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THE EULERIAN LIMIT AND THE SLIP BOUNDARY CONDITIONS—ADMISSIBLE IRREGULARITY OF THE BOUNDARY

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Abstract. We investigate the inviscid limit for the stationary Navier-Stokes equations in a two dimensional bounded domain with slip boundary conditions admitting nontrivial inflow across the boundary. We analyze admissible regularity of the boundary necessary to obtain convergence to a solution of the Euler system. The main result says that the boundary of the domain must be at least C^2-piecewise smooth with possible interior angles between regular components less than π.

1. Introduction. In this note we investigate a model of a two dimensional stationary flow of a viscous incompressible fluid. The motion is governed by the steady Navier-Stokes equations in a two dimensional bounded domain

$$\begin{array}{ll} v\cdot\nabla v - \operatorname{div}\mathbf{T}(v,p) = 0 & \text{in} \quad \Omega, \\ \operatorname{div} v = 0 & \text{in} \quad \Omega, \end{array} \tag{1.1}$$

where $v = (v^1, v^2)$ is the velocity of the fluid, p the pressure and $\mathbf{T}(v,p)$ is the stress tensor

$$\mathbf{T}(v,p) = \nu\mathbf{D}(v) - pId, \tag{1.2}$$

where ν is the constant positive viscous coefficient and Id the identity matrix; $\mathbf{D}(v)$ is the deformation tensor and

$$\mathbf{D}(v) = \nabla v + (\nabla v)^T = \{v^i_{,j} + v^j_{,i}\}_{i,j=1,2}, \tag{1.3}$$

where the comma denotes differentiation.

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As a supplement to equations (1.1) we take the slip boundary conditions

$$\begin{array}{ll} n\cdot \mathbf{T}(v,p)\cdot \tau + \nu f v\cdot \tau = 0 & \text{on} \quad \partial\Omega, \\ v\cdot n = d & \text{on} \quad \partial\Omega, \end{array} \tag{1.4}$$

where f is the friction function which in general must be nonnegative, n and τ are the normal and tangent vectors to the boundary and d describes the inflow/outflow data. By $(1.1)_2$ we require the following compatibility condition:

$$\int_{\partial\Omega} d\, d\sigma = 0. \tag{1.5}$$

An investigation of the above system has been done in [6], where the inviscid limit of solutions to (1.1)-(1.4) has been studied. The results say that we are able to find a suitable subsequence of solutions $v = v^{\nu_k}$, where v^{ν} denote a solution to (1.1)-(1.4) with viscous coefficient ν, such that it converges to a solution of the Euler system

$$\begin{array}{lll} v\cdot\nabla v + \nabla p = 0 & \text{in} & \Omega, \\ \text{div } v = 0 & \text{in} & \Omega, \\ n\cdot v = d & \text{on} & \partial\Omega. \end{array} \tag{1.6}$$

The connection between these two systems is a consequence of properties of the slip boundary conditions (1.4). Let us recall that for the Dirichlet boundary data, which are the most popular in models from the mathematical part of fluid mechanics, we cannot find a good estimate for solutions to the steady Navier-Stokes equations and we are not able to prove any results concerning the inviscid limit. It is possible only for the evolutionary equations [2, 5, 10, 11 and 12]. One point which should be underlined is that system (1.6), and even its evolutionary version, is under-completed. This follows from inhomogeneity of the boundary data $(1.6)_3$ which implies nonuniqueness of the Euler system, because of its hyperbolic character. To keep uniqueness there is a need to describe the vorticity at a part of the boundary, where $d < 0$. More precise analysis of the Euler system one can find in [13, 14, 15].

That is the reason the more precise analysis of system (1.1)-(1.4) seems to be interesting and worthwhile to investigate. Results from [6] work only for regular domains, i.e. the boundary $\partial\Omega$ is C^2-piecewise smooth and interior angles between smooth components are $\pi/2$. In the present paper we want to generalize this result to a larger class of domains.

The evolutionary case of our issue for the Navier-Stokes equations with slip boundary conditions has been investigated in [1]. The three dimensional version of the problem has been studied in [16, 17].

Throughout the paper we assume that the domain Ω is simply connected and C^2-piecewise smooth. Moreover we admit a finite number of irregular points and we denote them by

$$\mathcal{N} = \{w_1, \dots, w_{N_0}\}. \tag{1.7}$$

Additionally angles between smooth elements for each vertex w_k must be of positive measure. We exclude cases when the angle is zero. By θ_k we denote the angle at vertex w_k and by assumptions we have

$$0 < \theta_1, \theta_2, \dots, \theta_{N_0} < 2\pi \quad \text{and} \quad \theta_{max} = \{\theta_1, \dots, \theta_{N_0}\}. \tag{1.8}$$

Moreover we restrict our attention to domains such that for each $k = 1, \dots, N_0$ there exists a neighborhood $\mathcal{U}_k$ of w_k such that $\Omega \cap \mathcal{U}_k$ is a sector of angle θ_k. This assumption is important to control singularities which may appear at vertex w_k.

An example of the domain can be illustrated by the following picture.

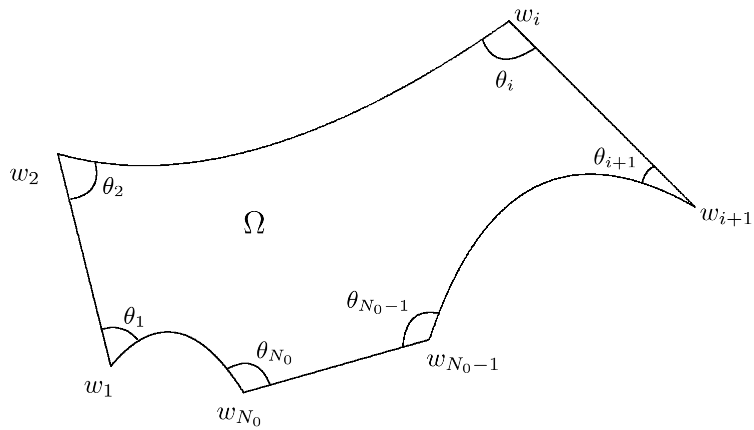

Fig. 1

To explain properties of system (1.1)-(1.4) we construct a reformulation of the system. Assuming existence of sufficiently smooth solutions to problem (1.1)-(1.4) we reformulate the system using the vorticity of the velocity which in the two dimensional case is a scalar function.

Introduce the vorticity of the velocity field

$$\alpha = \text{rot } v = v^2_{,1} - v^1_{,2}. \tag{1.9}$$

We obtain the following reformulation of problem (1.1)-(1.4)

$$\begin{array}{lll} v \cdot \nabla\alpha - \nu\Delta\alpha = 0 & \text{in} & \Omega, \\ \alpha = (2\chi - f)v \cdot \tau - 2d_{,s} & \text{on} & \partial\Omega \end{array} \tag{1.10}$$

and

$$\begin{array}{lll} \text{rot } v = \alpha & \text{in} & \Omega, \\ \text{div } v = 0 & \text{in} & \Omega, \\ n \cdot v = d & \text{on} & \partial\Omega. \end{array} \tag{1.11}$$

To obtain $(1.10)_1$ we take the rotation of $(1.1)_1$. Boundary condition $(1.10)_2$ follows from (1.4) by differentiation of $(1.4)_2$ with respect to the length parameter of the curve $\partial\Omega$ which we denote by s (see [8-Appendix]). To clarify the statement of boundary condition $(1.10)_2$ we specify the meaning of the quantity $d_{,s}$. We will assume that $d \in W^1_\infty(\partial\Omega)$, however by the character of condition $(1.4)_2$ we should not treat this inclusion in the regular sense. The differentiation is well defined only on smooth components of boundary $\partial\Omega$, hence we generalize space $W^1_\infty(\partial\Omega)$ as a sum of functions from W^1_∞ defined in each regular component of $\partial\Omega$. Then $d_{,s}$ can be treated as a regular function belonging to the L_∞-space defined on boundary $\partial\Omega$.

Since we are interested in the inviscid limit, our considerations should be realized on the level of weak solutions. We should not use only information about solutions connected with higher regularity, because it depends on the viscous coefficient.

We introduce the following definition of solutions to problem (1.1)-(1.4).

DEFINITION 1.1. We say that v is a *weak-$*$ solution* to problem (1.1)-(1.4) if and only if

$$\begin{array}{c} v \in C(\Omega), \qquad div\, v = 0, \\ \alpha = rot\, v \in L_\infty(\Omega), \qquad n \cdot v|_{\partial\Omega} = d \end{array} \tag{1.12}$$

and

$$\int_\Omega v \cdot \nabla\phi\alpha dx + \nu \int_\Omega \alpha\Delta\phi dx - \nu \int_{\partial\Omega} ((2\chi - f)v\cdot\tau - 2d_{,s})\frac{\partial\phi}{\partial n} d\sigma = 0 \tag{1.13}$$

for $\phi \in W_1^2(\Omega) \cap \{\phi|_{\partial\Omega} = 0\}$.

As in [6] we will study our system under a geometrical constraint. To specify this condition we introduce the following quantity.

DEFINITION 1.2. Let Ω be sufficiently regular, then we introduce

$$\gamma_\infty(\Omega) = \|\nabla\pi\|_{C(\Omega)}, \tag{1.14}$$

where π is the weak solution to the following problem

$$\begin{array}{lll} \Delta\pi = 1 & \text{in} & \Omega, \\ \pi = 0 & \text{on} & \partial\Omega. \end{array} \tag{1.15}$$

The weak solution is unique, however for a certain class of irregular domains Ω the quantity $\gamma_\infty(\Omega)$ may be infinite.

For domains with regular boundaries we state the following result from [6-Theorems 1.3 and 1.4].

THEOREM 1.1. *Let $\nu > 0$, $f \in L_\infty(\partial\Omega)$ and $d \in W^1_\infty(\partial\Omega)$. Additionally we assume that*

$$\|\gamma_\infty(\Omega)(2\chi - f)\|_{L_\infty(\partial\Omega)} < 1, \tag{1.16}$$

where $\gamma_\infty(\Omega)$ is given by (1.14) and χ is the curvature of $\partial\Omega$; then there exists at least one solution to (1.1)-(1.4) in the sense of Definition 1.1 such that

$$\|rot\, v^\nu\|_{L_\infty(\Omega)} \leq M\|d\|_{W^1_\infty(\partial\Omega)}, \tag{1.17}$$

where M is independent of ν. Additionally there exists a subsequence $\{rot\, v^{\nu_k}\}$ for $\nu_k \to 0$ such that

$$rot\, v^{\nu_k} \rightharpoonup rot\, v_E \quad weakly\text{-}* \quad in\ L_\infty(\Omega), \tag{1.18}$$

where v_E is a solution to the Euler system (1.6) such that

$$\|rot\, v_E\|_{L_\infty(\Omega)} \leq M\|d\|_{W^1_\infty(\partial\Omega)}. \tag{1.19}$$

In our paper we want to consider the irregularity of the boundary. To keep the well posedness of the weak formulation it is enough to assume that the boundary $\partial\Omega$ is C^2-piecewise smooth. Nevertheless this assumption seems to be insufficient. Considering Ω as a polygon, the curvature is zero almost everywhere, however convex and non convex examples of this type of domains seem to have not the same properties - see the picture below.

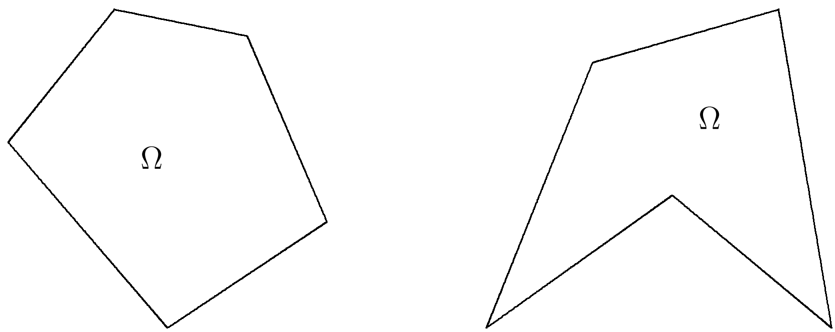

Fig. 2

We want to define admissible irregularity of the boundary. For this purpose we prescribe an approximation of the original problem by suitable globally smooth approximation of the original domain. We introduce the following definition.

DEFINITION 1.3. A domain Ω_0 is called *admissible* if and only if it is C^2-piecewise smooth and there exists a friction function f_0 such that the geometrical constraint is fulfilled, i.e.

$$\|\gamma_\infty(\Omega_0)(2\chi_0 - f_0)\|_{L_\infty(\partial\Omega)} < 1; \tag{1.20}$$

and there exists a sequence of domains Ω_ϵ and friction functions f_ϵ such that

$$\|2\chi_\epsilon - f_\epsilon\|_{L_\infty(\partial\Omega)} \leq \|2\chi_0 - f_0\|_{L_\infty(\partial\Omega)}; \tag{1.21}$$

and

$$\Omega_\epsilon \to \Omega_0 \quad \text{and} \quad f_\epsilon \to f_0 \qquad \text{as} \quad \epsilon \to 0, \tag{1.22}$$

where convergence (1.22) is defined as follows

$$\begin{aligned} &|\Omega_0 \setminus \Omega_\epsilon| + |\Omega_\epsilon \setminus \Omega_0| \to 0 && \text{as} \quad \epsilon \to 0, \\ &|\partial\Omega_0 \setminus \partial\Omega_\epsilon| + |\partial\Omega_\epsilon \setminus \partial\Omega_0| \to 0 && \text{as} \quad \epsilon \to 0, \\ &\|(f_\epsilon - f_0)|_{\partial\Omega_0 \cap \partial\Omega_\epsilon}\|_{L_\infty(\partial\Omega_0 \cap \partial\Omega_\epsilon)} \to 0 && \text{as} \quad \epsilon \to 0, \end{aligned} \tag{1.23}$$

where $|\cdot|$ denotes the Lebesgue measure of two and one dimensional sets, respectively.

The main result of the present paper is the following generalization of Theorem 1.1. Throughout the paper we always assume that $a < 1$.

THEOREM 1.2. *Let $0 < a \leq \pi/\theta_{max} - 1$ and $\partial\Omega$ be C^2-piecewise smooth and suppose interior angles between smooth elements are less than π and condition (1.16) is fulfilled by the domain Ω. Then there exists at least one weak-$*$ solution to problem (1.1)-(1.4) such that*

$$\alpha \in L_\infty(\Omega) \quad \textit{and} \quad v \in C^a(\Omega). \tag{1.24}$$

Moreover there exists a smooth approximation of problem (1.10) such that the approximation sequence tends to the original solution and the following bound is valid

$$\|rot\, v\|_{L_\infty(\Omega)} + \|v\|_{C^a(\Omega)} \leq M\|d\|_{W^1_\infty(\partial\Omega)}, \tag{1.25}$$

where M is independent of viscosity coefficient ν.

The above result extends Theorem 1.1 on a larger class of domains. However nontrivial angles can restrict regularity of velocity ($v \in C^{\pi/\theta_{\max}-1}(\Omega)$ only) which is a consequence of appearance of singular solutions to the elliptic problem in nonsmooth domains. Also it clarifies the sense of boundary condition $(1.10)_2$ for piecewise smooth $\partial\Omega$.

A consequence of Theorem 1.2 is the following result concerning the inviscid limit for solutions of (1.1)-(1.4).

THEOREM 1.3. *Let assumptions of Theorem 1.2 hold and $\{v^\nu, \alpha^\nu\}$ be a solution given by Theorem 1.2. Then there exists a subsequence $\{\nu_k\}_{k\in\mathbf{N}}$*

$$\nu_k \to 0 \qquad \textit{for} \;\; k \to +\infty \tag{1.26}$$

such that

$$\begin{aligned} v^{\nu_k} &\to v_E && \textit{strongly in } C^{a-\delta}(\Omega), \\ \alpha^{\nu_k} &\to rot\, v_E && \textit{weakly-}* \textit{ in } L_\infty(\Omega) \end{aligned} \tag{1.27}$$

as $k \to \infty$ for $\delta > 0$, where v_E is a solution to the Euler system (1.6) and the following bound is valid

$$\|rot\, v_E\|_{L_\infty(\Omega)} + \|v_E\|_{C^a(\Omega)} \le M\|d\|_{W^1_\infty(\partial\Omega)}. \tag{1.28}$$

Theorem 1.3 follows from Theorem 1.2 and the proof in our case is the same as in [6, section 4], hence we omit it here.

The paper is organized as follows. In section 2 we consider admissible singularities of the boundary. Next we prove the a priori bound for the solutions. And in Section 4 we show Theorem 1.2.

Throughout the paper we try to use the standard notations [4, 9].

2. Admissible points of irregularity. The first step in the proof of Theorem 1.2 is a construction of an approximation sequence for the original domain Ω_0. In this section we concentrate our attention on one part of the geometrical constraint. We examine the quantity

$$2\chi_\epsilon - f_\epsilon. \tag{2.1}$$

We introduce the following construction of the approximation for a given domain Ω_0. The definition of a collection Ω_ϵ will be prescribed by an image of a homeomorphism of domain Ω_0. We find a map

$$\Phi_\epsilon : \mathbf{R}^2 \to \mathbf{R}^2 \tag{2.2}$$

such that

$$\Phi_\epsilon(\Omega_0) = \Omega_\epsilon \tag{2.3}$$

and

$$\Phi_\epsilon(\partial\Omega_0) = \partial\Omega_\epsilon. \tag{2.4}$$

Moreover, let

$$\mathcal{N}_\epsilon = \bigcup_{i=1,\ldots,N_0} B(w_i, 2\epsilon), \tag{2.5}$$

where w_i are irregular point of the boundary and $B(w_i,\epsilon)$ denote balls with centers in w_i and radii equal to ϵ. Then

$$\partial\Omega_0 \setminus \mathcal{N}_\epsilon \tag{2.6}$$

is regular as a submanifold in $\mathbf{R}^2$ and we require that

$$\Phi_\epsilon|_{\partial\Omega_0\setminus\mathcal{N}_\epsilon} = id. \tag{2.7}$$

Now we investigate neighborhoods of irregular points. Looking at Fig. 2 we divide the considerations into two cases.

First we consider points between smooth parts with interior angles greater than π. Thus our domain is concave.

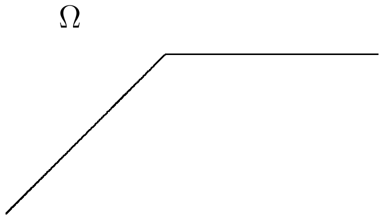

Fig. 3

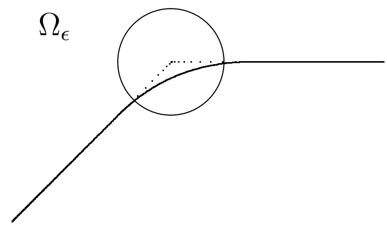

Fig. 4

Let us consider an approximation for this sector described by a diffeomorphism Φ_ϵ. Taking into account restriction (2.7) we obtain the following possibility of a shape of domain Ω_ϵ.

By the elementary properties we see that in a neighborhood of vertex w_k any approximation will have the following behavior at the limit near the examined point

$$\inf \chi_\epsilon \to -\infty \qquad \text{as} \;\; \epsilon \to 0. \tag{2.8}$$

Since the friction function must be nonnegative there is no possibility to fulfill the geometrical constraint. Quantity (2.1) will be negative and unbounded. It follows that irregular points cannot describe concave corners.

The second case consists of points between two smooth elements of the boundary with interior angles less than π.

We choose the following approximation of this sector describing by Fig. 6 below. Remembering about (2.7) we obtain.

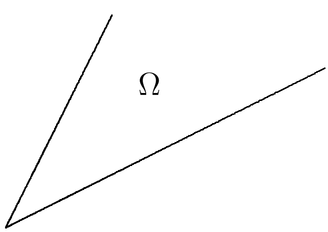

Fig. 5

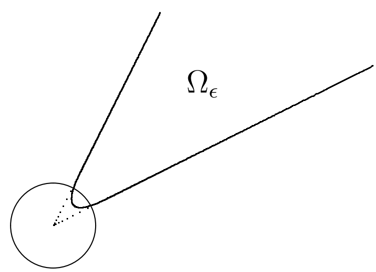

Fig. 6

By the properties of the above construction we are able to choose such map Φ_ϵ that the curvature of $\partial\Omega_\epsilon$ satisfies the following inequality

$$\chi_\epsilon|_{\partial\Omega_\epsilon\cap B(w_k,\epsilon)} \geq 0. \tag{2.9}$$

The singularity at point w_k causes that for any approximation in the neighborhood of w_k the curvature of the boundary blows up, i.e.

$$\sup \chi_\epsilon \to +\infty \quad \text{as} \quad \epsilon \to 0. \tag{2.10}$$

However, by the properties of this sector, (2.9) and (2.10) we are able to choose the following friction function

$$f_\epsilon = \begin{cases} \chi_\epsilon & \text{on} \quad \partial\Omega_\epsilon \cap B(w_k, \epsilon) \\ \max\{\chi_\epsilon, f_0(\Phi_\epsilon(\cdot))\} & \text{on} \quad \partial\Omega_\epsilon \cap (B(w_k, 2\epsilon) \setminus B(w_k, \epsilon)) \\ f_0 & \text{on} \quad \partial\Omega_\epsilon \setminus B(w_k, 2\epsilon) \end{cases} \tag{2.11}$$

This setting makes it possible to omit the irregular part, because

$$(2\chi_\epsilon - f_\epsilon)|_{B(w_k,\epsilon)} \equiv 0. \tag{2.12}$$

The above considerations lead to the following result. If the domain Ω_0 possesses only irregular points with angles less than π, then we are able to find a construction such the approximation will satisfy the following estimate

$$\|2\chi_\epsilon - f_\epsilon\|_{L_\infty(\partial\Omega_\epsilon)} \leq \|2\chi_0 - f_0\|_{L_\infty(\partial\Omega_0)} \tag{2.13}$$

for any $\epsilon > 0$. Inequality (2.13) can be fulfilled, hence convex corners will stay in our investigation.

Additionally our construction guarantees that

$$\Omega_\epsilon \subset \Omega \quad \text{and} \quad \Omega \setminus \mathcal{N}_\epsilon = \Omega_\epsilon \setminus \mathcal{N}_\epsilon \tag{2.14}$$

for $\epsilon \to 0$.

In the next steps of our technique (2.14) will be important, because we will be able to examine the approximation in the original domain Ω.

3. A priori bound. In this part we want to show the a priori bound for solutions to system (1.10)-(1.11) as well to control the behavior of quantity $\gamma_\infty(\Omega)$ for domains with irregular boundaries.

Taking the vorticity system (1.10) we are able to apply the maximum principle and get the following bound

$$\|\alpha\|_{L_\infty(\Omega)} \leq \|2\chi - f\|_{L_\infty(\partial\Omega)}\|v\|_{C(\Omega)} + 2\|d_{,s}\|_{L_\infty(\partial\Omega)}. \tag{3.1}$$

Since estimate (3.1) follows from the integration by parts, we need to require only the Lipschitz continuity of the boundary which is satisfied by our basic assumptions. For details we refer to [6 and 7].

To prove Theorem 1.2 we concentrate on possible singularities of solutions to system (1.11) coming from influences of boundary angles. As we will see the highest regularity of solutions v will be bounded by factor $\pi/\theta_{max} - 1$ where θ_{max} is given by (1.8).

The next step is to examine the elliptic system (1.11). The problem we split into two systems

$$\begin{array}{llll} \text{rot } v_1 = 0, & \text{rot } v_2 = \alpha & \text{in} & \Omega, \\ \text{div } v_1 = 0, & \text{div } v_2 = 0 & \text{in} & \Omega, \\ n \cdot v_1 = d, & n \cdot v_2 = 0 & \text{on} & \partial\Omega, \end{array} \tag{3.2}$$

where v is the solution to (1.11) is given by the following sum

$$v = v_1 + v_2. \tag{3.3}$$

The solvability of the first system from (3.2) is analogical to the second one, thus first we concentrate our study on the last one only.

By $(3.2)_2$ we find a stream function which describes the velocity as follows

$$v_2 = (-\partial_{x_2}\phi, -\partial_{x_1}\phi). \tag{3.4}$$

Also by the boundary condition $(3.2)_3$ and (3.4) we observe that

$$n \cdot v_2 = \frac{d}{ds}\phi = 0, \tag{3.5}$$

and since the function ϕ is given up to a constant and the domain is simply connected we may choose zero value at the boundary. Thus the second system from (3.2) takes the following form

$$\begin{array}{lll} \Delta\phi = \alpha & \text{in} & \Omega, \\ \phi = 0 & \text{on} & \partial\Omega. \end{array} \tag{3.6}$$

To close the estimation begun by (3.1) we need the following result for solutions to (3.6)

$$\|\nabla\phi\|_{C(\Omega)} \leq \gamma_\infty(\Omega)\|\alpha\|_{L_\infty(\Omega)}. \tag{3.7}$$

The above estimate is obvious as we consider a domain with regular boundary. However if the boundary possesses nontrivial corners, the solution may develop singularities. Let us consider a model case for a domain being a sector described by angle θ.

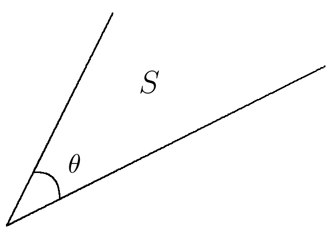

Fig. 7

For the above domain we consider the following system

$$\begin{array}{lll} \Delta\Psi = 0 & \text{in} & S, \\ \Psi = 0 & \text{on} & \partial S. \end{array} \tag{3.8}$$

Elementary considerations yield explicit solutions to problem (3.8) as follows

$$\Psi = \Psi_k(x) = r^{k\pi/\theta} \sin\left(k\pi/\theta\varphi\right) \tag{3.9}$$

for $k = 1, 2, \ldots$, where $r = \sqrt{x_1^2 + x_2^2}$ and $\varphi =$arc $\tan(x_2/x_1)$.

To analyze the structure of weak solution to (3.6) we need the following result [3] which will control singularities created by corners.

THEOREM 3.1. *Let $0 < a < 1$ and Ω be defined as for problem (1.1)-(1.4), then the weak solution to system (3.6) has the following form*

$$\phi = \phi_R + \sum_{i=1}^{N_0} \sum_{k=1}^{K_i} b_{ik} \eta_i \Psi_k^i, \tag{3.10}$$

where $\phi_R \in C^{1+a}(\Omega)$, Ψ_k^i are described by Φ_k from (3.9) with the origin at w_i and η_i are standard localizers of neighborhoods of w_i. In particular

$$\phi_R, \Psi_k^i \in H^1(\Omega) \tag{3.11}$$

for $i = 1, \ldots, N_0$ and $k = 1, \ldots, K_i$.

Theorem 3.1 will be applied as follows. We want to show that for a certain $a > 0$ we are able to restrict angles of corners such that the singular part of function ϕ vanishes, i.e. all coefficients b_{ik} in (3.10) are zero.

First we note that since ϕ is a weak solution, hence $\nabla\phi \in L_2(\Omega)$ which by (3.11) implies also that $\nabla\Psi_k^i \in L_2(\Omega)$. The last statement is valid if

$$r_i^{k\pi/\theta_i - 1} \in L_2(\Omega), \tag{3.12}$$

where $r_i = |x - w_i|$; which holds if (dim $\Omega = 2$)

$$k\pi/\theta > 0 \quad \text{for} \quad k \in \mathbf{N} \setminus \{0\}. \tag{3.13}$$

This restriction describes K_i as a function of angle θ_i.

The next step is to analyze the regularity $C^{1+a}(\Omega)$. We want to show that all functions Ψ_k belong to C^{1+a} and b_k are zero, which in our case is equivalent to the following inequality

$$k\pi/\theta \geq 1 + a. \tag{3.14}$$

By the analysis in section 2 we have already removed from our investigation angles greater than π. It follows that we examine only $\theta_i < \pi$, so in particular, for $k = 1$, we have

$$\pi/\theta \geq 1 + a. \tag{3.15}$$

It follows that number a has to be less or equal $\pi/\theta_{max} - 1$ - see (1.8). Then function Ψ_1 from (3.9) would belong to C^{1+a}, which implies that $b_1 = 0$. We conclude that $\phi = \phi_R$ in the studied case.

Thus, the solutions have no singular parts and we can consider all cases for $a \leq \pi/\theta_{max} - 1$.

It is worthwhile to underline that for interior angles greater than π we have the following inequality

$$\pi/\theta < 1. \tag{3.16}$$

It follows that expansion (3.10) is not trivial since

$$\Psi_1 \notin C^{1+a}(\Omega) \tag{3.17}$$

for any $a > 0$. Thus we would obtain a restriction on factor a, however considerations from section 2 excluded this case.

A result of the above considerations can be stated as the following theorem.

THEOREM 3.2. *Let* $0 < a \leq \pi/\theta_{max} - 1$, Ω *be defined as in Theorem 1.2 and* $f \in L_\infty(\Omega)$. *Then there exist finite numbers* $\gamma_\infty(\Omega)$ *and* $\gamma_\infty^a(\Omega)$ *such that the weak solution to problem*

$$\begin{aligned} &\Delta u = F \quad &in \quad &\Omega, \\ &u = 0 \quad &on \quad &\partial\Omega, \end{aligned} \tag{3.18}$$

fulfills the following estimates

$$\begin{aligned} \|\nabla u\|_{C(\Omega)} &\leq \gamma_\infty(\Omega)\|F\|_{L_\infty(\Omega)}, \\ \|\nabla u\|_{C^a(\Omega)} &\leq \gamma_\infty^a(\Omega)\|F\|_{L_\infty(\Omega)}. \end{aligned} \tag{3.19}$$

As a corollary of Theorem 3.2 we obtain the estimate for solutions to the second problem from (3.2). By (3.19) we conclude the following bounds

$$\begin{aligned} \|\nabla v_2\|_{C(\Omega)} &\leq \gamma_\infty(\Omega)\|\alpha\|_{L_\infty(\Omega)}, \\ \|\nabla v_2\|_{C^a(\Omega)} &\leq \gamma_\infty^a(\Omega)\|\alpha\|_{L_\infty(\Omega)}. \end{aligned} \tag{3.20}$$

Now we return to the first system from (3.2). To solve it we need the following elementary result.

PROPOSITION 3.1. *Let* Ω *fulfill conditions as in Theorem 1.2. If* $d \in W_\infty^1(\Omega)$ *and condition (1.5) is fulfilled then there exists a vector field* V *such that*

$$V \in W_\infty^1(\Omega), \qquad \|V\|_{W_\infty^1(\Omega)} \leq c\|d\|_{W_\infty^1(\partial\Omega)}, \qquad n \cdot V|_{\partial\Omega} = d. \tag{3.21}$$

The above proposition one can treat as a compatibility condition on datum d. We may just assume existence of field V satisfying (3.21).

Proposition 3.1 reduces our system to the following one

$$\begin{aligned} &\text{rot } u = -\text{rot}\, V \quad &\text{in} \quad &\Omega, \\ &\text{div } u = -\text{div}\, V \quad &\text{in} \quad &\Omega, \\ &n \cdot u = 0 \quad &\text{on} \quad &\partial\Omega, \end{aligned} \tag{3.22}$$

putting the following form of the solutions

$$v_1 = u + V \tag{3.23}$$

and V is the field given by Proposition 3.1.

We want to reduce (3.22) to the second system from (3.2). For this purpose we introduce a scalar function z defined as the solution to the following problem

$$\begin{aligned} &\Delta z = -\text{div } V \quad &\text{in} \quad &\Omega, \\ &\tfrac{\partial z}{\partial n} = 0 \quad &\text{on} \quad &\partial\Omega. \end{aligned} \tag{3.24}$$

By properties of V and condition (1.5) the compatibility condition to system (3.24) is fulfilled. Thus we are able to obtain the next simplification of system (3.22) which reads

$$\begin{aligned} &\text{rot } w = -\text{rot}\, V \quad &\text{in} \quad &\Omega, \\ &\text{div } w = 0 \quad &\text{in} \quad &\Omega, \\ &n \cdot w = 0 \quad &\text{on} \quad &\partial\Omega, \end{aligned} \tag{3.25}$$

where the solution to (3.22) has the following form

$$u = w + \nabla z. \tag{3.26}$$

For system (3.25) the theory [3] gives an analogical result as Theorem 3.1 for system (3.6), hence we have shown that the first system from (3.2) can be reduced to the second one, because

$$v_1 = V + \nabla z + w. \tag{3.27}$$

By the above considerations we conclude that

$$\begin{aligned} \|v_1\|_{C(\Omega)} &\le \kappa_\infty(\Omega)\|d\|_{W^1_\infty(\partial\Omega)}, \\ \|v_1\|_{C^a(\Omega)} &\le \kappa^a_\infty(\Omega)\|d\|_{W^1_\infty(\partial\Omega)} \end{aligned} \tag{3.28}$$

for finite $\kappa_\infty(\Omega)$ and $\kappa^a_\infty(\Omega)$ for $0 < a \le \pi/\theta_{max} - 1$.

The analysis of system (3.2), bounds (3.20), (3.28) and form (3.3) leads to the following estimate for solutions to problem (1.11)

$$\begin{aligned} \|v\|_{C(\Omega)} &\le \gamma_\infty(\Omega)\|\alpha\|_{L_\infty(\Omega)} + \kappa_\infty\|d\|_{W^1_\infty(\partial\Omega)}, \\ \|v\|_{C^a(\Omega)} &\le \gamma^a_\infty(\Omega)\|\alpha\|_{L_\infty(\Omega)} + \kappa^a_\infty\|d\|_{W^1_\infty(\partial\Omega)}. \end{aligned} \tag{3.29}$$

Combining $(3.29)_1$ with (3.1) we obtain the desired bound

$$\|\alpha\|_{L_\infty(\Omega)} \le (1 - \gamma_\infty(\Omega)\|2\chi\|_{L_\infty(\partial\Omega)})^{-1} B\|d\|_{W^1_\infty(\partial\Omega)}, \tag{3.30}$$

provided

$$\gamma_\infty(\Omega)\|2\chi\|_{L_\infty(\partial\Omega)} < 1. \tag{3.31}$$

The a priori estimate from Theorem 1.2 is proved.

4. Approximation. In this part we prove Theorem 1.2. We will construct a sequence of approximations for the formulation given by Definition 1.1. By (2.14) we are able to formulate the approximation on the whole domain Ω restricting only the supports of test functions.

For given $\epsilon > 0$ we define approximation of solutions.

DEFINITION 4.1. A pair $\{v_\epsilon, \alpha_\epsilon\}$ we call a *approximative solution* for problem (1.1)-(1.4) if and only if

$$\begin{aligned} v_\epsilon \in C^a(\Omega), \qquad & div\, v_\epsilon = 0, \\ \alpha_\epsilon = rot\, v_\epsilon \in L_\infty(\Omega), \qquad & n \cdot v_\epsilon|_{\partial\Omega} = d \end{aligned} \tag{4.1}$$

and

$$\int_\Omega \bar v_\epsilon \cdot \nabla\phi_\epsilon \bar\alpha_\epsilon dx + \nu\int_\Omega \bar\alpha_\epsilon \Delta\phi_\epsilon dx - \nu\int_{\partial\Omega}((2\chi_\epsilon - f_\epsilon)\bar v_\epsilon\cdot\tau - 2d_{,s})\frac{\partial\phi_\epsilon}{\partial n}d\sigma = 0 \tag{4.2}$$

for any $\phi_\epsilon \in W^2_1(\Omega)$ such that

$$supp\,\phi_\epsilon \subset \Omega_\epsilon \tag{4.3}$$

(see section 2); and

$$\bar v_\epsilon = v_\epsilon|_{\Omega_\epsilon}, \quad \bar\alpha_\epsilon = \alpha_\epsilon|_{\Omega_\epsilon} \quad \text{and} \quad \alpha|_{\Omega\setminus\Omega_\epsilon} = 0. \tag{4.4}$$

Definition 4.1 describes the weak formulation to the following system

$$\begin{aligned} &\bar v_\epsilon \cdot \nabla\bar\alpha_\epsilon - \nu\Delta\bar\alpha_\epsilon = 0 \quad \text{in } \Omega_\epsilon, \\ &\bar\alpha_\epsilon = \begin{cases} (2\chi_\epsilon - f_\epsilon)\bar v_\epsilon\cdot\tau - 2d_{,s} & \text{on} \quad \partial\Omega_\epsilon \cap \partial\Omega, \\ 0 & \text{on} \quad \partial\Omega_\epsilon \setminus \partial\Omega, \end{cases} \end{aligned} \tag{4.5}$$

This setting is correct, since by (2.12) we have

$$2\chi_\epsilon - f_\epsilon \equiv 0 \qquad \text{on} \quad \partial\Omega_\epsilon \cap \mathcal{N}_{\epsilon/2}. \tag{4.6}$$

Thus we control the behavior of boundary data in irregular points. Moreover α_ϵ is given by the following trivial extension

$$\alpha_\epsilon = \begin{cases} \bar{\alpha}_\epsilon & \text{for} \quad x \in \Omega_\epsilon, \\ 0 & \text{for} \quad x \in \Omega \setminus \Omega_\epsilon, \end{cases} \tag{4.7}$$

and vector v_ϵ fulfills the following system

$$\begin{array}{lll} \text{rot } v_\epsilon = \alpha_\epsilon & \text{in} & \Omega, \\ \text{div } v_\epsilon = 0 & \text{in} & \Omega, \\ n \cdot v_\epsilon = d & \text{on} & \partial\Omega, \end{array} \tag{4.8}$$

remembering that

$$\bar{v}_\epsilon = v_\epsilon|_{\Omega_\epsilon}. \tag{4.9}$$

Considerations from section 3 lead to the following estimate for solutions to the coupled system (4.5)-(4.8)

$$\|\alpha_\epsilon\|_{L_\infty(\Omega)} + \|v_\epsilon\|_{C^a(\Omega)} \leq B\|d\|_{W^1_\infty(\partial\Omega)}, \tag{4.10}$$

where B is independent of parameter ϵ and $0 < a \leq \pi/\theta_{max} - 1$.

Bound (4.10) finishes a part of the proof concerning the construction of the approximation and starts consideration about the convergence of the approximation sequence.

Since $\{v_\epsilon, \alpha_\epsilon\}_{\epsilon>0}$ are defined on the whole Ω and information given by (4.10) is independent of ϵ we are able to find a subsequence $\{v_{\epsilon_k}, \alpha_{\epsilon_k}\}_{k=0}^\infty$ such that

$$\epsilon_k \to 0 \qquad \text{as} \quad k \to +\infty \tag{4.11}$$

and

$$\begin{array}{ll} v_{\epsilon_k} \to v_* & \text{strongly in } C^{a-\delta}(\Omega), \\ \alpha_{\epsilon_k} \rightharpoonup \alpha_* & \text{weakly-}* \text{ in } L_\infty(\Omega) \end{array} \tag{4.12}$$

fro $\delta > 0$, where v_* and α_* fulfills the following bound

$$\|\alpha_*\|_{L_\infty(\Omega)} + \|v_*\|_{C^a(\Omega)} \leq B\|d\|_{W^1_\infty(\partial\Omega)}. \tag{4.13}$$

Now we show that $\{v_*, \alpha_*\}$ fulfills Definition 1.1. For this purpose we take $\phi \in W^2_1(\Omega) \cap \{\phi|_{\partial\Omega} = 0\}$ and consider the following quantities connected with formula (4.2)

$$L(v_*, \alpha_*, \phi) = \int_\Omega v_* \nabla\phi \alpha_* dx + \nu \int_\Omega \alpha_* \Delta\phi dx - \nu \int_{\partial\Omega} ((2\chi - f) v_* \cdot \tau - 2d_{,s}) \frac{\partial\phi}{\partial n} d\sigma. \tag{4.14}$$

It is required to show that for any ϕ as in (1.13) the following identity holds

$$L(v_*, \alpha_*, \phi) = 0. \tag{4.15}$$

First, let us note that for any fixed $\delta > 0$ we are able to find a function $\phi_\sigma \in W^2_1(\Omega)$ such that

$$\text{supp } \phi_\sigma \subset \Omega_\sigma, \tag{4.16}$$

where Ω_σ is defined as in (4.3); and

$$\|\phi - \phi_\sigma\|_{W^2_1(\Omega)} \leq \delta. \tag{4.17}$$

Hence by the form of (4.14) we have

$$L(v_*, \alpha_*, \phi) = L(v_*, \alpha_*, \phi_\sigma) + L(v_*, \alpha_*, \phi - \phi_\sigma). \tag{4.18}$$

By (4.14) and bound (4.13) we easily show the following estimate

$$|L(v_*, \alpha_*, \phi - \phi_\sigma)| \le c\delta. \tag{4.19}$$

It follows that we concentrate our attention on the first term of the r.h.s. of (4.18). For any ϵ_k such that $0 < \epsilon_k < \sigma$ we have

$$L(v_*, \alpha_*, \phi_\sigma) = L(v_\epsilon, \alpha_\epsilon, \phi_\sigma) + \left(L(v_*, \alpha_*, \phi_\sigma) - L(v_\epsilon, \alpha_\epsilon, \phi_\sigma)\right). \tag{4.20}$$

By properties of pair $(v_{\epsilon_k}, \alpha_{\epsilon_k})$ and Definition 4.1 we see that if $\sigma \ge \epsilon_k$, then

$$L(v_{\epsilon_k}, \alpha_{\epsilon_k}, \phi_\sigma) = 0. \tag{4.21}$$

To analyze the second term of the r.h.s. of (4.21) we recall (4.14). First we note that

$$\int_\Omega (\alpha_* - \alpha_{\epsilon_k})\Delta\phi_\sigma dx \to 0 \qquad \text{as} \;\; k \to +\infty \tag{4.22}$$

which follows from $(4.12)_2$; moreover if $\sigma \ge 2\epsilon_k$ by $(4.5)_2$ the next term can be treated as follows

$$\nu \int_{\partial\Omega} \left[(2\chi_{\epsilon_k} - f_{\epsilon_k})\bar{v}_{\epsilon_k} \cdot \tau - (2\chi - f)v_* \cdot \tau\right] \frac{\partial \phi_\sigma}{\partial n} d\sigma \to 0 \qquad \text{as} \;\; k \to +\infty \tag{4.23}$$

by $(4.12)_1$, (4.16) and (2.12) for $\sigma \ge 2\epsilon_k$.

The last term is connected to the nonlinearity from the equation. We have

$$\int_\Omega (v_* \nabla\phi_\sigma \alpha_* - v_\epsilon \nabla\phi_\sigma \alpha_\epsilon) dx = \int_\Omega v_* \nabla\phi_\sigma(\alpha_* - \alpha_\epsilon)dx + \int_\Omega (v_* - v_\epsilon)\nabla\phi_\sigma \alpha_\epsilon dx. \tag{4.24}$$

The first term of the r.h.s. of (4.24) satisfies

$$\int_\Omega v_* \nabla\phi_\sigma(\alpha_* - \alpha_\epsilon)dx \to 0 \qquad \text{as} \;\; k \to +\infty \tag{4.25}$$

by $(4.12)_2$ and the second one of the r.h.s. of (4.24) satisfies

$$\int_\Omega (v_* - v_\epsilon)\nabla\phi_\sigma \alpha_\epsilon dx \to 0 \qquad \text{as} \;\; k \to +\infty \tag{4.26}$$

by $(4.12)_1$. Thus by (4.22), (4.23), (4.25) and (4.26) we conclude

$$L(v_*, \alpha_*, \phi_\sigma) = 0 \tag{4.27}$$

for any $\sigma > 0$.

By (4.19) and (4.27) we proved

$$|L(v_*, \alpha_*, \phi)| \le c\delta \tag{4.28}$$

for any $\delta > 0$, hence we show identity (4.15). Theorem 1.2 is proved.

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