_j(\lambda) \bar{f} + G^3_j(\lambda),
\]

\[
\Pi(\lambda) \bar{f} = \lambda^{-\frac{n+1}{2}} G^1_\pi(\lambda) \bar{f} + (\lambda^{\frac{n}{2}} \log \lambda) G^2_\pi(\lambda) \bar{f} + G^3_\pi(\lambda),
\]

\[
(\lambda - \Delta) \bar{w} + \nabla \theta = \bar{f}, \quad \text{div} \bar{w} = 0 \quad \text{in} \ \mathbb{R}^n_+,
\]

\[
\bar{w}|_{x_n = 0} = 0,
\]
Theorem 5. Let $n \geq 2$. There exists a constant $\delta = \delta(\Omega, n) > 0$ with the following property: if $\vec{a} \in J^n(\Omega)$ satisfies $\|\vec{a}\|_{L_n(\Omega)} \leq \delta$, then the integral equation

$$
\vec{u}(t) = T(t)\vec{a} - \int_0^t T(t-s)P((\vec{u}(s) \cdot \nabla)\vec{u}(s))ds
$$

The Navier-Stokes flow in a perturbed half-space. Following the arguments due to Kato [24], Kozono [25], Hishida [22] and Wiegner [43] and using Theorem 1 we can show the following theorem.

**Theorem 5.** Let $n \geq 2$. There exists a constant $\delta = \delta(\Omega, n) > 0$ with the following property: if $\vec{a} \in J^n(\Omega)$ satisfies $\|\vec{a}\|_{L_n(\Omega)} \leq \delta$, then the integral equation

$$
\vec{u}(t) = T(t)\vec{a} - \int_0^t T(t-s)P((\vec{u}(s) \cdot \nabla)\vec{u}(s))ds
$$

in $B_R^+$ when $n \geq 2$ and $n$ is even; and

$$
R_j(\lambda)\vec{f} = \lambda^2 G_j^1(\lambda)\vec{f} + (\lambda^{\frac{n+1}{2}} \log \lambda) G_j^2(\lambda)\vec{f} + G_j^3(\lambda),
$$

Equation (17) $\Pi(\lambda)\vec{f} = \lambda^2 G_{\pi}^1(\lambda)\vec{f} + (\lambda^{\frac{n+1}{2}} \log \lambda) G_{\pi}^2(\lambda)\vec{f} + G_{\pi}^3(\lambda),$

in $B_R^+$ when $n \geq 3$ and $n$ is odd, provided that $\lambda \in U_{1/16} \setminus (-\infty, 0]$ and $\vec{f} \in L_{p,R}(\Omega)$.

**Theorem 4.** Let $1 < p < \infty$, $0 < \epsilon < \pi/2$, and let $R(\lambda)$ and $\Pi(\lambda)$ be the operators given in Theorem 3 for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. Let $\Sigma_\epsilon$ be the set in $\mathbb{C}$ defined by

$$
\Sigma_\epsilon = \{ \lambda \in \mathbb{C} \setminus \{0\} \mid \arg \lambda \leq \pi - \epsilon \}.
$$

Then, there exist operators $R(0) \in \mathcal{L}(L_{p,R}(\mathbb{R}^n_+), W^2_{p,loc}(\mathbb{R}^n_+))$ and $\Pi(0) \in \mathcal{L}(L_{p,R}(\mathbb{R}^n_+), W^1_{p,loc}(\mathbb{R}^n_+))$ which satisfy the following three conditions:

(i) Given $\vec{f} \in L_{p,R}(\mathbb{R}^n_+)$, $\vec{v} = R(0)\vec{f}$ and $\theta = \Pi(0)\vec{f}$ satisfy the equation:

$$
- \Delta \vec{v} + \nabla \theta = \vec{f}, \quad \text{div} \vec{v} = 0 \quad \text{in} \mathbb{R}^n_+, \quad \vec{v}|_{x_n=0} = 0.
$$

(ii) $\|R(\lambda)\vec{f} - R(0)\vec{f}\|_{W^1_{p,R}(\mathbb{R}^n_+)} + \|\Pi(\lambda)\vec{f} - \Pi(0)\vec{f}\|_{L_{p,R}(\mathbb{R}^n_+)} \leq C_{p,R,\epsilon} |\lambda|^\frac{1}{2} \|\vec{f}\|_{L_p(\mathbb{R}^n)}$

for any $\vec{f} \in L_{p,R}(\mathbb{R}^n_+)$ and $\lambda \in \Sigma_\epsilon$ with $|\lambda| \leq 1/16$, where $C_{p,R,\epsilon}$ is a constant independent of $\vec{f}$ and $\lambda$.

(iii) $[R(0)\vec{f}](x) \leq C_{p,R} |x|^{-(n-1)} \|\vec{f}\|_{L_p(\mathbb{R}^n)},$

$\|\nabla [R(0)\vec{f}](x)\| \leq C_{p,R} |x|^{-(n-1)} \|\vec{f}\|_{L_p(\mathbb{R}^n)},$

$[\Pi(0)\vec{f}](x) \leq C_{p,R} |x|^{-(n-1)} \|\vec{f}\|_{L_p(\mathbb{R}^n)},$

for any $\vec{f} \in L_{p,R}(\mathbb{R}^n_+)$ and $x \in \mathbb{R}^n_+$ with $|x| \geq 2\sqrt{2}R$, where $C_{p,R}$ is a constant independent of $\vec{f}$ and $x$.

Constructing a parametrix of the resolvent problem in a perturbed half-space, we can derive from Theorem 3 and Theorem 4 that the resolvent operator $(\lambda + A)^{-1}$ has the expansion formula of the same type near $\lambda = 0$ in the space $\mathcal{L}(L_{p,R}(\Omega), W^2_{p}(\Omega \cap B_R))$ as in the half-space case, which is applied to $(Rp)$ implies Theorem 2. The detailed proof of Theorems 1 and 2 is given in Kubo and Shibata [29] and that of Theorems 3 and 4 in Kubo-Shibata [28]. The fundamental idea of the proofs of Theorems 1 and 2 by using Theorems 3 and 4 goes back to a paper due to Shibata [34].
admits a unique strong solution $\bar{u}(t) \in BC([0, \infty); J_n(\Omega))$ with $\nabla \bar{u}(t) \in C^0((0, \infty); L_n(\Omega))$. Moreover as $t \to \infty$,
\[
\|\bar{u}(t)\|_{L_r(\Omega)} = o(t^{-\frac{1}{2} + \frac{n}{2r}}) \quad \text{for} \quad n \leq r \leq \infty,
\]
\[
\|\nabla \bar{u}(t)\|_{L_r(\Omega)} = o(t^{-1 + \frac{n}{2r}}) \quad \text{for} \quad n \leq r < \infty.
\]

**THEOREM 6.** Let $n \geq 2$. There exists a constant $\eta = \eta(\Omega, n) \in (0, \delta]$ with the following property: if $\bar{a} \in L_1(\Omega) \cap J_n(\Omega)$ satisfies $\|\bar{a}\|_{L_n(\Omega)} \leq \eta$, then the solution $\bar{u}(t)$ obtained in Theorem 5 satisfies the estimates:
\[
\|\bar{u}(t)\|_{L_r(\Omega)} = O(t^{-\frac{n-2}{2}}) \quad \text{for} \quad 1 < r \leq \infty,
\]
\[
\|\nabla \bar{u}(t)\|_{L_r(\Omega)} = O(t^{-\frac{n-2}{2} - \frac{1}{2}}) \quad \text{for} \quad 1 < r < \infty,
\]

as $t \to \infty$.

5. **On the periodic solution of the Navier-Stokes equation in the perturbed half-space.** We can show that if the incompressible fluid in the perturbed half-space is governed by the periodic external force, the Navier-Stokes equations have a periodic strong solution with the same period as the external force. Let $\Omega$ be a domain in $\mathbb{R}^n(n \geq 3)$. Let us consider the following Navier-Stokes equations in $\Omega$:
\[
\begin{aligned}
\frac{\partial \bar{u}}{\partial t} - \Delta \bar{u} + \bar{u} \cdot \nabla \bar{u} + \nabla \pi = \bar{f}, & \quad x \in \Omega, \; t \in \mathbb{R}, \\
\operatorname{div} \bar{u} = 0, & \quad x \in \Omega, \; t \in \mathbb{R}, \\
\bar{u}|_{\partial \Omega} = 0.
\end{aligned}
\]

(NSP)

Applying the projection operator $P_r$ to both sides of the first equation of (NSP), we have
\[
\frac{d\bar{u}}{dt} + A_r \bar{u} + P_r(\bar{u} \cdot \nabla \bar{u}) = P_r \bar{f}
\]
on $J_r(\Omega)$ for $t \in \mathbb{R}$. The above (E) can be further transformed to the following integral equation:
\[
\bar{u}(t) = \int_{-\infty}^{t} T(t-s)P_r \bar{f}(s)ds - \int_{-\infty}^{t} T(t-s)P_r((\bar{u} \cdot \nabla)\bar{u})(s)ds.
\]

Concerning the external force $\bar{f}$, we impose the following assumption:

**ASSUMPTION 1.** Let the exponents $r$ and $q$ be such as $2 < r < n$, $\frac{n}{2} < q < n$. When $n \geq 4$, we assume that
\[
\bar{f} \in BC(\mathbb{R}; L_p(\Omega) \cap L_\ell(\Omega))
\]
for $1 < p, \ell < \infty$ with $1/r + 2/n < 1/p, 1/q < 1/\ell < 1/q + 1/n$.

When $n = 3$, we assume that
\[
P_p \bar{f}(s) = A_p^* \bar{g}(s) \quad (s \in \mathbb{R}) \quad \text{with some} \; \bar{g} \in BC(\mathbb{R}; D(A_p^*))
\]
for $1 < p < \min(r, q)$ and $\delta > 0$ satisfying $3/2p + \delta \geq \max(1 + 3/2r, 1/2 + 3/2q)$ and that
\[
\bar{f} \in BC(\mathbb{R}; L_\ell(\Omega))
\]
for $1/q < 1/\ell < 1/q + 1/3$. 


Using the method due to Kozono and Nakao [26], we can show the following two theorems in the perturbed half-space case.

**Theorem 7.** Let $\Omega$ and $\vec{f}$ satisfy Assumption 1. Suppose that $\vec{f}(t) = \vec{f}(t + \omega)$ for all $t \in \mathbb{R}$ with some $\omega > 0$. Then there is a constant $\eta = \eta(n, r, q, p, \ell, \delta) > 0$ such that if

$$\sup_{s \in \mathbb{R}} \| P_f \vec{f}(s) \|_{L^p(\Omega)} + \sup_{s \in \mathbb{R}} \| P_\ell \vec{f}(s) \|_{L^\ell(\Omega)} \leq \eta$$

when $n \geq 4$, we have a periodic solution $\vec{u}$ of (I-E) with the same period $\omega$ as $\vec{f}$ in the class $BC(\mathbb{R}; J_r(\Omega))$ with $\nabla \vec{u} \in BC(\mathbb{R}; L_q(\Omega))$.

Such a solution $\vec{u}$ is unique within this class provided $\sup_{s \in \mathbb{R}} \| \vec{u}(s) \|_{L^r(\Omega)} + \sup_{s \in \mathbb{R}} \| \nabla \vec{u}(s) \|_{L^q(\Omega)}$ is sufficiently small.

**Theorem 8.** In addition to the hypotheses of Theorem 7, let us assume furthermore that $\vec{f}$ is a Hölder continuous function on $\mathbb{R}$ with values in $L_n(\Omega)$. Then the periodic solution $\vec{u}$ given by Theorem 7 has the following additional properties:

1. $\vec{u} \in BC(\mathbb{R}; J_n(\Omega)) \cap C^1(\mathbb{R}; J_n(\Omega))$;
2. $\vec{u}(t) \in D(A_n)$ for all $t \in \mathbb{R}$ and $A_n \vec{u} \in C^0(\mathbb{R}; J_n(\Omega))$;
3. $\vec{u}$ satisfies (E) in $J_n(\Omega)$ for all $t \in \mathbb{R}$.

References


