ORLICZ CENTENARY VOLUME
BANACH CENTER PUBLICATIONS, VOLUME 64
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2004

ORLICZ AND UNCONDITIONALLY CONVERGENT SERIES IN L^1

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Abstract. We revisit Orlicz's proof of the square summability of the norms of the terms of an unconditionally convergent series in L^1. The result is then used to motivate abstract generalizations and concrete improvements.

From the earliest days of Banach space theory there was a fascination with unconditionally convergent series.

In his thesis, Banach noted (and took great advantage of) the unconditional convergence of absolutely convergent series in *complete* normed linear spaces. That this phenomenon *characterizes* the Banach spaces among all normed linear spaces hints at the wisdom of spending one's analytic life in complete spaces.

Banach, Mazur and their Lwów colleagues were intrigued by the relationship of unconditionally convergent series and absolutely convergent series and conjectured (in the Scottish book no less) that the equivalence of these two notions was characteristic of finite-dimensional spaces.

The mysteries of unconditional convergence were too attractive to remain hidden for long. Already in 1929, Orlicz had sensed the need to understand unconditional convergence while studying orthogonal series. In fact, in the second of his series of papers on the subject of orthogonal series, he takes special note of the fact that *if X is a weakly sequentially complete Banach space and if $\sum_n x_n$ is a series composed of terms in X for which $\sum_n |x^*(x_n)| < \infty$ for each continuous linear functional x^* on X, then $\sum_n x_n$ is unconditionally convergent in X.* A close inspection of Orlicz's proof ought to convince anyone who wonders of the fairness of including Orlicz's name in the 'Orlicz-Pettis' Theorem!

An aside: we would be remiss if we didn't comment on Orlicz's natural instinct to home in on the condition: *if $\sum_n |x^* x_n| < \infty$ for each $x^* \in X^*$, then $\sum_n x_n$ is (unconditionally)*

2000 *Mathematics Subject Classification*: 43, 46, 47.

Key words and phrases: Rademacher functions, uniform convexity, cotype, tensor products, Lebesgue spaces.

The paper is in final form and no version of it will be published elsewhere.

convergent. It would be a quarter of a century before two young Poles, Bessaga and Pełczyński, would show that *it is precisely in Banach spaces X which do not contain any isomorphic copy of c_0 that the condition that $\sum_n |x^*(x_n)| < \infty$ for each $x^* \in X^*$ suffices for the unconditional convergence of the series $\sum_n x_n$ in X.*

This theorem of Bessaga and Pełczyński is an excellent example of a result whose depth is due to its clarity of formulation and utility, rather than the complexity of its proof. It is an isomorphic invariant of great theoretical import.

The Lebesgue spaces $L^p(0,1)$, and their cousins $\ell^p, 1 \leq p < \infty$, were the principal examples of spaces that were known to be weakly sequentially complete at the time of Orlicz's early work (soon to be joined, by the way, by Orlicz spaces generated by functions satisfying the Δ_2-condition). That $L^p(0,1)$ and ℓ^p were weakly sequentially complete was due to F. Riesz in case $1 < p < \infty$, to Schur for ℓ^1 and to Steinhaus for $L^1(0,1)$. Studying unconditional convergence of series in these Lebesgue spaces led Orlicz to discover some remarkably sharp results; these results appeared in two short notes published in Studia in 1933. They conclude to the following:

THEOREM (Orlicz). *Let $\sum_n f_n$ be an unconditionally convergent series in $L^p(0,1)$.*

(1) If $1 \leq p \leq 2$, then $\sum_n ||f_n||_p^2 < \infty$.
(2) If $2 \leq p \leq \infty$, then $\sum_n ||f_n||_p^p < \infty$.

We find the case $p = 1$ particularly fascinating and believe a detailed exposition of Orlicz's proof of it insightful, so here it goes.

Because $\sum_n f_n$ is unconditionally convergent there is an $M > 0$ so that for all n and all choices $\sigma_i = \pm 1$ of signs,

$$\Big\| \sum_{i \leq n} \sigma_i f_i \Big\|_1 \leq M.$$

Suppose (r_n) is the Rademacher sequence. Then for any n we have

$$\Big(\sum_{i\leq n} \|f_i\|_1^2\Big)^{\frac{1}{2}} = \Big(\sum_{i\leq n}\Big(\int_0^1 |f_i(s)|ds\Big)^2\Big)^{\frac{1}{2}} = \Big\|\Big(\int_0^1 |f_1(s)|ds, \cdots, \int_0^1 |f_n(s)|ds\Big)\Big\|_{\ell_n^2} \quad (1)$$

$$= \Big\| \int_0^1 (|f_1(s)|, \cdots, |f_n(s)|)ds \Big\|_{\ell_n^2} \leq \int_0^1 \|(|f_1(s)|, \cdots, |f_n(s)|)\|_{\ell_n^2} ds \quad (2)$$

$$= \int_0^1 \Big(\sum_{i\leq n} |f_i(s)|^2\Big)^{\frac{1}{2}} ds = \int_0^1 \Big(\int_0^1 \Big|\sum_{i\leq n} f_i(s) r_i(t)\Big|^2 dt\Big)^{\frac{1}{2}} ds \quad (3)$$

$$\leq \int_0^1 \Big(K \int_0^1 \Big|\sum_{i\leq n} f_i(s) r_i(t)\Big| dt\Big) ds = K \int_0^1 \Big(\int_0^1 \Big|\sum_{i\leq n} r_i(t) f_i(s)\Big| ds\Big) dt \quad (4)$$

$$= K \int_0^1 \Big\|\sum_{i\leq n} r_i(t) f_i\Big\|_{L^1(0,1)} dt \leq K \int_0^1 M dt = KM, \quad (5)$$

where K is the constant in Khinchin's inequality that assures us that regardless of

$a_1, \cdots, a_n$,

$$\Big(\int_0^1 \Big|\sum_{i\le n} a_i r_i(t)\Big|^2 dt\Big)^{\frac{1}{2}} \le K \int_0^1 \Big|\sum_{i\le n} a_i r_i(t)\Big| dt.$$

We've stated/proved Orlicz's Theorem for $L^1(0,1)$ but it holds for any $L^1(\mu)$ by the slightest of modifications. Indeed, no proof arising in this note calls on any subtlety of measure theory and if we choose to present the proofs for $L^1(0,1)$ it's purely for our convenience; everything clearly holds for any $L^1(\mu)$-space.

Orlicz's proof was the first use in functional analysis of the inequalities now universally ascribed to Khinchin. In fact, their formulation in $L^p(0,1)$-terms was fresh-off-the-press in Orlicz's day coming from a remarkable sequence of papers of Paley and Zygmund. To be sure, recall what these inequalities say (and this formulation seems to have first been made in Zygmund's famous treatise on 'Trigonometric Series'): for any $1 \le p < \infty$ there are constants A_p, $B_p > 0$ such that, regardless of scalars $a_1, \cdots, a_n$ we have

$$B_p\Big(\int_0^1 \Big|\sum_{i\le n} a_i r_i(t)\Big|^p dt\Big)^{\frac{1}{p}} \le \Big(\sum_{i\le n}|a_i|^2\Big)^{\frac{1}{2}} \le A_p\Big(\int_0^1 |\sum_{i\le n} a_i r_i(t)|^p dt\Big)^{\frac{1}{p}}.$$

Orlicz's proof offers much if we but pay close attention. For instance, one can easily glean from it that given $f_1, \cdots, f_n \in L^1(0,1)$, then

$$\Big(\sum_{i\le n}\|f_i\|_1^2\Big)^{\frac{1}{2}} \le K \int_0^1 \Big\|\sum_{i\le n} r_i(t) f_i\Big\|_{L^1(0,1)} dt.$$

It was roughly 40 years before the language of Banach space theory included 'cotype' and 'type' but Orlicz proved that '$L^1(0,1)$ has cotype 2', regardless!

Quantites like

$$\Big(\int_0^1 \Big\|\sum_{i\le n} r_i(t) x_i\Big\|^p dt\Big)^{\frac{1}{p}}$$

arise frequently in modern abstract analysis and it's a remarkable fact that such quantities grow asymptotically alike, independent of $1 \le p < \infty$; indeed, Kahane discovered that whenever $1 \le p$, $q < \infty$ there is a constant $K_{p,q} > 0$ so that regardless of the Banach space X,

$$\Big(\int_0^1 \Big\|\sum_{i\le n} r_i(t) x_i\Big\|^p dt\Big)^{\frac{1}{p}} \le K_{p,q}\Big(\int_0^1 \Big\|\sum_{i\le n} r_i(t) x_i\Big\|^q dt\Big)^{\frac{1}{q}}$$

for any $x_1, \cdots, x_n \in X$. For future reference we call $Rad(X)$ the closed linear span of the collection of members of $L^p_X(0,1)$ of the form $r_n \otimes x, n \in N$ and $x \in X$. No index p is necessary so long as we're using isomorphic language.

Another tid-bit to be gleaned from Orlicz's proof is that if the 'Rademacher averages'

$$\int_0^1 \Big\|\sum_{i\le n} r_i(t) f_i\Big\|_{L^1(0,1)} dt$$

are bounded in n, then $\|(\sum_{i\le n}|f_i(t)|^2)^{\frac{1}{2}}\|_{L^1(0,1)}$ is also bounded in n; after all,

$$\Big\|\Big(\sum_{i\le n}|f_i(t)|^2\Big)^{\frac{1}{2}}\Big\|_{L^1(0,1)} \le K\int_0^1 \Big\|\sum_{i\le n} r_i(t)f_i\Big\|_{L^1(0,1)}dt.$$

Naturally, this says that $(\sum_n |f_n(\cdot)|^2)^{\frac{1}{2}} \in L^1(0,1)$, thanks to Fatou's lemma. In other words, the boundedness of the 'Rademacher average' ensures the integrability of the square function $(\sum_n |f_n(\cdot)|^2)^{\frac{1}{2}}$. We'll return to this later.

Though this is neither the time nor the place to discuss in great detail the remarkable inequalities of Khinchin, we would be derelict in our duties if we didn't at least mention further Polish connections. In a stunning piece of extremal analysis, Szarek showed that the best K in the real case of Khinchin's comparison of $L^1(0,1)$ and $L^2(0,1)$ is $\sqrt{2}$; later, two students of Kwapień, Latała and Oleszkiewicz, gave a stunning new proof of Szarek's result which also showed that $\sqrt{2}$ works for the vector-valued case (Kahane's inequality) as well. We highly recommend an enjoyable reading of their paper or of Kwapień's rearrangement of their proof as found in the Notes and Remarks (pp. 227-228) of the book 'Absolutely Summing Operators'. No serious discussion of best constants in Khinchin can fail to marvel at Haagerup's tour-de-force of classical analysis in finding best constants in case $1 < p < \infty$ or of Sawa's proof that $\sqrt{\pi/2}$ is best possible constant in case of complex coefficients in the $L^1(0,1)$ vs. $L^2(0,1)$ Khinchin inequality.

It's natural to discuss cotype 2 further but we delay such a discussion for a bit so as to mention other work related to Orlicz's theorem.

There are really at least two aspects of Orlicz's theorem that need be discussed: The reflexive case $(1 < p < \infty)$ and the $L^1(0,1)$-situation.

$L^p(0,1)$ is more than reflexive, if $1 < p < \infty$; it's uniformly convex. Recall with Clarkson that a Banach space X is *uniformly convex* if given $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ so that regardless of $x, y \in X$ with $\|x\| = 1 = \|y\|$ we have that whenever $\|x - y\| \ge \epsilon$ then $\|\frac{x+y}{2}\| \le 1 - \delta$. If we define the modulus of convexity of X by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = 1 = \|y\|, \|x-y\| = \epsilon\right\}, \qquad 0 < \epsilon \le 2,$$

then $\delta_X(\epsilon) > 0$ whenever $\epsilon > 0$ signals X's uniform convexity. Refining Clarkson's inequalities, Hanner estimated the moduli of convexity of $L^p(0,1)$ for $1 < p < \infty$. He showed that

$$\delta_{L^p(0,1)}(\epsilon) \ge \begin{cases} c\epsilon^2 & \text{if } 1 < p \le 2, \\ c\epsilon^p & \text{if } 2 < p \le \infty. \end{cases}$$

The relevance? Well, a wonderful abstract theorem of father Kadets goes as follows.

THEOREM (Kadets). *If $\sum_n x_n$ is an unconditionally convergent series in a uniformly convex space X, then $\sum_n \delta_X(\|x_n\|) < \infty$.*

For Banach-space-geometry enthusiasts the only *possible* drawback to Kadets's theorem is that to be uniformly convex entails reflexivity. This 'drawback' aside, we cannot be more enthusiastic in our own recommendation to read the lovely book of the Kadets familia on 'Series in Banach Spaces'.

Curiously, Kadets's *idea* has merit even in the case of L^1; right idea, wrong scalar field. We say a complex Banach space X is *complex uniformly convex* if given $\epsilon > 0$ there is a $\delta > 0$ so that whenever $\|x + \mu y\| \leq 1$ for all $|\mu| \leq 1$ and $\|y\| \geq \epsilon$ then $\|x\| \leq 1 - \delta$; alternatively, X is complex uniformly convex whenever $H_\infty^X(\epsilon) > 0$ for any $\epsilon > 0$, where

$$H_\infty^X(\epsilon) = \inf_{\|x\|=1, \|y\|=\epsilon} \{\sup\{\|x + e^{i\theta} y\| : 0 \leq \theta \leq 2\pi\} - 1\}.$$

Naturally,

$$1 + H_\infty^X(\epsilon) = \inf_{x \neq 0, \|y\|=\epsilon} \sup_{0 \leq \theta \leq 2\pi} \left\| \frac{x}{\|x\|} + e^{i\theta} y \right\|. \qquad (*)$$

Globevnik showed that in case $X = L^1(0,1)$ then the function inverse to $H_\infty^{L_1}(\epsilon)$ is $\leq 10\delta^{\frac{1}{2}}$ so $H_\infty^{L^1} \geq \epsilon^2/10$, making the following theorem of Dilworth extremely satisfying.

THEOREM (Dilworth). *If $\sum_n x_n$ is an unconditionally convergent series in the complex uniformly convex space X, then $\sum_n H_\infty^X(\|x_n\|) < \infty$.*

Proof. We start with a bit of harmless but helpful normalization: suppose $\| \sum_{j \leq n} \alpha_j x_j \| \leq 1$, for any n and any scalars $\alpha_1, \cdots, \alpha_n$ with $|\alpha_j| \leq 1$. In particular, $\|x_j\| \leq 1$ for all j. We'll also assume none of the x_j's is zero.

Notice that $(*)$ tells us that

$$\|x_1\|(1 + H_\infty^X(\|x_2\|)) \leq \max_{0 \leq \theta \leq 2\pi} \|x_1 + e^{i\theta}\|x_1\| x_2\|,$$

so that there is a $\theta_2 : 0 \leq \theta_2 \leq 2\pi$ so that

$$\|x_1\|(1 + H_\infty(\|x_2\|)) \leq \|x_1 + e^{i\theta_2}\|x_1\| x_2\|.$$

Similarly, we find a $\theta_3 : 0 \leq \theta_3 \leq 2\pi$ so that

$$1 + H_\infty(\|x_3\|) \leq \left\| \frac{x_1 + e^{i\theta_2}\|x_1\| x_2}{\|x_1 + e^{i\theta_2}\|x_1\| x_2\|} + e^{i\theta_3} x_3 \right\|$$

and so

$$\|x_1 + e^{i\theta_2}\|x_1\| x_2\| \, (1 + H_\infty(\|x_3\|)) \leq \|x_1 + e^{i\theta_2}\|x_1\| x_2 + e^{i\theta_3}\|x_1 + e^{i\theta_2}\|x_1\| x_2\| x_3\| \leq 1,$$

thanks to our normalizing of relations before starting this fray. Repeating and letting $\mu_2 = \|x_1\|, \mu_3 = \|x_1 + e^{i\theta_2}\|x_1\| x_2\|, \cdots$, we find θ_n's $(n \geq 2)$ so $0 \leq \theta_n \leq 2\pi$ and μ_n's $(n \geq 2)$ so that $|\mu_n| \leq 1$ so that

$$\begin{aligned} &\|x_1 + e^{i\theta_2}\mu_2 x_2 + \cdots + e^{i\theta_n}\mu_n x_n\|(1 + H_\infty(\|x_{n+1}\|)) \leq \\ &\|x_1 + e^{i\theta_2}\mu_2 x_2 + \cdots + e^{i\theta_n}\mu_n x_n + e^{i\theta_{n+1}}\mu_{n+1} x_{n+1}\| \leq 1. \end{aligned}$$

Collect terms with the proper amount of love and care and discover that

$$\|x_1\| \prod_{k=2}^{n} (1 + H_\infty(\|x_k\|)) \leq \|x_1 + e^{i\theta_2}\mu_2 x_2 + \cdots + e^{i\theta_{n+1}}\mu_{n+1} x_{n+1}\| \leq 1.$$

It follows that for each n

$$\sum_{k=2}^{n} H_\infty(\|x_n\|) \leq \|x_1\|^{-1}.$$

Tra la, Tra la.

THEOREM (Globevnik). *If $f, g \in L^1(0,1)$ with $\|f\|_1 = 1$ and $\|f + zg\|_1 \le 1 + \delta$ whenever $|z| \le 1$, then $\|g\| \le \sqrt{\delta}(4 + 2\sqrt{1+2\delta})$.*

Proof. We can, and do, assume that $|f(t)| < \infty$ for all t. Everything that's meaningful happens inside the set

$$P = [g(t) \neq 0];$$

on P set

$$h(t) = \frac{f(t)}{g(t)}.$$

Let $c > 0$. Look at the events

$$P_1 = [|h| > c] \cap P, \quad P_2 = [|h| \le c] \cap P.$$

Of course,

$$(*) \quad \int_{P_1} |g| = \int_{P_1} \left|\frac{f}{h}\right| \le \frac{1}{c} \int_{P_1} |f| \le \frac{1}{c}.$$

Also our hypotheses guarantee that

$$\int (|f+g| + |f-g| + |f+ig| + |f-ig| - 4|f|) \le 4(1+\delta) - 4 = 4\delta.$$

Consider the function

$$Q(r,\theta) = |re^{i\theta} + i| + |re^{i\theta} - i| - 2r$$

for $r \ge 0$. For a fixed r, $\theta \to Q(r,\theta)$ is continuous and nonnegative. From $-\pi/2$ to 0 it's increasing while from 0 to $\pi/2$ it's decreasing. Regardless, $Q(r,-\theta) = Q(r,\theta)$ and $Q(r,\theta+\pi) = Q(r,\theta)$. Alas, it follows that

$$Q(r,\theta) + Q(r,\theta+\pi/2) \ge Q(r,\pi/4).$$

Now, about $Q(r,\pi/4)$:

$$Q(r,\pi/4) = (r^2 + 2^{\frac{1}{2}}r + 1)^{\frac{1}{2}} + (r^2 - 2^{\frac{1}{2}}r + 1)^{\frac{1}{2}} - 2r$$

and so

$$(Q(r,\pi/4) + 2r)^2 \ge 4r^2 + 2$$

Since $Q(r,\pi/4) + 2r \ge 0$ we conclude that

$$Q(r,\pi/4) + 2r \ge (4r^2+2)^{\frac{1}{2}}.$$

Next look at the function

$$F(t) = \begin{cases} |h(t)+1| + |h(t)-1| + |h(t)+i| + |h(t)-i| - 4|h(t)| & \text{if } t \in P, \\ 0 & \text{if } t \notin P. \end{cases}$$

Notice that $F(t) \ge 0$ and

$$\int F|g| = \int (|f+g| + |f-g| + |f+ig| + |f-ig| - 4|f|) \le 4\delta.$$

But F also has the form

$$F(t) = Q(|h(t)|, \arg h(t)) + Q(|h(t)|, \arg(h(t) + \pi/2))$$

and $|h(t)| \le c$ for $s \in P_2$. So for $s \in P_2$ we have

$$F(s) \ge (4|h(s)|^2 + 2)^{\frac{1}{2}} - 2|h(s)| \ge (4c^2+2)^{\frac{1}{2}} - 2c$$

because, after all, $(4r^2+2)^{\frac{1}{2}} - 2r$ is positive and *decreasing* in r. Now we're 'in business':

$$4\delta \geq \int_{P_2} F|g| \geq \int_{P_2} |g|[(4c^2+2)^{\frac{1}{2}} - 2c]$$

and so

$$\frac{4\delta}{(4c^2+2)^{\frac{1}{2}} - 2c} \geq \int_{P_2} |g|.$$

In sum

$$\|g\|_1 = \int_{P_1} + \int_{P_2} |g| \leq \frac{1}{c} + \frac{4\delta}{(4c^2+2)^{\frac{1}{2}} - 2c}.$$

Choose c *judiciously*, say $= 1/2\delta^{\frac{1}{2}}$, and the result is

$$\|g_1\|_1 \leq 2\delta^{\frac{1}{2}} + \frac{4\delta}{(2+1/\delta)^{\frac{1}{2}} - \delta^{-\frac{1}{2}}} = \delta^{\frac{1}{2}}(4+2(1+2\delta)),$$

if one can but believe in the algebra we learned eons ago.

To proceed further, we need to enter the enigmatic world of tensor products. For our present purposes only two tensor norms need exposure: the injective and projective tensor norms, the most classical of all tensor norms.

Let X and Y be Banach spaces (over the same scalar field). For $u \in X \otimes Y$ we define the injective tensor norm $\|u\|_\vee$ of u by

$$\|u\|_\vee = \sup\{|(x^* \otimes y^*)(u)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\}$$

and the projective tensor norm $\|u\|_\wedge$ of u by

$$\|u\|_\wedge = \inf\Big\{\sum_{i\leq n} \|x_i\|\ \|y_i\| : u = \sum_{i\leq n} x_i \otimes y_i\Big\}.$$

Each of these norms is reasonable ($\|x\otimes y\|_\wedge = \|x\|\ \|y\| = \|x\otimes y\|_\vee$) and uniform (if $u_1 : X_1 \to X_2$ and $u_2 : Y_1 \to Y_2$ are bounded linear operators then $u_1 \otimes u_2$ is a bounded linear operator with bound $\leq \|u_1\|\ \|u_2\|$). They're symmetric (so $X \otimes Y$ and $Y \otimes X$ are naturally isometrically isomorphic). Fortunately, unless everything in sight is finite dimensional we cannot hope for completeness so we complete $X \otimes Y$ and obtain the injective tensor product $X \check{\otimes} Y$ of X and Y and the projective tensor product $X \hat{\otimes} Y$ of X and Y.

In case of the injective tensor product, a few choice identifications are worth mentioning: for any compact Hausdorff space S, $C(S)\check{\otimes}X$ *is isometrically isomorphic to the space* $C_X(S)$ *of* X*-valued continuous functions defined in* S *with the usual supremum norm* and $\ell^1\check{\otimes}X$ *is isometrically isomorphic to the space of unconditionally convergent series in* X *with norm* $\|(x_n)\| = \sup_{\|x^*\|\leq 1}\left\{\sum_n |x^* x_n|\right\}$. The injective product is injective so $X_0\check{\otimes}Y$ is a subspace of $X\check{\otimes}Y$ if X_0 is a subspace of X.

The projective tensor product is (as one can see from the form of its norm) the biggest tensor norm on $X \otimes Y$ and a description of members u of $X\hat{\otimes}Y$ is possible (and due to Grothendieck): $u \in X\hat{\otimes}Y$ *precisely when there are sequences* $(x_n) \subseteq X$ *and* $(y_n) \subseteq Y$ *so that* $\sum_n \|x_n\|\ \|y_n\| < \infty$ *and* $u = \sum_n x_n \otimes y_n$; *in such a case* $\|u\|_\wedge = \inf\left\{\sum_n \|x_n\|\ \|y_n\| : u = \sum_n x_n \otimes y_n\right\}$.

THEOREM (Grothendieck). *For any measure μ, $L^1(\mu)\hat{\otimes}X$ is isometrically isomorphic to $L^1_X(\mu)$, the space of (equivalence classes of) Bochner μ-integrable X-valued functions. In particular, $\ell^1\hat{\otimes}X$ is isometrically isomorphic to $\ell^1(X)$.*

The norm $\|\cdot\|_\wedge$ *is* projective so whenever X_0 is a closed linear subspace of X and $q : X \to X/X_o$ the canonical quotient map then $q \otimes id_Y$ is a metric linear quotient map of $X\hat{\otimes}Y$ onto $(X/X_o)\hat{\otimes}Y$. It is *not* injective despite the above theorem; indeed, Grothendieck also showed the following remarkable result.

THEOREM (Grothendieck). *If Z is a Banach space such that $Z\hat{\otimes}X$ is a closed linear subspace of $Z\hat{\otimes}Y$ whenever X is a closed linear subspace of Y, then Z is isometrically isomorphic to an $L^1(\mu)$-space.*

An important aspect of the projective tensor product is the Universal Mapping Property: *$(X\hat{\otimes}Y)^*$ is identifiable with the space $\mathcal{B}(X,Y)$ of continuous bilinear functionals on $X \times Y$, with $\tau^* \in (X\hat{\otimes}Y)^*$ corresponding to $Q \in \mathcal{B}(X,Y)$ via the formula $\tau^*(x \otimes y) = Q(x,y)$.*

If E and F are finite dimensional Banach spaces then so are $(E \otimes F, \|\quad\|_\vee)$ and $(E \otimes F, \|\quad\|_\wedge)$; What's more, in this case, $(E\check{\otimes}F)^* = E^*\hat{\otimes}F^*$ and $(E\hat{\otimes}F)^* = E^*\check{\otimes}F^*$. For infinite dimensional spaces life is not so kind; however, *if Y is a Banach space where dual Y^* has the Radon-Nikodym property and the approximation property, then $(X\check{\otimes}Y)^*$ is $X^*\hat{\otimes}Y^*$, regardless of X.*

Where's this leading? Here's a theorem found in the Résumé and ascribed by Grothendieck to Littlewood.

THEOREM. *If $\sum_n f_n$ is an unconditionally convergent series in $L^1(\mu)$, then $(f_n) \in \ell^2\hat{\otimes}L^1(\mu)$.*

To be sure, in a paper following his Résumé, Grothendieck made an incisive analysis of the Dvoretzky-Rogers lemma and among other delicious discoveries he showed that *if $1 < p < \infty$ and X is an infinite dimensional Banach space then*

$$\ell^p\hat{\otimes}X \;\subset\; \ell^p(X) \;\subset\; \ell^p_{\text{weak}}(X),$$

where $\ell^p_{\text{weak}}(X)$ is the space of sequences (x_n) in X for which $\sum_n |x^*(x_n|^p < \infty$ for each $x^* \in X^*$ and $\ell^p(X)$ is the space of sequences (x_n) in X for which $\sum_n \|x_n\|^p < \infty$, and $\subset$ indicates proper containment. So the above theorem is stronger than is Orlicz's. But what does it actually say? In his analysis of $\ell^p\hat{\otimes}X$, Grothendieck side-steps any description of which sequences are in $\ell^p\hat{\otimes}X$ (except for $p = 1$). However we *can* say something because we're dealing with $\ell^2\hat{\otimes}L^1(\mu)$ which is just $L^1(\mu)\hat{\otimes}\ell^2$ in reversed order; $L^1(\mu)\hat{\otimes}\ell^2$ is $L^1_{\ell^2}(\mu)$ so what's it take for a sequence (f_n) to be in $\ell^2\hat{\otimes}L^1(\mu)$?

It must be that $(\sum_n |f_n|^2)^{\frac{1}{2}} \in L^1(\mu)$. Look familiar? It should: in Orlicz's proof we saw that if $\sum_n f_n$ is unconditionally convergent then there is a constant $K \cdot M$ ('K' from Khinchin and 'M' from bounded multiplier) so that for each $n \in \mathbb{N}$, $\|(\sum_{j\le n} |f_j|^2)^{\frac{1}{2}}\|_{L^1} \le KM$. The sequence $(\sum_{j\le n} |f_j|^2)^{\frac{1}{2}}_n$ converges almost everywhere to $(\sum_n |f_n|^2)^{\frac{1}{2}}$ and so Fatou's lemma assures us that looking closely at Orlicz's proof will also lead us to membership in $\ell^2\hat{\otimes}L^1(\mu)$.

Grothendieck wondered if X is a Banach space such that $\ell^1\check{\otimes}X \hookrightarrow \ell^2\hat{\otimes}X$ need X be an $L^1(\mu)$-space. In the late seventies, Kisliakov and Pisier discovered by different techniques that if R is a reflexive subspace of $L^1(\mu)$, then $L^1(\mu)/R$ also solves the inclusion $\ell^1\check{\otimes}X \hookrightarrow \ell^2\hat{\otimes}X$. Then in the early eighties, Bourgain showed that if T denotes the unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$ then $L^1(T)/H_0^1$ is another solution, where H_0^1 is the subspace of the Hardy space H^1 consisting of functions with vanishing value at 0. Since then some progress has been made *in the abstract.* Here's one such result.

THEOREM (Arregui-Blasco, Bu). *Suppose X has cotype 2. Then the following statements regarding X are equivalent.*

1. *Every operator $u : X \to \ell^2$ takes unconditionally convergent series in X to absolutely convergent series in ℓ^2.*
2. *$\ell^1\check{\otimes}X \hookrightarrow \ell^2\hat{\otimes}X$.*
3. *$Rad(X) = \ell^2\hat{\otimes}X$.*

It is unknown if the hypothesis that X have cotype 2 is necessary in the above theorem.

The property that every unconditionally convergent series in X be square summable was aptly coined by Lindenstrauss and Pełczyński as 'X has the Orlicz property'; if $\ell^1\check{\otimes}X \hookrightarrow \ell^2\hat{\otimes}X$ we'll say 'X has the Littlewood-Orlicz property'. Naturally, adding Littlewood to the team leads to a stronger property. Like the Orlicz property, the Littlewood-Orlicz property is special and conditions for its possession precious. Suppose $i : \ell^2 \hookrightarrow c_0$ and $j : \ell^1 \hookrightarrow \ell^2$ are the natural inclusions; of course, $i^* = j$.

THEOREM. *A necessary and sufficient condition that X^* has the Littlewood-Orlicz property is that $i \otimes id_X$ takes $\ell^2\check{\otimes}X$ continuously into $c_0\hat{\otimes}X$.*

Suppose $i \otimes id_X$ takes $\ell^2\check{\otimes}X$ continuously into $c_0\hat{\otimes}X$. Let's compare the duals (that is the natural codomain and domain of $(i \otimes id_X)^*$. Because ℓ^2 has both the approximation property and the Radon-Nikodym property, the domain of $(i \otimes id_X)^*$ is $\ell^2\hat{\otimes}X^* = (\ell^2\check{\otimes}X)^*$. The Universal Mapping property ensures that the dual of $c_0\hat{\otimes}X$ is the space $\mathcal{B}(c_0, X)$ of bounded bilinear forms on $c_0 \times X$ which is easily seen to be $\mathcal{L}(c_0; X^*)$, the space of continuous linear operators from c_0 into X^*. Now it's plain (and easy-to-see) that $\ell^1\check{\otimes}X^*$ embeds inside $\mathcal{L}(c_0; X^*)$ and on $\ell^1\check{\otimes}X^*$ the action of $(i \otimes id_X)^*$ is precisely that of $j \otimes id_{X^*}$ with values in $\ell^2\hat{\otimes}X^*$.

On the other hand, if $j \otimes id_{X^*}$ takes $\ell^1\check{\otimes}X^*$ continuously into $\ell^2\hat{\otimes}X^*$, then computing duals leaves us with $(\ell^1\check{\otimes}X^*)^*$ being identifiable with the space $\mathcal{B}^\wedge(\ell^1, X^*)$ of integral bilinear forms on $\ell^1 \times X^*; c_0\hat{\otimes}X$ lies isometrically inside $\mathcal{B}^\wedge(\ell^1, X^*)$ thanks to c_0 having the metric approximation property. In turn, $(\ell^2\hat{\otimes}X^*)^*$ is identifiable, thanks to the Universal Mapping property with the space $\mathcal{B}^\wedge(\ell^2, X^*)$ which contains $\ell^2\check{\otimes}X^*$ isometrically and naturally. A quick check of $(j \otimes id_{X^*})^*$ on elementary tensors shows $(j \otimes id_{X^*})^*$ takes $\ell^2\check{\otimes}X$ into $c_0\hat{\otimes}X$ much as $i \otimes id_X$ does. It follows that $i \otimes id_X$ maps $\ell^2\check{\otimes}X$ continuously into $c_0\hat{\otimes}X$, being as it is – the restriction of $(j \otimes id^*_{X^*}$ to $\ell^2\check{\otimes}X$.

Many are the topics that are attractive and relate to the Orlicz Theorem, the Orlicz property, the Littlewood-Orlicz, property, cotype 2; however, we have to finish this paper somewhere so why not on a high note with an all-too-brief discussion of one of our favorite results? It has its origins in the wonderous thesis of Maurey and the beautiful paper of

Kwapień and Pełczyński on the main triangle projection. Its relationship with what we've been discussing is established once we recall a crucial ingredient to the result's proof, a theorem of Maurey and Rosenthal.

THEOREM (Maurey, Rosenthal). *Let X be a Banach space and $v : X \to L^1(0,1)$ be a bounded linear operator. Then a necessary and sufficient condition that there be a $K > 0$ such that for any $x_1, \cdots, x_n \in X$ we have*

$$\Big\| \Big(\sum_{k \leq n} |vx_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^1} \leq K \Big(\sum_{k \leq n} \|x_k\|^2 \Big)^{\frac{1}{2}}$$

is that v admits a factorization

$$\begin{tikzcd} X \arrow[rr, "v"] \arrow[dr, "d"'] & & L^1(0,1) \\ & L^2(0,1) \arrow[ur, "M"'] & \end{tikzcd}$$

where $d : X \to L^2(0,1)$ is a bounded linear operator of norm $\leq K$ and $M : L^2(0,1) \to L^1(0,1)$ is the multiplication operator $Mf = f \cdot g$ for some $g \in L^2(0,1)$.

A stunning consequence of this was drawn independently by Bennett and by Maurey and Nahoum, a consequence certainly appreciated by the workers of the orthogonal-functions vineyards.

THEOREM (Bennett-Maurey-Nahoum). *Suppose $\sum_n f_n$ is an unconditionally convergent series in $L^1(0,1)$. Then there is a sequence $(a_n) \in \ell^2$, a $g \in L^2(0,1)$ and an orthonormal sequence (g_n) in $L^2(0,2)$ such that for each $n \in \mathbb{N}$ and almost all $t \in [0,1]$,*

$$f_n(t) = a_n g(t) g_n(t).$$

P. Ørno has given a titillating proof of this arousing result and one can find his proof fully exposed in 'Absolutely Summing Operators' on pages 253–254.

Acknowledgements. I wrote most of this paper while visiting the Department of Pure and Applied Mathematics at the University of Pretoria; once again I take great pleasure in extending my thanks to Professors Maepa, Swart and Stroh for their hospitality and comradeship during my visit. I also wish to thank the organizers of the 'Orlicz Centennial Conference' in Poznań for ensuring a pleasant atmosphere to hear and talk about so much beautiful mathematics. I am grateful to G. Saluzzo for making a molehill out of a mountain in the proof of the 'characterization' of duals with the Littlewood-Orlicz property. Finally, I'd like to express my appreciation to Ms. Misty Tackett for decoding my handwriting and turning it into this manuscript.

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