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ON THE STOKES EQUATION WITH NEUMANN BOUNDARY CONDITION

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Abstract. In this paper, we study the nonstationary Stokes equation with Neumann boundary condition in a bounded or an exterior domain in $\mathbb{R}^n$, which is the linearized model problem of the free boundary value problem. Mainly, we prove L_p-L_q estimates for the semigroup of the Stokes operator. Comparing with the non-slip boundary condition case, we have the better decay estimate for the gradient of the semigroup in the exterior domain case because of the null force at the boundary.

1. Introduction. Let Ω be a bounded or an exterior domain in $\mathbb{R}^n$ $(n \geq 2)$ with boundary $\partial\Omega$ which is a $C^{2,1}$ hypersurface. We consider the nonstationary Stokes problem with Neumann boundary condition:

$$(1.1)\qquad \begin{cases} \partial_t u - \operatorname{Div} \mathrm{T}(u,\pi) = 0, \quad \operatorname{div} u = 0 & \text{in } \Omega,\ t > 0,\\ \mathrm{T}(u,\pi)\nu = 0 & \text{on } \partial\Omega,\ t > 0,\\ u|_{t=0} = u_0 & \text{in } \Omega, \end{cases}$$

where u is the unknown velocity vector, π is the unknown pressure, and u_0 is a given

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velocity vector. T is the stress tensor whose (j,k) component is given by

$$\mathrm{T}_{jk}(u,\pi) = D_{jk}(u) - \delta_{jk}\pi, \quad j,k = 1,\dots,n,$$
$$D_{jk}(u) = \partial u_j/\partial x_k + \partial u_k/\partial x_j, \quad \delta_{jk} = 1\ (j=k),\ = 0\ (j \neq k).$$

For simplicity, we assume that the viscous coefficient $\mu = 1$. Under the condition $\operatorname{div} u = 0$, $\operatorname{Div}\mathrm{T}(u,\pi) = \Delta u - \nabla\pi$.

(1.1) is a model problem of the free boundary value problem (cf. Solonnikov [16] and Abels [1]). Let us consider the region $\Omega(t) \in \mathbb{R}^n$ occupied by the fluid which is given only at the initial time $t = 0$, while for $t > 0$ it is to be determined. In this model the effect of surface tension is neglected.

$$(1.2) \qquad \begin{cases} \partial_t v + (v\cdot\nabla)v - \Delta v + \nabla q = f(x,t) & \text{in } \Omega(t),\ t>0, \\ \nabla\cdot v = 0 & \text{in } \Omega(t),\ t>0, \\ \mathrm{T}(v,q)\nu_t + p_0(x,t)\nu_t = 0 & \text{on } \partial\Omega(t),\ t>0, \\ v|_{t=0} = v_0 & \text{in } \Omega(0), \end{cases}$$

where ν_t is the unit outer normal to $\partial\Omega(t)$ at the point x, v_0 is a given initial velocity, $\Omega(0)$ is the initial domain filled by the fluid, and $f(x,t)$ and $p_0(x,t)$ are the external mass force vector and the pressure defined on the whole space. Below we assume that $p_0(x,t) = 0$, since we can arrive at this case by replacing $p(x,t)$ by $p + p_0$.

Following the approach due to Solonnikov [16], we reduce (1.2) to the problem as an initial boundary value problem in the given region $\Omega(0) = \Omega$. A kinematic condition for $\partial\Omega(t)$ is satisfied, which gives $\partial\Omega(t)$ as a set of points $x = x(\xi,t)$, $\xi \in \partial\Omega$, where $x(\xi,t)$ is the solution of the Cauchy problem

$$(1.3) \qquad \frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi.$$

We can rewrite (1.2) as an initial boundary value problem in Ω, if we go over the Euler coordinates $x \in \Omega(t)$ to the Lagrange coordinates $\xi \in \Omega$ connected with x by (1.3). If a velocity vector field $u(\xi,t)$ is known as a function of the Lagrange coordinates ξ, then this connection can be written in the form

$$x = \xi + \int_0^t u(\xi,\tau)\,d\tau := X_u(\xi,t).$$

Passing to the Lagrange coordinates in (1.2) and setting $v(X_u(\xi,t),t) = u(\xi,t)$ and $\tilde q(X_u(\xi,t),t) = \pi(\xi,t)$, we obtain

$$(1.4) \qquad \begin{cases} \partial_t u - \Delta_u u + \nabla_u\pi = f(X_u(\xi,t),t), \quad \operatorname{div}_u u = 0 & \text{in } \Omega,\ t>0, \\ \mathrm{T}_u(u,\pi)\nu_u = 0 & \text{on } \partial\Omega,\ t>0, \\ u|_{t=0} = u_0 & \text{in } \Omega, \end{cases}$$

where using $A(u) = {}^t(D_\xi X_u)^{-1}(\xi,t)$,

$$\nabla_u = A(u)\nabla,\ \operatorname{div}_u u = \nabla_u\cdot u = \operatorname{tr}(A(u)\nabla u),$$
$$\Delta_u = \operatorname{div}_u\nabla_u,\ \nu_u\cdot T_u(u,\pi) = \nu_u\cdot(\nabla_u u + {}^t(\nabla_u u)) - \pi\nu_n,$$
$$\nu_u(\xi,t) = A(u)\nu_\xi/|A(u)\nu_\xi|,$$

ν_ξ denotes the unit outer normal at $\xi \in \partial\Omega$. If t is small, then the operators Δ_u, ∇_u, div_u and T_u are closed to Δ, ∇, div and T. Therefore we write (1.4) as a fixed point problem:

$$\begin{cases} \partial_t u - \mu\Delta u + \nabla\pi = -\mu(\Delta - \Delta_u)u & \\ \qquad\qquad + (\nabla - \nabla_u)\pi + f(X_u(\xi,t),t) & \text{in } \Omega,\ t>0, \\ \mathrm{div} u = (\mathrm{div} - \mathrm{div}_u)u & \text{in } \Omega,\ t>0, \\ T(u,\pi)\nu = (T\nu - T_u\nu_u)(u,\pi) & \text{on } \partial\Omega,\ t>0, \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases}$$

Our final goal is to prove a globally in time existence of solutions of (1.2) for small initial data by using the analytic semigroup approach. To do this, we have the following plan of analysis:

1° Analysis of the resolvent problem corresponding to (1.1).
2° Analytic semigroup approach to (1.1).
3° L_p-L_q estimate of (1.1).
4° Maximal regularity of the linearized problem with inhomogeneous right members.

In this paper, we report on the results about 1°, 2° and 3°.

The free boundary value problem (1.2) was already solved by Solonnikov [16] in the bounded domain case. The linear problem (1.1) was already studied by using the theory of pseudo-differential operators with parameter (cf. Grubb and Solonnikov [10] and Grubb [8] and [9]). Our approach is completely different from [16], [10], [8] and [9].

2. Analysis of the resolvent problem to (1.1).

The resolvent problem corresponding to (1.1) is:

$$\begin{cases} \lambda u - \Delta u + \nabla\pi = f, \quad \mathrm{div}\, u = 0 \quad \text{in } \Omega, \\ \mathrm{T}(u,\pi)\nu|_{\partial\Omega} = 0. \end{cases} \tag{2.1}$$

As the space for the pressure, we set

$$\hat{W}^1_p(\Omega) = \{\pi \in L_{p,\mathrm{loc}}(\overline{\Omega}) \mid \nabla\pi \in L_p(\Omega)^n\},$$
$$X_p(\Omega) = \{\pi \in \hat{W}^1_p(\Omega) \mid \|\pi\|_{X_p(\Omega)} < \infty\}.$$

When Ω is a bounded domain, $\|\pi\|_{X_p(\Omega)} = \|\pi\|_{W^1_p(\Omega)}$ and $W^1_p(\Omega) = X_p(\Omega)$. When Ω is an exterior domain,

$$\|\pi\|_{X_p(\Omega)} = \begin{cases} \|\nabla\pi\|_{L_p(\Omega)} + \|\pi/d\|_{L_p(\Omega)}, & n \le p < \infty, \\ \|\nabla\pi\|_{L_p(\Omega)} + \|\pi/d\|_{L_p(\Omega)} + \|\pi\|_{L_{\frac{np}{n-p}}(\Omega)}, & 1 < p < n, \end{cases}$$

$$d(x) = \begin{cases} 2 + |x|, & p \ne n, \\ (2+|x|)\log(2+|x|), & p = n. \end{cases}$$

Concerning (1.1), we have the following theorem proved by Shibata and Shimizu [15], which is the base of our analytic semigroup approach to (1.1).

THEOREM 2.1. *Let $1 < p < \infty$, $0 < \epsilon < \pi/2$ and $\delta > 0$. We set*

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \le \pi - \epsilon\}.$$

For every $\lambda \in \Sigma_\epsilon$ *and* $f \in L_p(\Omega)^n$*, there exists a unique solution* $(u,\pi) \in W^2_p(\Omega)^n \times X_p(\Omega)$ *of* (1)*. Moreover, the* (u,π) *satisfies the estimate:*

$$|\lambda|\|u\|_{L_p(\Omega)} + \|u\|_{W^2_p(\Omega)} + \|\pi\|_{X_p(\Omega)} \le C_{\epsilon,\delta,p}\|f\|_{L_p(\Omega)}$$

for any $\lambda \in \Sigma_\epsilon$ *with* $|\lambda| \ge \delta$*.*

3. Analytic semigroup approach to (1.1)**.** In order to formulate (1.1) in the analytic semigroup framework, first of all we have to introduce the 2nd Helmholtz decomposition:

$$L_p(\Omega)^n = J_p(\Omega) \oplus G_p(\Omega)$$

where we have set

$$\begin{aligned} J_p(\Omega) &= \{u \in L_p(\Omega)^n \mid \nabla\cdot u = 0 \quad \text{in } \Omega\},\\ G_p(\Omega) &= \{\nabla\pi \mid \pi \in \dot X_p(\Omega)\},\\ \dot X_p(\Omega) &= \{\pi \in X_p(\Omega) \mid \pi|_{\partial\Omega} = 0\}. \end{aligned}$$

To prove the 2nd Helmholtz decomposition and also the unique solvability of the Laplace equation with Dirichlet condition, we use the following theorem which is proved by letting $\lambda \to \infty$ in (2.1) and using Theorem 2.1.

Lemma 3.1. (A) *Given* $f \in L_p(\Omega)^n$*, there exist unique* $g \in J_p(\Omega)$ *and* $\pi \in \dot X_p(\Omega)$ *such that* $f = g + \nabla\pi$ *in* Ω*.*

(B) *If* $\pi \in \hat W^1_p(\Omega)$ *satisfies* $\Delta\pi = 0$ *in* Ω *and* $\pi|_{\partial\Omega} = 0$*, then* $\pi = 0$*.*

(C) *Given* $h \in W_p^{1-1/p}(\partial\Omega)$*, there exists a* $\pi \in X_p(\Omega)$ *which solves the equation:*

$$\Delta\pi = 0 \quad \text{in } \Omega, \quad \pi|_{\partial\Omega} = h.$$

Let $P_p : L_p(\Omega)^n \to J_p(\Omega)$ be the solenoidal projection, and then there exists a unique $\theta \in \dot X_p(\Omega)$ such that $f = P_p f + \nabla\theta$. Inserting this formula into (2.1) and noting that $\theta|_{\partial\Omega} = 0$, (2.1) is reduced to the equation:

$$\begin{aligned} &\lambda u - \Delta u + \nabla(\pi - \theta) = P_p f, \;\; \operatorname{div} u = 0 \quad \text{in } \Omega,\\ &\mathrm{T}(u, \pi-\theta)\nu|_{\partial\Omega} = 0. \end{aligned}$$

Therefore we consider (2.1) for $f \in J_p(\Omega)$, below.

Now, we shall introduce the reduced Stokes equation corresponding to (2.1). Given $f \in J_p(\Omega)$, let $(u,\pi) \in W^2_p(\Omega)^n \times X_p(\Omega)$ be a solution of the equation:

$$\begin{aligned} &\lambda u - \Delta u + \nabla\pi = f, \quad \nabla\cdot u = 0 \;\text{ in } \Omega,\\ &(\mathrm{T}(u,\pi)\nu)_i|_{\partial\Omega} = \sum_{j=1}^n \nu_j\,(\partial_j u_i + \partial_i u_j) - \nu_i\pi|_{\partial\Omega} = 0 \;\; (i = 1, \ldots, n), \end{aligned}$$

where $(\mathrm{T}(u,\pi)\nu)_i$ denotes the i-th component of the n-vector $\mathrm{T}(u,\pi)\nu$. Applying the divergence to the first equation implies that $\Delta\pi = 0$ in Ω. Multiplying the boundary condition by ν_i and using $\sum_{i=1}^n \nu_i^2 = 1$ on $\partial\Omega$ and $\operatorname{div} u = 0$ in Ω, we have

$$\pi|_{\partial\Omega} = \sum_{i,j=1}^n \nu_i\nu_j D_{ij}(u) - \operatorname{div} u\,|_{\partial\Omega}.$$

In view of Lemma 3.1, there exists a solution operator $K : W_p^{1-1/p}(\partial\Omega)^n \to X_p(\Omega)$ associated with the equation:

$$\Delta K(u) = 0 \text{ in } \Omega, \quad K(u)|_{\partial\Omega} = \sum_{i,j=1}^{n} \nu_i \nu_j D_{ij}(u) - \operatorname{div} u\,|_{\partial\Omega}$$

such that there holds the estimate:

$$\|K(u)\|_{X_p(\Omega)} \le C_p \|u\|_{W_p^{1-1/p}(\partial\Omega)}.$$

Using the operator K, we see that when $f \in J_p(\Omega)$, the problem:

$$\lambda u - \Delta u + \nabla \pi = f, \quad \nabla \cdot u = 0 \text{ in } \Omega,$$
$$\sum_{j=1}^{n} \nu_j (\partial_j u_i + \partial_i u_j) - \nu_i \pi|_{\partial\Omega} = 0 \ (i = 1, \ldots, n)$$

is equivalent to the reduced Stokes resolvent problem

$$\begin{aligned} &\lambda u - \Delta u + \nabla K(u) = f \quad \text{in } \Omega, \\ &\mathrm{T}(u, K(u))\nu|_{\partial\Omega} = 0. \end{aligned} \tag{3.1}$$

The reason why we insert $\operatorname{div} u$ into the boundary condition is to prove that the solution u of (3.1) satisfies the condition: $\operatorname{div} u = 0$ in Ω. Theorem 2.1 implies the following theorem immediately.

THEOREM 3.2. *Let $1 < p < \infty$, $0 < \epsilon < \pi/2$ and $\delta > 0$. Given $\lambda \in \Sigma_\epsilon$ and $f \in L_p(\Omega)^n$,* (3.1) *admits a unique solution $u \in W_p^2(\Omega)^n$ satisfying the estimate:*

$$|\lambda| \| u \|_{L_p(\Omega)} + \| u \|_{W_p^2(\Omega)} \le C_{\epsilon,\delta,p} \| f \|_{L_p(\Omega)}$$

for any $\lambda \in \Sigma_\epsilon$ with $|\lambda| \ge \delta$.

Let us define the reduced Stokes operator A_p by the relations:

$$A_p u = -\Delta u + \nabla K(u) \quad \text{for } u \in \mathcal{D}(A_p),$$
$$\mathcal{D}(A_p) = \{u \in J_p(\Omega) \cap W_p^2(\Omega)^n \mid \mathrm{T}(u, K(u))\nu|_{\partial\Omega} = 0\}.$$

Then (3.1) is formulated as $\lambda u + A_p u = f$ in Ω and $u \in \mathcal{D}(A_p)$. Letting $\lambda \to \infty$ in (3.1), by Theorem 3.2 we obtain the following lemma.

LEMMA 3.3. *Let $1 < p < \infty$. Then, A_p is a densely defined closed operator.*

Combining Theorem 3.2 and Lemma 3.3, we obtain the following theorem.

THEOREM 3.4. *Let $1 < p < \infty$. Then, A_p generates an analytic semigroup $\{T(t)\}_{t\ge 0}$ on $J_p(\Omega)$.*

Moreover, we can also prove the following theorem concerning the dual space and the adjoint operator.

THEOREM 3.5. *Let $1 < p < \infty$ and $p' = p/(p-1)$. Then, $J_p(\Omega)^* = J_{p'}(\Omega)$ and $A_p^* = A_{p'}$.*

4. L_p-L_q estimate of (1.1)

4.1. *The bounded domain case.* Let Ω be a $C^{2,1}$-class bounded domain in $\mathbb{R}^n$ $(n \geq 2)$. Let us set

$$\mathcal{R} = \{Ax + b \mid \ A \text{ is an anti-symmetric matrix and } b \in \mathbb{R}^n\}.$$

Let $p_1, \ldots, p_M$ $(M = n(n-1)/2 + n)$ be an orthogonal basis of $\mathcal{R}$ in Ω such that $(p_j, p_k)_\Omega = \delta_{jk}$. Let us set

$$\dot{L}_p(\Omega) = \{u \in L_p(\Omega)^n \mid (u, p_k)_\Omega = 0, \quad k = 1, \ldots, M\}.$$

Then, we have the following exponential stability of the semigroup $\{T(t)\}_{t\geq 0}$ in the bounded domain case.

THEOREM 4.1. *Given any $f \in J_p(\Omega) \cap \dot{L}_p(\Omega)$, we have*

$$\|\nabla^j T(t)f\|_{L_q(\Omega)} \leq C_{p,q} e^{-ct} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j}{2}} \|f\|_{L_p(\Omega)}$$

for $1 \leq p \leq q \leq \infty$ $(p \neq \infty,\ q \neq 1)$, $t > 0$ and $j = 0, 1$, where $c = c_{p,q}$ is a positive constant.

To prove this theorem, the key is the solvability of the following problem:

$$\begin{aligned} -\mathrm{Div}\, \mathrm{T}(u, \pi) &= f, \ \ \mathrm{div}\, u = 0 \ \text{ in } \Omega, \\ \mathrm{T}(u, \pi)\nu|_{\partial\Omega} &= g. \end{aligned} \tag{4.1}$$

In fact, we have the following theorem concerning this equation.

THEOREM 4.2. *Let $1 < p < \infty$. Given $f \in L_p(\Omega)^n$ and $g \in W_p^{1-1/p}(\partial\Omega)^n$ satisfying the condition:*

$$(f, p_j)_\Omega + (g, p_j)_{\partial\Omega} = 0, \quad j = 1, \ldots, M,$$

(4.1) *admits a unique solution*

$$(u, \pi) \in (W_p^2(\Omega)^n \cap \dot{L}_p(\Omega)) \times W_p^1(\Omega).$$

Combining this theorem with Theorem 3.2, we have the following theorem.

THEOREM 4.3. *Let $1 < p < \infty$ and $0 < \epsilon < \pi/2$. Then, there exists a $\sigma > 0$ such that given $f \in J_p(\Omega) \cap \dot{L}_p(\Omega)$ and $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leq \sigma\}$, we have*

$$|\lambda| \|(\lambda + A_p)^{-1} f\|_{L_p(\Omega)} + \|(\lambda + A_p)^{-1} f\|_{W_p^2(\Omega)} \leq C_p \|f\|_{L_p(\Omega)}.$$

By Theorem 4.3, we have immediately

$$\|T(t)f\|_{W_p^j(\Omega)} \leq C_p e^{-ct} t^{-\frac{j}{2}} \|f\|_{L_p(\Omega)}, \quad j = 0, 2. \tag{4.2}$$

By using the complex interpolation:

$$(L_p(\Omega), W_p^2(\Omega))_\theta = W_p^s(\Omega), \quad \theta = s/2,$$

the real interpolation:

$$[L_p(\Omega), W_p^2(\Omega)]_{\theta,1} = B_{p,1}^{n/p}(\Omega), \quad \theta = n/2p,$$

the embedding theorems:

$$W_p^s(\Omega) \subset L_q(\Omega),\ s = n\left(\frac{1}{p} - \frac{1}{q}\right)\ (q \neq \infty),$$
$$B_{p,1}^{n/p}(\Omega) \subset L_\infty(\Omega),$$

the semigroup property: $T(t)f = T(t/2)T(t/2)f$ and the dual argument, we can show Theorem 4.1 from (4.2).

4.2. *The exterior domain case.* Let Ω be an exterior domain in $\mathbb{R}^n$ $(n \geq 3)$, whose boundary $\partial\Omega$ is a $C^{2,1}$ hypersurface. Then, we have the following theorem.

THEOREM 4.4.

$$\|T(t)f\|_{L_q(\Omega)} \leq C_{p,q} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L_p(\Omega)} \tag{4.3}$$

for $1 \leq p \leq q \leq \infty$ $(p \neq \infty,\ q \neq 1)$, $t > 0$ *and* $f \in J_p(\Omega)$, *and*

$$\|\nabla T(t)f\|_{L_q(\Omega)} \leq C_{p,q} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|f\|_{L_p(\Omega)} \tag{4.4}$$

for $1 \leq p \leq q \leq \infty$ $(p \neq \infty,\ q \neq 1)$, $t > 0$ *and* $f \in J_p(\Omega)$.

REMARK 4.5. If we consider the non-slip boundary condition $u|_{\partial\Omega} = 0$ instead of the Neumann boundary condition, to obtain (4.4) we have to assume that $1 \leq p \leq q \leq n$ $(q \neq 1)$ (cf. [11], [12], [14], [4], [5] and [6]).

5. A sketch of proof of Theorem 4.4

5.1. *1st step. Construction of a solution operator $R(\lambda)$.* The following theorem is concerned with the solution operator to (2.1).

THEOREM 5.1. *Let* $1 < p \leq q \leq \infty$ *and set*

$$L_{p,R}(\Omega) = \{f \in L_p(\Omega)^n \mid f(x) = 0 \quad x \notin B_R\}.$$

Then, there exists an $\epsilon > 0$ *and an operator* $R(\lambda) = (R_0(\lambda), R_1(\lambda))$ *for* $\lambda \in \dot{U}_\epsilon = \{\lambda \in \mathbb{C} \setminus (-\infty, 0] \mid |\lambda| < \epsilon\}$ *having the following properties:*

(1) *If we set* $u = R_0(\lambda)f$ *and* $\pi = R_1(\lambda)f$, *then* (u, π) *solves the problem:*

$$\lambda u - \operatorname{Div} \mathrm{T}(u, \pi) = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \qquad \mathrm{T}(u, \pi)\nu|_{\partial\Omega} = 0.$$

(2) *There holds the relation:* $R_0(\lambda)f = (\lambda + A)^{-1} P_p f$ *for any* $\lambda \in \dot{U}_\epsilon$ *and* $f \in L_{p,R}(\Omega)$.

(3) *There holds the estimate:*

$$\|R_0(\lambda)f\|_{L_q(\Omega)} \leq C_{p,q} |\lambda|^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-1} \|f\|_{L_p(\Omega)}$$

for $1 < p \leq q \leq \infty$, $\lambda \in \dot{U}_\epsilon$ *and* $f \in L_{p,R}(\Omega)$.

(4) *There holds the estimate:*

$$\|\nabla R_0(\lambda)f\|_{L_p(\Omega)} \leq C_p |\lambda|^{-\min\left(\frac{1}{2}, \frac{n}{2p}\right)} \|f\|_{L_p(\Omega)}$$

for $\lambda \in \dot{U}_\epsilon$ *and* $f \in L_{p,R}(\Omega)$.

(5) *There holds the expansion formula:*

$$R(\lambda) = \lambda^{\frac{n}{2}-1}(\log\lambda)^{\sigma(n)}H_0 + \lambda^{\frac{n}{2}-1}H_1(\lambda) + H_2(\lambda)$$

for $\lambda \in \dot{U}_\epsilon$ *on* $\Omega_R = \Omega \cap B_R$,

where

$$\begin{aligned}
&\sigma(n) = 1\ (n \geq 4, even), \sigma(n) = 0\ (n \geq 3, odd);\\
&H_0 \in \mathcal{L}(L_{p,R}(\Omega), W_p^2(\Omega_R)^n \times W_p^1(\Omega_R));\\
&H_1(\lambda) \in \mathcal{BA}(\dot{U}_\epsilon, \mathcal{L}(L_{p,R}(\Omega), W_p^2(\Omega_R)^n \times W_p^1(\Omega_R)));\\
&H_2(\lambda) \in \mathcal{BA}(U_\epsilon, \mathcal{L}(L_{p,R}(\Omega), W_p^2(\Omega_R)^n \times W_p^1(\Omega_R)));\\
&U_\epsilon = \{\lambda \in \mathbb{C} \mid |\lambda| < \epsilon\},
\end{aligned}$$

and $\mathcal{BA}(U, W)$ *is the set of all bounded analytic functions on* U *with their values in* W.

Using Theorem 5.1, we can show (4.3) and also (4.4) under the assumption: $1 \leq p \leq q \leq n$ ($q \neq 1$) in Theorem 4.4. To prove Theorem 5.1, we use the solution operator (E_λ, Π) of the Stokes resolvent equation in $\mathbb{R}^n$, which gives the solutions $u = E_\lambda f$ and $\pi = \Pi f$ of the equation:

$$(\lambda - \Delta)u + \nabla\pi = f, \quad \operatorname{div} u = 0 \text{ in } \mathbb{R}^n.$$

Since $E_\lambda f$ is given by the modified Bessel function of order $(n-2)/2$, applying the Young inequality we have

$$\|\nabla^j E_\lambda f\|_{L_q(\mathbb{R}^n)} \leq C_{p,q}|\lambda|^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-1+\frac{j}{2}}\|f\|_{L_p(\mathbb{R}^n)} \qquad j = 0, 1 \tag{5.1}$$

for $1 < p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$), $\lambda \in \Sigma_\epsilon = \{\lambda \in \mathbb{C}\setminus\{0\} \mid |\lambda| \leq \pi - \epsilon\}$ and $f \in L_p(\mathbb{R}^n)$. By using the expansion formula of the modified Bessel function near the origin, we have

$$E_\lambda f = \lambda^{\frac{n}{2}-1}(\log\lambda)^{\sigma(n)}G_1(\lambda)f + G_2(\lambda)f \text{ in } B_R \tag{5.2}$$

for $f \in L_{p,R}(\mathbb{R}^n) = \{f \in L_p(\mathbb{R}^n)^n \mid f(x) = 0 \text{ for } x \notin B_R\}$ and $\lambda \in \dot{U}_{\frac{1}{2}}$, where

$$G_j(\lambda) \in \mathcal{BA}(U_{\frac{1}{2}}, \mathcal{L}(L_{p,R}(\mathbb{R}^n), W_p^2(B_R))).$$

And also, we use the solution operator (A, B) which gives solutions $u = Af$ and $\pi = Bf$ of the interior problem:

$$\begin{aligned}
&-\operatorname{Div}\mathrm{T}(u,\pi) = f, \ \operatorname{div} u = 0 \text{ in } \Omega_R,\\
&\mathrm{T}(u,\pi)\nu|_{\partial\Omega} = 0,\\
&\mathrm{T}(u,\pi)\nu_0|_{S_R} = \mathrm{T}(E_0 f_0, \Pi f_0)\nu_0|_{S_R},
\end{aligned}$$

where $\nu_0 = x/|x|$, $S_R = \{|x| = R\}$, $\Omega_R = \Omega \cap B_R$, $\partial\Omega_R = \partial\Omega \cup S_R$, $f_0 = f$ ($x \in \Omega$) and $f_0 = 0$ ($x \notin \Omega$). Since there holds the compatibility condition:

$$(f, p_j)_{\Omega_R} + (\mathrm{T}(E_0 f_0, \Pi f_0)\nu_0, p_j)_{S_R} = 0$$

for $j = 1, \ldots, M$, we can find A and B. Moreover, since $D(p_j) = 0$ and $\operatorname{div} p_j = 0$, we may assume that

$$(Af - E_0 f_0, p_j)_{\Omega_R} = 0, \ j = 1, \ldots, M.$$

To define our parametrix for (2.1), we choose a cut-off function φ in such a way that

$$0 \le \varphi \le 1, \;\; \varphi(x) = 1 \;(|x| \le R-2), \;\; \varphi(x) = 0 \;(|x| \ge R-1)$$

where R is a number such that $B_R \supset \Omega^c$. As the parametrix for (2.1), we set

$$\begin{aligned} \Phi_\lambda f &= (1-\varphi)E_\lambda f_0 + \varphi A f + \mathbb{B}[(\nabla\varphi)(E_\lambda f - Af)], \\ \Psi f &= (1-\varphi)\Pi f_0 + \varphi B f, \end{aligned}$$

where $\mathbb{B}$ is the usual Bogovskiĭ operator (cf. [2], [3], [13], [7]). Then, there exists a compact operator T_λ of $L_{p,R}(\Omega)$ such that

$$\begin{aligned} &\lambda\Phi_\lambda f - \operatorname{Div}\mathrm{T}(\Phi_\lambda f_0, \Psi f) = (I + T_\lambda)f, \quad \operatorname{div}\Phi_\lambda f = 0 \text{ in } \Omega, \\ &T(\Phi_\lambda f, \Psi f)\nu|_{\partial\Omega} = 0. \end{aligned}$$

The uniqueness of the solution to the homogeneous equation:

$$-\operatorname{Div}\mathrm{T}(u,\pi) = 0, \;\; \operatorname{div} u = 0 \text{ in } \Omega, \quad \mathrm{T}(u,\pi)\nu|_{\partial\Omega} = 0$$

in the class of functions satisfying the radiation condition:

$$\begin{aligned} &u(x) = O(|x|^{-(n-2)}), \;\; \nabla u(x) = O(|x|^{-(n-1)}), \\ &\pi(x) = O(|x|^{-(n-1)}) \quad \text{as } |x| \to \infty, \end{aligned}$$

and Fredholm's alternative theorem imply the existence of the inverse operator:

$$(I + T_\lambda)^{-1} \in \mathcal{BA}(\dot{U}_\epsilon, \mathcal{L}(L_{p,R}(\Omega))).$$

Therefore, we can define $R(\lambda)$ by the relations:

$$R_0(\lambda) = \Phi_\lambda(I+T_\lambda)^{-1}, \;\; R_1(\lambda) = \Psi(I+T_\lambda)^{-1}.$$

By this, (5.1) and (5.2), we can show Theorem 5.1.

5.2. *2nd step. Modification of* $R(\lambda)$. By using the special structure of Neumann boundary condition, we modify $R(\lambda)$ to prove Theorem 4.4, especially (4.4). In order to do this, we use the following reduction: Given $f \in L_p(\Omega)^n$, let u and π be solutions to the resolvent problem:

$$\begin{aligned} &\lambda u - \operatorname{Div}\mathrm{T}(u,\pi) = f, \;\; \operatorname{div} u = 0 \text{ in } \Omega, \\ &\mathrm{T}(u,\pi)\nu|_{\partial\Omega} = 0. \end{aligned}$$

We set

$$u = E_\lambda f_0 + v \text{ and } \pi = \Pi f_0 + \theta.$$

Then, v and θ enjoy the equation:

$$\begin{aligned} &\lambda v - \operatorname{Div}\mathrm{T}(v,\theta) = 0, \;\; \operatorname{div} v = 0 \text{ in } \Omega, \\ &\mathrm{T}(v,\theta)\nu|_{\partial\Omega} = -\mathrm{T}(E_\lambda f_0, \Pi f_0)\nu|_{\partial\Omega}. \end{aligned}$$

Since

$$(\mathrm{T}(E_\lambda f_0, \Pi f_0)\nu, p_j)_{\partial\Omega} = -(\operatorname{Div}\mathrm{T}(E_\lambda f_0, \Pi f_0), p_j)_{\Omega^c} = -(\lambda E_\lambda f_0, p_j)_{\Omega^c}$$

for $j = 1, \dots, M$, there exists (w, τ) which solves the equation:

$$\begin{aligned} &\lambda w - \mathrm{Div}\, \mathrm{T}(w,\tau) = g_\lambda, \ \ \mathrm{div}\, w = 0 \ \text{ in } \Omega_R, \\ &\mathrm{T}(w,\tau)\nu|_{\partial\Omega} = -\mathrm{T}(E_\lambda f_0, \Pi f_0)\nu|_{\partial\Omega}, \\ &\mathrm{T}(w,\tau)\nu_0|_{S_R} = 0, \end{aligned}$$

where

$$g_\lambda = \sum_{j=1}^{M} (\lambda E_\lambda f_0, p_j)_{\Omega^c} p_j.$$

We set

$$v = \varphi w + z - \mathbb{B}[(\nabla\cdot\varphi)w] \text{ and } \theta = \varphi\tau + \omega,$$

and then z and ω enjoy the equation:

$$\lambda z - \mathrm{Div}\, \mathrm{T}(z,\omega) = h_\lambda, \ \ \mathrm{div}\, z = 0 \ \text{ in } \Omega, \quad \mathrm{T}(z,\omega)\nu|_{\partial\Omega} = 0,$$

where

$$\begin{aligned} h_\lambda = &-\varphi g_\lambda + 2(\nabla\varphi) : \nabla w + (\Delta\varphi) w \\ &- \lambda \mathbb{B}[(\nabla\varphi)\cdot w] + \mathrm{Div} D(\mathbb{B}[(\nabla\varphi)\cdot w]) - (\nabla\varphi)\tau. \end{aligned}$$

We can divide h_λ into two parts : $h_\lambda = h^1_\lambda + \lambda h^2_\lambda$, where

$$\begin{aligned} &\mathrm{supp}\, h^j_\lambda \subset D_{R-2,R-1} = \{x \in \mathbb{R}^n \mid R-2 \le |x| \le R-1\}, \\ &(h^1_\lambda, p_j)_{\mathbb{R}^n} = 0, \ \ j = 1, \dots, M. \end{aligned}$$

Finally, we set

$$z = z^1 + \lambda R_0(\lambda) h^2_\lambda \ \text{ and } \omega = \omega^1 + \lambda R_1(\lambda) h^2_\lambda,$$

and then z^1 and ω^1 enjoy the equation:

$$\lambda z^1 - \mathrm{Div}\, \mathrm{T}(z^1,\omega^1) = h^1_\lambda, \ \ \mathrm{div}\, z^1 = 0 \ \text{ in } \Omega, \quad \mathrm{T}(z^1,\omega^1)\nu|_{\partial\Omega} = 0.$$

Now, let us set

$$\mathcal{I} = \{f \in L_p(\mathbb{R}^n)^n \mid \mathrm{supp}\, f \subset D_{R-2,R-1}, \ (f, p_j)_{\mathbb{R}^n} = 0 \ (j = 1, \dots, M)\}.$$

Since $h^1_\lambda \in \mathcal{I}$, we consider the problem :

$$\begin{aligned} &\lambda u - \mathrm{Div}\, \mathrm{T}(u,\pi) = f, \quad \ \mathrm{div}\, u = 0 \ \text{ in } \Omega, \\ &\mathrm{T}(u,\pi)\nu|_{\partial\Omega} = 0 \end{aligned}$$

with $f \in \mathcal{I}$. Recall that

$$\begin{aligned} &\lambda\Phi_\lambda f - \mathrm{Div}\, \mathrm{T}(\Phi_\lambda f_0, \Psi f) = (I + T_\lambda) f, \qquad \mathrm{div}\, \Phi_\lambda f = 0 \ \text{ in } \Omega, \\ &T(\Phi_\lambda f, \Psi f)\nu|_{\partial\Omega} = 0. \end{aligned}$$

The point is that we can divide T_λ into two parts: $T_\lambda = A_\lambda + \lambda B_\lambda$, where

$$\begin{aligned} &A_\lambda \text{ is a compact operator on } \mathcal{I}; \\ &\|A_\lambda f - A_0 f\|_{L_p} \le C|\lambda|^{1/2} \|f\|_{L_p}; \\ &B_\lambda \text{ is a bounded operator from } \mathcal{I} \text{ into } L_{p,R}(\Omega). \end{aligned}$$

Therefore, if we set

$$\begin{aligned} U_\lambda f &= \Phi_\lambda f - \lambda R_0(\lambda) B_\lambda f, \\ \Theta_\lambda f &= \Psi f - \lambda R_1(\lambda) B_\lambda f, \end{aligned}$$

then we see that

$$\begin{aligned} &\lambda U_\lambda f - \operatorname{Div} \mathrm{T}(U_\lambda f, \Theta_\lambda f) = f + A_\lambda f, \quad \operatorname{div} U_\lambda f = 0 \ \text{ in } \Omega, \\ &\mathrm{T}(U_\lambda f, \Theta_\lambda f)\nu|_{\partial\Omega} = 0. \end{aligned}$$

By using the uniqueness of the solution to the Stokes equation with Neumann boundary condition and the Fredholm alternative theorem, we can show that there exists an $\epsilon > 0$ such that

$$(I + A_\lambda)^{-1} \in \mathcal{BA}(\dot{U}_\epsilon, \mathcal{L}(\mathcal{I})).$$

From these consideration, by using not only (5.1) and Theorem 5.1 but also the relation:

$$E_\lambda f = \lambda^{\frac{n}{2}} (\log \lambda)^{\sigma(n)} G_1'(\lambda) f + G_2(\lambda) f, \quad f \in \mathcal{I},$$

on B_R with some $G_1'(\lambda) \in \mathcal{BA}(U_{1/2}, \mathcal{L}(\mathcal{I}, W_p^2(B_R)^n \times W_p^1(B_R)))$, we can show the following proposition.

Proposition 5.2. *There exist operators $Y(\lambda)$ and $Z(\lambda)$ such that for any $f \in L_p(\Omega)^n$*

$$\begin{aligned} &(\lambda + A)^{-1} P_p f = Y(\lambda) f + Z(\lambda) f, \ \ \lambda \in \Sigma_\epsilon \cap U_\epsilon, \\ &\|Y(\lambda) f\|_{L_q(\Omega)} \le C_{p,q} |\lambda|^{\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - 1} \|f\|_{L_p(\Omega)}, \\ &\|\nabla Y(\lambda) f\|_{L_q(\Omega)} \le C_{p,q} |\lambda|^{\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{1}{2}} \|f\|_{L_p(\Omega)}, \end{aligned}$$

for any $1 < p \le q \le \infty$ $(p \ne \infty)$, $\lambda \in \dot{U}_\epsilon$ and

$$Z(\lambda) f \in \mathcal{BA}(U_\epsilon, \mathcal{L}(L_p(\Omega), W_\infty^2(\Omega))), \quad \operatorname{supp} Z(\lambda) f \subset B_R.$$

If we write

$$T(t) f = \frac{1}{2\pi} \int_{\Gamma_1} e^{\lambda t} (\lambda I + A)^{-1} f \, d\lambda + \frac{1}{2\pi} \int_{\Gamma_2} e^{\lambda t} (Y(\lambda) + Z(\lambda)) f \, d\lambda$$

where

$$\Gamma_1 = \{s e^{\pm i\theta_0} \mid \epsilon \le s < \infty\}, \ \frac{\pi}{2} < \theta_0 < \pi, \quad \Gamma_2 = \{\epsilon e^{i\theta} \mid -\theta_0 \le \theta \le \theta_0\},$$

then by Proposition 5.2 and Theorem 3.2 we can show Theorem 4.4.

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