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A SHORT NOTE ON REGULARITY CRITERIA FOR THE NAVIER–STOKES EQUATIONS CONTAINING THE VELOCITY GRADIENT

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Abstract. We review several regularity criteria for the Navier–Stokes equations and prove some new ones, containing different components of the velocity gradient.

1. Introduction. We consider the Navier–Stokes equations

$$\left.\begin{array}{c}\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla p = \boldsymbol{0}\\ \operatorname{div}\boldsymbol{u} = 0\end{array}\right\}\ \text{in } I\times\Omega \tag{1}$$

describing the flow of a viscous incompressible fluid. For simplicity, we put the constant density as well as the viscosity equal to 1. We also take the right-hand side equal to zero. It is an easy matter to formulate all the criteria stated below for sufficiently smooth right-hand side.

To simplify the situation further, we have $\Omega = \mathbb{R}^3$. The same results can be obtained for $\Omega = (0, L)^3$ with the space-periodic boundary conditions. On the other hand, for Ω a smooth domain with homogeneous Dirichlet boundary conditions some proofs do not work. The main reason is a missing control on the pressure in the case of Dirichlet boundary conditions.

Finally, let the initial condition $\boldsymbol{u}_0$ be a sufficiently smooth ($\boldsymbol{u}_0 \in W^{1,2}(\mathbb{R}^3)^3$ at least) divergence-free vector field.

Even though system (1) looks very innocently, in three space dimensions, the well-

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posedness in the sense of Hadamard is an open question since the work of Leray [13]. It is well known that for any $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)^3$, $\operatorname{div} \boldsymbol{u}_0 = 0$ in $\mathcal{D}'(\mathbb{R}^3)$ there exists at least one weak solution to (1) which satisfies the energy inequality

$$\|\boldsymbol{u}(t)\|_2^2 + 2\int_0^t \|\nabla \boldsymbol{u}\|_2^2 d\tau \le \|\boldsymbol{u}_0\|_2^2 \qquad \text{for all } t \in I \tag{2}$$

and thus also $\lim_{t\to 0^+} \|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_2 = 0$. Such solutions are called Leray–Hopf weak solutions.

On the other hand, the uniqueness and smoothness of the solution is known only either locally in time or for sufficiently small initial velocity. If $\Omega \subseteq \mathbb{R}^2$, the situation changes; we get both uniqueness of the weak solution and its regularity provided the data are smooth enough.

First regularity (and uniqueness) criteria were formulated already at the end of fifties and beginning of sixties; cf. [21], [24]. If

$$\boldsymbol{u} \in L^t(I; L^s(\mathbb{R}^3)^3), \qquad \frac{2}{t} + \frac{3}{s} \le 1, \quad s > 3$$

then the solution to (1) is smooth on I and thus unique in the class of all Leray–Hopf weak solutions. Due to the scaling, the conditions $\frac{2}{t} + \frac{3}{s} \le a$ will appear quite often. To simplify the notation, we will say that $\boldsymbol{u} \in (PS)_a$ if $\boldsymbol{u} \in L^t(I; L^s(\mathbb{R}^3))$ for some s, t satisfying $\frac{2}{t} + \frac{3}{s} = a$, $s \in [\frac{3}{a}, \infty]$ and $\boldsymbol{u} \in \widetilde{(PS)}_a$ if $s \in (\frac{3}{a}, \infty]$ only. Thus the classical Prodi–Serrin condition can be formulated that $\boldsymbol{u} \in \widetilde{(PS)}_1$ implies regularity and uniqueness of the Leray–Hopf weak solutions.

This result was improved later on; first, it was shown (see [12]) that the case $\boldsymbol{u} \in L^\infty(I; L^3(\mathbb{R}^3)^3)$ implies uniqueness, later in a series of papers [8], [9] and [23] that this condition is also sufficient for smoothness, at least for $\Omega = \mathbb{R}^3$ and $\Omega = \mathbb{R}^3_+$.

Another improvement of the classical result can be found in [1]. The authors showed that for the regularity it is enough to have a certain information on two velocity components. The third component is estimated by the other two basically due to the divergence-free condition. A natural question appeared whether a certain regularity of one velocity component leads necessarily to the regularity of the whole vector field. This question was positively answered in the paper [16], where the authors showed that the needed regularity is

$$u_3 \in (PS)_{\frac{1}{2}}.$$

(This result was proved as local regularity criterion for suitable weak solutions, however, it is an easy matter to transform it for the Cauchy problem.)

A very interesting combination of both criteria mentioned above can be found in [17]:

$$\begin{aligned} &u_1, u_2 \in (PS)_a,\ 2 \le t_1 \le \infty,\ 2 \le s_1 \le \infty, \\ &u_3 \in (PS)_b,\ 2 \le t_2 \le \infty,\ 3 \le s_2 \le \infty, \\ &a + b \le 2,\ 2/t_1 + 2/t_2 \le 1,\ 2/s_1 + 2/s_2 < 1. \end{aligned}$$

Further criteria can be formulated on the pressure p. Let us mention here only three such results. In [4] the authors showed that

$$p \in \widetilde{(PS)}_2$$

is enough to ensure the regularity. On the other hand, Seregin and Šverák showed that the boundedness of the pressure from below (i.e. without loss of generality, $p \geq 0$) is also sufficient. A local criterion for the non-negative part of the pressure was proved in [15]; however, both $p_- \in \widetilde{(PS)}_2$ and a certain regularity of the velocity in an arbitrarily small neighborhood of the point were needed.

A natural candidate for regularity criteria is the gradient of the velocity. One may guess that

$$\nabla \boldsymbol{u} \in (PS)_2$$

could yield the regularity. At least for $s < 3$, it immediately follows from the classical Prodi–Serrin conditions and the results by Escauriaza, Seregin and Šverák due to the Sobolev imbedding theorem. Even though such a result has probably been known for a longer time, it was firstly published in [2] in 1995 (for $s > \frac{3}{2}$).

Evidently, if we replace $\nabla \boldsymbol{u}$ by the vorticity $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}$, we get the same criterion.

Similarly as for the velocity itself, there is a question whether one can formulate regularity criteria only for certain components of the velocity gradient or for certain components of the vorticity.

For the vorticity, it was shown in [5] that two components belonging to $\widetilde{(PS)}_2$ ensure the regularity. See also [17] for the discussion of the role of the vorticity. On the other hand, as was shown in [6] and further improved in [3], also certain smoothness of the vorticity direction guarantees the regularity.

The velocity gradient has nine components and the situation is much more complex. First result from [5] claimed that ∇u_1 and ∇u_2 belonging to $(PS)_1$ imply the regularity. This result is evidently not optimal in comparison to the vorticity. We will see that the condition $\widetilde{(PS)}_2$ is indeed sufficient, even for only two components of the velocity gradient. An interesting open question is whether the smoothness of only one vorticity component implies the regularity.

Next expected result,

$$\nabla u_3 \in (PS)_{\frac{3}{2}}$$

guaranteeing the regularity was shown in [20]; independently also in [25]; particular case $s = 3$ also in [24].

Further criteria will be mentioned in the following section. It was not our intention to mention all the criteria implying the regularity. One may replace the Lebesgue (Sobolev) spaces by the BMO space, or even the Besov spaces, see e.g. [11] or [7]. Other interesting results can be found in [17], [18], where the regularity criteria are expressed via the eigenvalues or eigenfunctions of the symmetric part of the velocity gradient.

Note that, except for a certain logarithmic improvement of the Prodi–Serrin conditions (see [14]), the "optimal" condition for the velocity is $(PS)_1$ while for the pressure and the velocity gradient it is $(PS)_2$.

2. Regularity criteria containing components of the velocity gradient. We have already announced that $\nabla u_3 \in (PS)_{\frac{3}{2}}$ guarantees the regularity. (In fact, we will slightly improve this result). It is not very surprising that $\frac{\partial \boldsymbol{u}}{\partial x_3} \in (PS)_{\frac{3}{2}}$ yields the same result.

On the other hand, it might be more surprising that only two components, $\frac{\partial u_3}{\partial x_3}$ and $\frac{\partial u_2}{\partial x_2} \in \widetilde{(PS)}_2$ are also sufficient; but in this case, due to the divergence-free condition, we tacitly assume that the same regularity also holds for $\frac{\partial u_1}{\partial x_1}$. The result mentioned above implies the expected result that ∇u_2 and $\nabla u_3 \in \widetilde{(PS)}_2$ is sufficient for the regularity. It is possible to formulate also conditions for one component of the velocity gradient ($\frac{\partial u_3}{\partial x_3} \in L^\infty(I \times \mathbb{R}^3)$, i.e. $\frac{\partial u_3}{\partial x_3} \in (PS)_0$). All the results mentioned above were proved in [19]. We will include all of them in Theorem 1 below, together with some new criteria whose sketch of the proof will be given in the last section.

THEOREM 1. *Let* $\boldsymbol{u}$ *be a Leray–Hopf weak solution to the Navier–Stokes equations (1) corresponding to the initial velocity* $\boldsymbol{u}_0 \in W^{1,2}(\mathbb{R}^3)^3$, $\operatorname{div} \boldsymbol{u}_0 = 0$. *Let one of the following conditions be satisfied:*

(a) three components

- *(i)* $\frac{\partial u_3}{\partial x_3}, \frac{\partial u_2}{\partial x_2} \in \widetilde{(PS)}_2$ *(then also* $\frac{\partial u_1}{\partial x_1} \in \widetilde{(PS)}_2$*)*
- *(ii)* $\frac{\partial \boldsymbol{u}}{\partial x_3} \in (PS)_{\frac{3}{2}}$
- *(iii)* $\frac{\partial u_3}{\partial x_3} \in (PS)_1$, $\frac{\partial u_2}{\partial x_2}, \frac{\partial u_1}{\partial x_1} \in \widetilde{(PS)}_2$
- *(iv)* $\frac{\partial u_3}{\partial x_3} \in (PS)_2$, $2 \le s \le 3$, $\frac{\partial u_3}{\partial x_3} \in (PS)_{\frac{5}{3}+\frac{1}{s}}$, $s > 3$, $\frac{\partial u_3}{\partial x_3} \in (PS)_{\frac{11}{4}-\frac{3}{2s}}$, $s \in [\frac{18}{11}, 2]$ *and* $\frac{\partial u_3}{\partial x_1}, \frac{\partial u_3}{\partial x_2} \in (PS)_{\frac{3}{2}}$
- *(v)* $\frac{\partial u_3}{\partial x_3} \in (PS)_2$, $2 \le s \le 3$, $\frac{\partial u_3}{\partial x_3} \in (PS)_{\frac{5}{3}+\frac{1}{s}}$, $s > 3$, $\frac{\partial u_3}{\partial x_3} \in (PS)_{\frac{11}{4}-\frac{3}{2s}}$, $s \in [\frac{18}{11}, 2]$ *and* $\frac{\partial u_2}{\partial x_3}, \frac{\partial u_3}{\partial x_2} \in (PS)_{\frac{3}{2}}$

(b) two components

- *(i)* $\frac{\partial u_1}{\partial x_3}, \frac{\partial u_2}{\partial x_3} \in (PS)_1$
- *(ii)* $\frac{\partial u_2}{\partial x_3} \in (PS)_2$, $s \in [2, 3]$, $\frac{\partial u_2}{\partial x_3} \in (PS)_{\frac{5}{3}+\frac{1}{s}}$, $s > 3$, $\frac{\partial u_2}{\partial x_3} \in (PS)_{\frac{11}{4}-\frac{3}{2s}}$, $s \in [\frac{18}{11}, 2]$ *and* $\frac{\partial u_3}{\partial x_3} \in (PS)_1$

(c) one component

- *(i)* $\frac{\partial u_3}{\partial x_3} \in (PS)_0$.

Then $\boldsymbol{u}$ *is a smooth solution to the Navier–Stokes equations, unique in the class of all Leray–Hopf weak solutions.*

A very interesting problem are intermediate results between the special criteria. Unfortunately, the methods of the proof do not seem to cover these “interpolation” results. We leave this as a very interesting open problem.

3. Proof of new regularity criteria. We will sketch the proof of new criteria $(a)_{(v)}$ and $(b)_{(i)}$–$(b)_{(ii)}$. The proof of the remaining ones can be found in [19] or [20]. We only show a priori estimates for smooth solutions. The reader can find arguments how to deal with weak solutions e.g. in the papers mentioned a few lines above.

STEP 1: Proof of $(a)_{(v)}$. We will follow the proof of $(a)_{(iv)}$ given in [20]. There, the stronger condition $\frac{\partial u_3}{\partial x_3} \in (PS)_{\frac{3}{2}}$ was assumed. We will point out the argument how we improve the result. First, we have

Lemma 1. *Denote by ω_3 the third component of the vorticity, $\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$. Assume that $\frac{\partial u_3}{\partial x_3} \in \widetilde{(PS)}_2$ and $\frac{\partial u_3}{\partial x_2}$, $\frac{\partial u_2}{\partial x_3} \in (PS)_{\frac{3}{2}}$. Then there exist C_1 and C_2 depending only on the initial condition and the norms mentioned above such that*

$$\|\omega_3\|^2_{L^\infty(I;L^2(\mathbb{R}^3))\cap L^2(I;L^6(\mathbb{R}^3))} \le C_1 + C_2\|\boldsymbol{\omega}\|_{L^\infty(I;L^2(\mathbb{R}^3))\cap L^2(I;L^6(\mathbb{R}^3))}.$$

Remark 1. Lemma 1 holds if we replace $\frac{\partial u_2}{\partial x_3}$ by $\frac{\partial u_3}{\partial x_1}$.

Proof. We have

$$\frac{1}{2}\frac{d}{dt}\|\omega_3\|_2^2 + \|\nabla\omega_3\|_2^2 = \int_{\mathbb{R}^3} (\boldsymbol{\omega}\cdot\nabla)u_3\omega_3$$
$$= \int_{\mathbb{R}^3} \omega_3^2\frac{\partial u_3}{\partial x_3} + \int_{\mathbb{R}^3} \omega_2\omega_3\frac{\partial u_3}{\partial x_2} + \int_{\mathbb{R}^3}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\omega_3\frac{\partial u_3}{\partial x_1}.$$

The Hölder inequality applied in a different way on the first, and on the last two terms together with the Gronwall inequality (see the estimates in [20]) yield the result. ■

Next, consider the momentum equation written in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} - \Delta\boldsymbol{u} + (\boldsymbol{\omega}\times\boldsymbol{u}) + \nabla\left(p + \frac{1}{2}|\boldsymbol{u}|^2\right) = \mathbf{0},$$

multiply it by $\Delta\boldsymbol{u}$ and integrate over $\mathbb{R}^3$. Using the Green theorem the equality yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}\|_2^2 + \|\nabla^2\boldsymbol{u}\|_2^2 = \int_{\mathbb{R}^3}(\boldsymbol{\omega}\times\boldsymbol{u})\cdot\Delta\boldsymbol{u}.$$

Recall that

$$(\boldsymbol{\omega}\times\boldsymbol{u})\cdot\Delta\boldsymbol{u} = (\omega_2u_3 - \omega_3u_2)\Delta u_1 + (\omega_3u_1 - \omega_1u_3)\Delta u_2 + (\omega_1u_2 - \omega_2u_1)\Delta u_3.$$

Terms containing ω_1 and ω_3 can be estimated as I_{22} and I_{23} in [20]. Thus we have to control

$$\int_{\mathbb{R}^3}(\omega_2u_3\Delta u_1 - \omega_2u_1\Delta u_3) = \int_{\mathbb{R}^3}\left(\frac{\partial\omega_2}{\partial x_k}u_1\frac{\partial u_3}{\partial x_k} - \frac{\partial\omega_2}{\partial x_k}u_3\frac{\partial u_1}{\partial x_k}\right)$$
$$= -\int_{\mathbb{R}^3}\frac{\partial^2u_1}{\partial x_3\partial x_k}u_3\frac{\partial u_1}{\partial x_k} + \int_{\mathbb{R}^3}\frac{\partial^2u_1}{\partial x_3\partial x_k}u_1\frac{\partial u_3}{\partial x_k} + \int_{\mathbb{R}^3}\frac{\partial^2u_3}{\partial x_1\partial x_k}u_3\frac{\partial u_1}{\partial x_k} - \int_{\mathbb{R}^3}\frac{\partial^2u_3}{\partial x_1\partial x_k}u_1\frac{\partial u_3}{\partial x_k}.$$

The first term, $\frac{1}{2}\int_{\mathbb{R}^3}\frac{\partial u_1}{\partial x_k}\frac{\partial u_1}{\partial x_k}\frac{\partial u_3}{\partial x_3}$, can be estimated as I_{21} in [20] for $\frac{\partial u_3}{\partial x_3} \in \widetilde{(PS)}_2$. The second term contains for $k = 2$ $\frac{\partial u_3}{\partial x_2}$ and for $k = 3$ $\frac{\partial u_3}{\partial x_3}$—here we need the stronger assumptions for $s < 2$ and $s > 3$, see the estimates below. If $k = 1$, we use the continuity equation and

$$\int_{\mathbb{R}^3}\frac{\partial^2u_1}{\partial x_3\partial x_1}u_1\frac{\partial u_3}{\partial x_1} = -\int_{\mathbb{R}^3}\frac{\partial^2u_2}{\partial x_3\partial x_2}u_1\frac{\partial u_3}{\partial x_1} - \int_{\mathbb{R}^3}\frac{\partial^2u_3}{\partial^2x_3}u_1\frac{\partial u_3}{\partial x_1}$$
$$= \int_{\mathbb{R}^3}\frac{\partial u_2}{\partial x_3}\left(\frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1} + u_1\frac{\partial^2u_3}{\partial x_2\partial x_1}\right) + \int_{\mathbb{R}^3}\frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1} - \frac{1}{2}\int_{\mathbb{R}^3}\frac{\partial u_3}{\partial x_3}\frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_1}.$$

Similarly can also be treated the other terms. Now, if $s < 2$,

$$\left|\int_{\mathbb{R}^3} \frac{\partial u_3}{\partial x_3} u_i \frac{\partial^2 u_j}{\partial x_k \partial x_l}\right| \leq C\left\|\frac{\partial u_3}{\partial x_3}\right\|_2 \|\nabla \boldsymbol{u}\|_2^{\frac{1}{2}} \|\nabla^2 \boldsymbol{u}\|_2^{\frac{3}{2}} \leq C\left\|\frac{\partial u_3}{\partial x_3}\right\|_s^{\frac{2s}{6-s}} \|\nabla \boldsymbol{u}\|_2{}^{\frac{1}{2}} \|\nabla^2 \boldsymbol{u}\|_2^{\frac{30-9s}{2(6-s)}}$$

$$\leq \frac{1}{2}\|\nabla^2 \boldsymbol{u}\|_2^2 + C\left\|\frac{\partial u_3}{\partial x_3}\right\|_s^{\frac{8s}{5s-6}} \|\nabla \boldsymbol{u}\|_2^{2+\frac{24-12s}{5s-6}}$$

$$\leq \frac{1}{2}\|\nabla^2 \boldsymbol{u}\|_2^2 + C\|\nabla \boldsymbol{u}\|_2^2\left(\left\|\frac{\partial u_3}{\partial x_3}\right\|_s^{\frac{8s}{11s-18}} + \|\nabla \boldsymbol{u}\|_2^2\right).$$

If $s > 3$,

$$\left|\int_{\mathbb{R}^3} \frac{\partial u_3}{\partial x_3} u_i \frac{\partial^2 u_j}{\partial x_k \partial x_l}\right| \leq \left\|\frac{\partial u_3}{\partial x_3}\right\|_s^{\frac{3s}{5s-6}} \|\nabla \boldsymbol{u}\|_2^{2\frac{s-3}{5s-6}} \|\nabla^2 \boldsymbol{u}\|_2 \|\boldsymbol{u}\|_{\frac{2}{3}\frac{5s-6}{s-2}}$$

$$\leq C\left\|\frac{\partial u_3}{\partial x_3}\right\|_s^{\frac{3s}{5s-6}} \|\nabla \boldsymbol{u}\|_2 \|\nabla^2 \boldsymbol{u}\|_2 \|\boldsymbol{u}\|_2^{2\frac{s-3}{5s-6}} \leq \frac{1}{2}\|\nabla^2 \boldsymbol{u}\|_2^2 + C\left\|\frac{\partial u_3}{\partial x_3}\right\|_s^{\frac{6s}{5s-6}} \|\nabla \boldsymbol{u}\|_2^2.$$

These two estimates correspond exactly to conditions $(a)_{(iv)}$ and $(a)_{(v)}$ for $\frac{\partial u_3}{\partial x_3}$.

STEP 2: Proof of $(b)_{(i)}$. The proof is relatively simple. We will show that $(b)_{(i)}$ implies $\frac{\partial \boldsymbol{u}}{\partial x_3} \in L^\infty(I; L^2(\mathbb{R}^3)^3)$, i.e. $\frac{\partial \boldsymbol{u}}{\partial x_3} \in (PS)_{\frac{3}{2}}$. We have

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial \boldsymbol{u}}{\partial x_3}\right\|_2^2 + \left\|\nabla \frac{\partial \boldsymbol{u}}{\partial x_3}\right\|_2^2 = -\int_{\mathbb{R}^3} \frac{\partial u_i}{\partial x_3}\frac{\partial u_j}{\partial x_3}\frac{\partial u_i}{\partial x_j}.$$

The only term which does not contain $\frac{\partial u_i}{\partial x_3}$, $i = 1, 2$ is

$$-\int_{\mathbb{R}^3}\left(\frac{\partial u_3}{\partial x_3}\right)^3 = 2\int_{\mathbb{R}^3} u_3 \frac{\partial u_3}{\partial x_3}\frac{\partial^2 u_3}{\partial x_3^2} = -2\int_{\mathbb{R}^3} u_3 \frac{\partial u_3}{\partial x_3}\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \frac{\partial^2 u_2}{\partial x_2 \partial x_3}\right)$$

$$= 2\int_{\mathbb{R}^3}\left(\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_3} + u_3 \frac{\partial^2 u_3}{\partial x_1 \partial x_3}\right)\frac{\partial u_1}{\partial x_3} + 2\int_{\mathbb{R}^3}\left(\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3} + u_3 \frac{\partial^2 u_3}{\partial x_2 \partial x_3}\right)\frac{\partial u_2}{\partial x_3}.$$

The first and the third term can be estimated by

$$\left\|\frac{\partial u_i}{\partial x_3}\right\|_s \left\|\frac{\partial u_3}{\partial x_3}\right\|_{\frac{2s}{s-2}} \|\nabla \boldsymbol{u}\|_2,$$

while the second and the fourth one by

$$\left\|\frac{\partial u_i}{\partial x_3}\right\|_s \|\boldsymbol{u}\|_{\frac{2s}{s-2}} \left\|\nabla \frac{\partial \boldsymbol{u}}{\partial x_3}\right\|_2.$$

Altogether

$$\left|\int_{\mathbb{R}^3} \frac{\partial u_i}{\partial x_3}\frac{\partial u_j}{\partial x_3}\frac{\partial u_i}{\partial x_j}\right| \leq \frac{1}{2}\left\|\nabla \frac{\partial \boldsymbol{u}}{\partial x_3}\right\|_2^2$$

$$+C(\|\boldsymbol{u}\|_{L^\infty(I;L^2(\mathbb{R}^3)^3)})\left(\left\|\frac{\partial u_3}{\partial x_3}\right\|_s^{\frac{2s}{s-3}} + \|\nabla \boldsymbol{u}\|_2^2\right)\left(1 + \left\|\frac{\partial u_3}{\partial x_3}\right\|_2^{2\frac{s-3}{2s-3}}\right)$$

which implies the result.

Step 3: Proof of $(b)_{(ii)}$. One important tool is a modification of Theorem 1 from [19]. We have

Lemma 2. *Let* $\boldsymbol{u}$ *be a Leray–Hopf weak solution to the Navier–Stokes equations (1) corresponding to the initial velocity* $\boldsymbol{u}_0 \in W^{1,2}(\mathbb{R}^3)^3$, $\operatorname{div}\boldsymbol{u}_0 = 0$. *Let* $u_3 \in \widetilde{(PS)}_1$ *and* $\frac{\partial u_2}{\partial x_3}$, $\frac{\partial u_3}{\partial x_3}$ *belong to* $(PS)_2$, $s \in [2,3]$, *to* $(PS)_{\frac{5}{3}+\frac{1}{s}}$, $s > 3$ *and to* $(PS)_{\frac{11}{4}-\frac{3}{2s}}$, $s \in [\frac{18}{11}, 2]$. *Then* $\boldsymbol{u}$ *is a smooth solution to the Navier–Stokes equations, unique in the class of all Leray–Hopf weak solutions.*

Proof. We proceed as in the proof of Theorem 1 mentioned above. We have

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\omega}\|_2^2 + \|\nabla\boldsymbol{\omega}\|_2^2 = \int_{\mathbb{R}^3} \omega_i \frac{\partial u_i}{\partial x_j}\omega_j.$$

The right-hand side can be written as

$$\int_{\mathbb{R}^3} \frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_1} - \int_{\mathbb{R}^3} \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_3} + \int_{\mathbb{R}^3} \frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_2}$$
$$- \int_{\mathbb{R}^3} \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_3} + \int_{\mathbb{R}^3} c_{ijklm} u_3 \frac{\partial u_i}{\partial x_j}\frac{\partial^2 u_k}{\partial x_l \partial x_m}.$$

The only term which cannot be handled as in the article mentioned above is

$$\int_{\mathbb{R}^3} \frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_2} = -\int_{\mathbb{R}^3} \frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_1} - \int_{\mathbb{R}^3} \frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_3}.$$

The last term can be easily estimated. We apply the Green theorem in the first term on the right-hand side and use once more the continuity equation. Then

$$-\int_{\mathbb{R}^3} \frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_1} = -2\int_{\mathbb{R}^3} \frac{\partial}{\partial x_3}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) u_1 \frac{\partial u_1}{\partial x_3}$$
$$= 2\int_{\mathbb{R}^3} \frac{\partial u_2}{\partial x_3}\left(u_1 \frac{\partial^2 u_1}{\partial x_2 \partial x_3} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3}\right) + 2\int_{\mathbb{R}^3} \frac{\partial u_3}{\partial x_3}\left(u_1 \frac{\partial^2 u_1}{\partial x_3^2} + \frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}\right).$$

We have

$$\left|\int_{\mathbb{R}^3} \frac{\partial u_i}{\partial x_3} u_1 \frac{\partial^2 u_1}{\partial x_j \partial x_k}\right| \le \left\|\frac{\partial u_i}{\partial x_3}\right\|_s \|\boldsymbol{u}\|_{\frac{2s}{s-2}} \|\nabla^2\boldsymbol{u}\|_2 \le \frac{1}{2}\|\nabla\boldsymbol{\omega}\|_2^2 + C\|\boldsymbol{\omega}\|_2^2 \left\|\frac{\partial u_2}{\partial x_3}\right\|_s^{\frac{2s}{2s-3}},$$

provided $s \in [2,3]$. For $s > 3$ or $s < 2$ we may use the estimates at the end of Step 1. ∎

Now, we proceed as in the proof of $(a)_{(iii)}$ in [19]. Assume that $\frac{\partial u_2}{\partial x_3}$ satisfy $(b)_{(ii)}$ and $\frac{\partial u_3}{\partial x_3} \in (PS)_1$. Then, as $\frac{\partial u_3}{\partial x_3} \in L^2(I \times \mathbb{R}^3)$, it is an easy matter to see that $\frac{\partial u_3}{\partial x_3} \in L^2(I; L^3(\mathbb{R}^3))$ and thus it satisfies the assumption of Lemma 2. It remains to estimate u_3 in $\widetilde{(PS)}_1$. We have

$$\frac{1}{3}\frac{d}{dt}\|u_3\|_3^3 + \frac{8}{9}\|\nabla|u_3|^{\frac{3}{2}}\|_2^2 = -\int_{\mathbb{R}^3} \frac{\partial p}{\partial x_3}|u_3|u_3.$$

We apply the Green theorem in the term on the right-hand side and, using standard estimates of the pressure, we get

$$\left|\int_{\mathbb{R}^3} \frac{\partial p}{\partial x_3}|u_3|u_3\right| \le C\left(\left\|\frac{\partial u_3}{\partial x_3}\right\|_s^{\frac{2s}{s-3}} + \|\boldsymbol{u}\|_6^2\right)\|\boldsymbol{u}\|_2^{\frac{s-3}{s}}\|u_3\|_3,$$

i.e. $u_3 \in \widetilde{(PS)}_1$ provided $\frac{\partial u_3}{\partial x_3} \in (PS)_1$. The proof of $(b)_{(ii)}$ is complete.

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