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CLASSICAL PLS-SPACES: SPACES OF DISTRIBUTIONS, REAL ANALYTIC FUNCTIONS AND THEIR RELATIVES

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Abstract. This paper is an extended version of an invited talk presented during the Orlicz Centenary Conference (Poznań, 2003). It contains a brief survey of applications to classical problems of analysis of the theory of the so-called PLS-spaces (in particular, spaces of distributions and real analytic functions). Sequential representations of the spaces and the theory of the functor Proj^1 are applied to questions like solvability of linear partial differential equations, existence of a solution depending linearly and continuously on the right hand side of the equation and existence of a solution depending analytically on parameters.

In the 1930's, the golden age of the Polish Mathematical School in Lvov (Banach, Mazur, Orlicz and others), the main emphasis in functional analysis was put on metric linear spaces. Only later, motivated by emerging applications, a new study of non-metrizable locally convex spaces started with ingenious discoveries of Grothendieck, especially with his theory of nuclear spaces.

This survey paper is devoted to a study of a special class of non-metrizable locally convex spaces, so-called PLS-spaces. The class contains many natural examples from analysis like the space of real analytic functions, the space of distributions and various spaces of ultradifferentiable functions and ultradistributions which turned out to be important for the theory of partial differential equations. Instead of a systematic "academic" presentation we preferred the more application oriented approach. We emphasize the use

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of two quite abstract tools, namely, sequential representations of the spaces in question (or an absence of such a representation!) and the machinery of the functor Proj^1. We will apply them to *non-abstract* classical problems like existence of solutions of linear partial differential equations or convolution equations, splitting of differential complexes (or, existence of linear continuous right inverses for linear differential or convolution operators) and analytic dependence of solutions of linear partial differential equations on parameters. One could say that the main message of this survey is that the presented part of the theory of locally convex spaces is extraordinarily useful in interesting questions of classical origin in spite of the prevailing scepticism towards the abstract machinery of functional analysis in general and of the theory of locally convex spaces in particular.

We try to explain (or, at least, to give a precise reference for explanation) all the necessary background from the theory of locally convex spaces in order to make the survey accessible for a wider audience. For non-explained notions see any standard book, for instance, [76] or [47].

1. PLS-spaces—examples.

Let us start with the precise definition.

DEFINITION 1.1. A locally convex space X is a *PLS-space* if it is a projective limit of a sequence of strong duals of Fréchet-Schwartz spaces (i.e., DFS-spaces). If we consider strong duals of nuclear Fréchet spaces instead (i.e., DFN-spaces) then X is called a *PLN-space.*

Roughly speaking, PLS-spaces are "regular" spaces of the form $\bigcap_{N\in\mathbb{N}}\bigcup_{n\in\mathbb{N}} X_{N,n}$, $X_{N,n}$ Banach spaces, with the natural topology. In the examples given below, one can give explicitly seminorms defining the topology of the considered spaces but from the application point of view it is better to look at them as intersections of unions of Banach spaces. It follows from the definition that every PLS-space is automatically complete and Schwartz. PLN-spaces are even nuclear and they have the approximation property. Let us note that every Fréchet-Schwartz space is automatically a PLS-space.

It is easily seen that any PLS-space $X = \mathrm{proj}_{N\in\mathbb{N}} \mathrm{ind}_{n\in\mathbb{N}} X_{N,n}$, where $X_{N,n}$ are Banach spaces, $\mathrm{ind}_{n\in\mathbb{N}} X_{N,n}$ denotes the locally convex inductive limit (direct limit) of a sequence $(X_{N,n})_{n\in\mathbb{N}}$ where the linking maps are compact, $\mathrm{proj}_{N\in\mathbb{N}} Y_N$ denotes the topological projective limit (inverse limit) of a sequence $(Y_N)_{N\in\mathbb{N}}$ of locally convex spaces (see [47], for the modern theory of locally convex inductive limits see [4]).

Let us denote by $\Omega \subseteq \mathbb{R}^d$ an open domain and by $(K_N)_{N\in\mathbb{N}}$, $K_1 \Subset K_2 \Subset \cdots \Subset \Omega$, a compact exhaustion, i.e., $\bigcup_{N\in\mathbb{N}} K_N = \Omega$. Here $\Subset$ means that one set is compact and contained in the interior of the other one.

EXAMPLE 1.2 (*The space of distributions* $\mathscr{D}'(\Omega)$). It is well known that $\mathscr{D}'(\Omega)$ is defined as the strong dual of the space $\mathscr{D}(\Omega) = \mathrm{ind}_{N\in\mathbb{N}} \mathscr{D}_{K_N}$ of test functions, where $\mathscr{D}_{K_N}$ is the nuclear Fréchet space of smooth functions with the support contained in K_N (equipped with the topology of uniform convergence with respect to all derivatives). Since the inductive limit $\mathscr{D}(\Omega)$ is strict [88, II.6.5], $\mathscr{D}'(\Omega) = \mathrm{proj}_{N\in\mathbb{N}} \mathscr{D}'_{K_N}$ and it is a PLN-space (comp. [76, 28.9(2)]).

EXAMPLE 1.3 (*The space of real analytic functions* $\mathscr{A}(\Omega) := \{f : \Omega \to \mathbb{C} : f \text{ analytic}\}$). The space $\mathscr{A}(\Omega)$ is equipped with the unique locally convex topology such that for any $U \subseteq \mathbb{C}^d$ open, $\mathbb{R}^d \cap U = \Omega$, the restriction map $R : H(U) \to \mathscr{A}(\Omega)$ is continuous and for any compact set $K \subseteq \Omega$ the restriction map $r : \mathscr{A}(\Omega) \to H(K)$ is continuous. As usual we equip the space $H(U)$ of holomorphic functions on U with the compact-open topology and the space $H(K)$ of germs of holomorphic functions on K with its natural topology:

$$H(K) = \operatorname{ind}_{n \in \mathbb{N}} H^\infty(U_n),$$

where $(U_n)_{n\in\mathbb{N}}$ is a basis of $\mathbb{C}^d$-neighbourhoods of K.

THEOREM 1.4 (Martineau, [64]). *There is exactly one topology on* $\mathscr{A}(\Omega)$ *satisfying the condition above and endowed with this topology one has*

$$\mathscr{A}(\Omega) = \operatorname{proj}_{N\in\mathbb{N}} H(K_N).$$

The topology defined above is *the* natural topology on $\mathscr{A}(\Omega)$, it is very well adapted to applications and to the structure of the space. Moreover it has several useful properties (see [64], the introductory parts of [6] and [7], [49], [34], [35], [36], [29], compare also the beautiful book on real analytic functions [50]). The topology on the space of real analytic functions plays an important role in applications (see, for instance, [51], [52], [45], [78], [55], [56], [60], [30], [31], [37]). Since for any two domains of holomorphy $\Omega_1 \Subset \Omega_2$ the restriction map $\rho : H^\infty(\Omega_2) \to H^\infty(\Omega_1)$ is nuclear (see the proof of [83, 6.4.2]), the space $H(K_N)$ is a DFN-space and $\mathscr{A}(\Omega)$ is a PLN-space (therefore, it is nuclear and has the approximation property). Using the classical result that Ω has a basis of $\mathbb{C}^d$-neighbourhoods which are domains of holomorphy (for elementary presentation of this result see [37]) one proves easily that polynomials are dense in $\mathscr{A}(\Omega)$, so $\mathscr{A}(\Omega)$ is separable.

EXAMPLE 1.5 (*The Roumieu class of ultradifferentiable functions* $\mathscr{E}_{\{\omega\}}(\Omega)$). We define this class as

$$\mathscr{E}_{\{\omega\}}(\Omega) := \{f \in C^\infty(\Omega) : \ \forall\, N \in \mathbb{N}\ \exists\, m \in \mathbb{N} : \|f\|_{N,m} < \infty\},$$

where

$$\|f\|_{N,m} := \sup_{x\in K_N} \sup_{\alpha\in\mathbb{N}^d} |f^{(\alpha)}(x)| \exp\left(-\frac{1}{m}\varphi^*\left(|\alpha| m\right)\right),$$

$$\varphi^*(t) := \sup_{x\geq 0}(xt - \varphi(t)) \quad \text{(the Young conjugate of } \varphi), \qquad \varphi(t) := \omega(e^t)$$

and $\omega : [0,\infty[\to [0,\infty[$ is a continuous increasing function (a so-called *weight*) satisfying the following conditions:

(α) $\omega(2t) = O(\omega(t))$;
(β) $\omega(t) = O(t)$;
(γ) $\log t = o(\omega(t))$;
(δ) φ is a convex function.

Let us note that for partial derivatives we use the typical multiindex notation where $|\alpha| := \alpha_1 + \cdots + \alpha_d$ for $\alpha \in \mathbb{N}^d$. The considered classes were introduced in [87] and systematically studied in [15]. The motivation comes from work of Holmgren and Gevrey who observed that the functions in the kernel of hypoelliptic linear partial differential operators are better smooth than expected. This observation led to the definition of the so-called Gevrey classes (i.e., $\mathscr{E}_{\{\omega\}}$ with $\omega(t) = t^{1/p}$, $p \in (0,1)$).

Clearly,

$$\mathscr{E}_{\{\omega\}}(\Omega) = \mathrm{proj}_{N\in\mathbb{N}}\, \mathrm{ind}_{n\in\mathbb{N}}\, \mathscr{E}_{\{\omega\},N,n}(\Omega) \quad \text{and} \quad \mathscr{E}_{\{\omega\},N,n}(\Omega) = \{f \in C^\infty(\Omega) : \|f\|_{N,n} < \infty\}$$

are Banach spaces with norms $\|f\|_{N,n}$. It is proved in [49, Prop. 2.4] (comp. [15, Cor. 3.6, Lemma 4.5] and [86, 1.16]), that $\mathscr{E}_{\{\omega\}}(\Omega)$ are PLN-spaces. If

$$\int_0^\infty \frac{\omega(t)}{1+t^2}\, dt = \infty$$

then the class (or the weight) is *quasianalytic* (i.e., there are no elements with compact support in $\mathscr{E}_{\{\omega\}}(\Omega)$). Otherwise the class is *non-quasianalytic*. The spaces $\mathscr{E}_{\{\omega\}}$ were considered, for instance, in [9], [10], [13], [14], [24], [25], [26], [57], [66], [72] and [77] .

EXAMPLE 1.6. The class of ultradistributions of Beurling type $\mathscr{D}'_{(\omega)}(\Omega)$, ω a non-quasianalytic weight, is a PLN-space (see [15] for the definition and properties).

EXAMPLE 1.7 (*The Köthe type PLS-sequence spaces* $\Lambda(A)$). Let $A = (a_{j,N,n})$ be a matrix of positive elements satisfying the following conditions:

(i) $a_{j,N,n} \geq a_{j,N,n+1}$;
(ii) $a_{j,N,n} \leq a_{j,N+1,n}$;
(iii) $\lim_{j\to\infty} \frac{a_{j,N,n+1}}{a_{j,N,n}} = 0$.

We define

$$\Lambda(A) := \{x = (x_j) : \ \forall\, N \in \mathbb{N}\ \exists\, n \in \mathbb{N} : \quad \|x\|_{N,n} < \infty\},$$

where

$$\|x\|_{N,n} := \sum_j |x_j| a_{j,N,n}.$$

Clearly,

$$\Lambda(A) = \mathrm{proj}_{N\in\mathbb{N}}\, \mathrm{ind}_{n\in\mathbb{N}}\, l_1(a_{j,N,n}),$$

where $l_1(a_{j,N,n})$ denotes the weighted l_1-space equipped with the norm $\|\cdot\|_{N,n}$. The condition (iii) implies that $\Lambda(A)$ is a PLS-space. If instead of (iii)

(iv) $\sum_j \frac{a_{j,N,n+1}}{a_{j,N,n}} < \infty$

holds then $\Lambda(A)$ is even a PLN-space.

EXAMPLE 1.8. Every closed subspace and every complete quotient of a PLS-space (PLN-space) is of the same type (see [32, Prop. 1.2, Theorem 1.3]).

2. Sequential representations. Let us recall that a sequence of elements $(f_n)_{n\in\mathbb{N}}$ of a locally convex space X is a *basis* if every element $f \in X$ can be represented uniquely as

$$f = \sum_{n=0}^{\infty} a_n f_n, \qquad (a_n) \text{ a sequence of scalars (see [47, Sec. 14.2]).}$$

There is an extensive theory of (Schauder) bases in a variety of function spaces (see the section on special bases in function spaces due to Figiel and Wojtaszczyk in [43, pp. 561–

597], see also [76, 29.5], [47, Sec. 14 and 21.10], [85, Chapter 8]). Of course, a sequence of explicitly given nice basis vectors is the most helpful object which allows us to decompose every element of the space into its elementary building blocks and, therefore, provides us with information on the individual elements considered. Nevertheless, even if the form of the basis vectors is unknown, the mere existence of a basis is extraordinarily useful since it allows us to represent elements of the space by numerical sequences giving more flexibility. Of course, the unit vectors form a basis in every Köthe type PLS-space $\Lambda(A)$. Surprisingly enough, for a large class of PLS-spaces it is somehow the only possible basis as the following version of the classical Dynin-Mityagin theorem (comp. [47, 21.10.1]) shows:

THEOREM 2.1 (Domański-Vogt [34, Th. 2.1]). *If an ultrabornological PLN-space has a basis then it is isomorphic to a Köthe type PLN-space $\Lambda(A)$ and the basis corresponds to the unit vector basis in $\Lambda(A)$.*

REMARK 2.2. In order to get the above reformulation of the result in [34] one has to apply the weak basis theorem [47, Th. 14.3.4] and inspect the original proof of [34, Th. 2.1]. See also [35].

Let us recall that a locally convex space X is *ultrabornological* if every linear map $T : X \to Y$, Y a locally convex space, is continuous whenever for every Banach space E and an operator $S : E \to X$ the composition $T \circ S$ is continuous [76, 24.14]. In Section 3 below we will explain how one can decide in practice when a given PLS-space is ultrabornological.

Before we present some examples of sequential representations, we need a special sequence space. Let $\alpha = (\alpha_j)$, $\alpha_j \nearrow \infty$, be a sequence of positive numbers. By the *power series space* [76, Sec. 29] we mean the Köthe sequence space:

$$\Lambda_r(\alpha) := \{x = (x_j)_{j\in\mathbb{N}} : \ \forall\, t < r \quad \|x\|_t := \sum_j |x_j| e^{t\alpha_j} < \infty\}.$$

It is a Fréchet-Schwartz space and its strong dual is isomorphic to the DFS-space

$$\Lambda_r'(\alpha) := \{x = (x_j)_{j\in\mathbb{N}} : \ \exists\, t < r \quad \|x\|_t^* := \sup_j |x_j| e^{-t\alpha_j} < \infty\}.$$

THEOREM 2.3. *(a)* (Vogt [93]) *If ω is a non-quasianalytic weight, $\alpha_j := \omega(j^{1/d})$ and $\Omega \subseteq \mathbb{R}^d$ is an open domain, then*

$$\mathscr{E}_{\{\omega\}}(\Omega) \simeq [\Lambda_0'(\alpha)]^{\mathbb{N}}, \qquad \mathscr{D}'_{(\omega)}(\Omega) \simeq [\Lambda_\infty'(\alpha)]^{\mathbb{N}}.$$

(b) (Valdivia [92], Vogt [93]) *In particular,*

$$\mathscr{D}'(\Omega) \simeq [\Lambda_\infty'(\log j)]^{\mathbb{N}}.$$

REMARK 2.4. It is well known that $s := \Lambda_\infty(\log j)$ is isomorphic to the Schwartz test space $\mathscr{S}$ for the space of tempered distributions, similarly $\Lambda_\infty(\log j) \simeq C^\infty[-1,1]$ (see [76, 29.5]).

Theorem 2.3 was proved by Vogt via the following three steps:

- $f_n(x) := e^{-i\langle n,x\rangle}$, $n \in \mathbb{Z}^d$, form a basis in the periodic part of $\mathscr{E}_{\{\omega\}}(\mathbb{R}^d)$ and the space of coefficients can be explicitly identified.

- (periodic part of $\mathscr{E}_{\{\omega\}}(\mathbb{R}^d))^{\mathbb{N}} \subseteq \mathscr{E}_{\{\omega\}}(\Omega) \subseteq$ (periodic part of $\mathscr{E}_{\{\omega\}}(\mathbb{R}^d))^{\mathbb{N}}$ and both topological embeddings can be made onto complemented subspaces.
- (Pełczyński decomposition) If $X = H^{\mathbb{N}}$, $X \underset{\text{comp}}{\subseteq} Y \underset{\text{comp}}{\subseteq} X$, then $X \simeq Y$.

Vogt's approach was used also to prove existence of bases in kernels of non-elliptic systems of partial differential equations defined on various spaces, see [54].

In the quasianalytic case the situation turns out to be completely different.

THEOREM 2.5 (Domański-Vogt [34], [35]). *A complemented subspace of $\mathscr{A}(\Omega)$ with basis must be a DFS-space. In particular, $\mathscr{A}(\Omega)$ never has a basis.*

REMARK 2.6. The space $\mathscr{A}(\Omega)$ is nuclear, separable, complete and it has the approximation property. To my best knowledge, $\mathscr{A}(\Omega)$ is the first example of a separable complete locally convex space without a basis which is natural, i.e., not constructed on purpose (see the historical notes on the basis problem in the introduction of [34], comp. [43]).

The proof of Theorem 2.5 is based on a deep analysis of the structure of $\mathscr{A}(\Omega)$, in particular, it is proved that any Fréchet complemented subspace of $\mathscr{A}(\Omega)$ is finite dimensional. On the other hand, by some combinatorial arguments, any ultrabornological PLS-space $\Lambda(A)$ which is not a DFS-space contains an infinite dimensional complemented Fréchet subspace (here Example 3.4 (c) below intervenes)—the conclusion follows from Theorem 2.1 and Example 3.4 (b) below.

PROBLEM 2.7. *Let ω be a quasianalytic weight. Is there any basis in $\mathscr{E}_{\{\omega\}}(\Omega)$?*

The proof of Theorem 2.3 is a highly non-constructive method of finding bases. As a consequence, even in $\mathscr{D}'(\Omega)$ we have no nice explicit basis known and there is no known explicit isomorphism of $\mathscr{D}'(\Omega)$ and $(s')^{\mathbb{N}}$. The second method of proving a sequential representation was introduced by Meise [65] (comp. also [66], [67]) and we explain it on the following example.

Let
$$\|f\|_{K,n} := \sup_{\mathbb{C}} |f(z)| \exp\left(-K|\operatorname{Im} z| - \frac{|z|}{n}\right).$$

THEOREM 2.8 (Langenbruch [55]). *Let $F(z) := \sum_{n=0}^{\infty} a_n z^n$ be an entire function such that $\|F\|_{0,n} < \infty$ for every $n \in \mathbb{N}$. The kernel of the partial differential operator of infinite order*

$$F(D) : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega), \qquad F(D)(f) := \sum_{n=0}^{\infty} a_n (-i)^n \frac{d^n}{dx^n} f$$

has a basis and it is isomorphic to $\Lambda(A)$ with an explicitly calculable matrix A.

REMARK 2.9. A similar result in the non-quasianalytic case $F(D) : \mathscr{E}_{\{\omega\}}(\mathbb{R}) \to \mathscr{E}_{\{\omega\}}(\mathbb{R})$ was obtained in [66]. For quasianalytic case see [77], [86]. For kernels of convolution operators on $\mathscr{D}'(\mathbb{R})$ or $\mathscr{D}'_{(\omega)}(\mathbb{R})$ the same method was applied in [39]. Meise's method was also applied in [24], [25], [29] and [55].

The proof is based on the following ingredients (for simplicity we assume that F has only simple zeros $(z_n)_{n\in\mathbb{N}}$).

- We define

$$H_{\infty,0}(\mathbb{C}) := \{f \in H(\mathbb{C}) : \ \exists\, K < \infty \ \forall\, n \in \mathbb{N} \quad \|f\|_{K,n} < \infty\},$$

$$Y := \{x = (x_j) \in \mathbb{C}^{\mathbb{N}} : \ \exists\, K \ \ \forall\, n \in \mathbb{N} \ \ \|x\|_{K,n} := \sup_j |x_j| \exp\left(-K|\operatorname{Im} z_j| - \frac{|z_j|}{n}\right) < \infty\},$$

$$M_F : H_{\infty,0}(\mathbb{C}) \to H_{\infty,0}(\mathbb{C}), \qquad M_F(f) := F \cdot f,$$

$$R : H_{\infty,0}(\mathbb{C}) \to Y, \qquad R(f) := (f(z_n))_{n\in\mathbb{N}}.$$

- We prove the “division” property: $\operatorname{im} M_F = \ker R$ because F is “slowly decreasing”.
- Using the interpolation method of Berenstein and Taylor [3] based on solving of the $\overline{\partial}$-equation, we prove that $\operatorname{im} R = Y$.
- Since via the Fourier-Laplace transform $H_{\infty,0} \simeq \mathscr{A}(\mathbb{R})'$ and $(F(D))' = M_F$, we observe that $\ker F(D) \simeq \operatorname{im} R' \simeq Y'$.

To show how useful the existence of the basis can be, let us sketch the proof (in the version given in [35, Th. 4.7]) of the following result.

COROLLARY 2.10 (Langenbruch [55]). *Under the assumptions of Theorem 2.8, if $F(D)$ has a linear continuous right inverse then for every $\varepsilon > 0$ there is a constant C such that for every zero $z_j = \xi_j + i\eta_j$ of F we have*

$$|\eta_j| \le C + \varepsilon|\xi_j|.$$

Let us observe that the operator $F(D)$ has a linear continuous right inverse if and only if $\ker F(D)$ is complemented in $\mathscr{A}(\mathbb{R}) = \operatorname{proj}_{N\in\mathbb{N}} \operatorname{ind}_{n\in\mathbb{N}} X_{N,n}$. By Theorem 2.5 and 2.8, it is a DFS-space, which implies certain estimates between norms on $X_{N,n}$. Applying these estimates to functions $\exp(-i\langle\cdot, z_j\rangle)$ we obtain the conclusion of Corollary 2.10.

3. Open mapping, closed graph and ultrabornologicity versus Proj^1 **functor.** Usually the open mapping theorem and the closed graph theorem are the corner stones of any reasonable theory of locally convex spaces. It turns out that ultrabornologicity, which was so important for sequential representations, also plays a profound role here. The following result is an immediate consequence of the classical De Wilde’s theory see [76, 24.30, 24.31].

THEOREM 3.1. *Let X be a PLS-space and Y be an ultrabornological locally convex space.*

(a) Every surjective operator $T : X \to Y$ is open.
(b) Every linear map $T : Y \to X$ with closed graph is continuous.

Now, it becomes urgent to find ready-for-application criteria for ultrabornologicity of PLS-spaces. Let $X = \operatorname{proj}_{N\in\mathbb{N}} X_N$, $X_N = \operatorname{ind}_{n\in\mathbb{N}} X_{N,n}$, where $(X_{N,n}, \|\cdot\|_{N,n})$ are Banach spaces and X_N are DFS-spaces. Let us consider a so-called *projective spectrum representing* X:

$$\dots \longrightarrow X_{N+1} \xrightarrow{i_N^{N+1}} X_N \xrightarrow{i_{N-1}^{N}} \dots \xrightarrow{i_1^2} X_1 \xrightarrow{i_0^1} X_0,$$

and let $i_N : X \to X_N$ be canonical maps. Let us define

$$j : X \to \prod_{N\in\mathbb{N}} X_N, \qquad j(x) := (i_N x)_{N\in\mathbb{N}}$$

and

$$\sigma : \prod_{N\in\mathbb{N}} X_N \to \prod_{N\in\mathbb{N}} X_N, \qquad \sigma((x_N)_{N\in\mathbb{N}}) := (i_N^{N+1} x_{N+1} - x_N)_{N\in\mathbb{N}}.$$

Clearly, $\operatorname{im} j = \ker \sigma$ but σ is not necessarily surjective. We define

$$\operatorname{Proj}^1 X := \Big(\prod_{N\in\mathbb{N}} X_N\Big) / \operatorname{im} \sigma,$$

in fact, it depends not only on X but on the whole spectrum, nevertheless Proj^1 is equal for all spectra with $\overline{i_N(X)} = X_N$ for every $N \in \mathbb{N}$.

The functor Proj^1 originates from the homological algebra and it was introduced in the theory of locally convex spaces by Palamodov [79], [80] and Vogt [96] (comp. [97]). Its official name sounds very abstract: the first derived functor of the functor of the projective limit of linear spaces. By now it has become an indispensable and powerful tool in the theory which has beautiful applications to classical problems of analysis. I recommend a nice and extensive survey due to Wengenroth [101] for the homological definition of the functor, its equivalence to our definition, various characterizations of the vanishing of it and plenty of other results.

The following theorem summarizes the long development of the theory and provides us with calculable criteria of ultrabornologicity.

THEOREM 3.2. *For a spectrum of DFS-spaces like above such that* $\overline{i_N X} = X_N$ *for every* $N \in \mathbb{N}$, *the following assertions are equivalent:*

(a) $\operatorname{Proj}^1 X = 0$*;*
(b) X *is ultrabornological;*
(c) X *is reflexive;*
(d) $\forall\, N\ \exists\, B$ *bounded absolutely convex subset of* X_N $\exists\, M > N$

$$i_N^M X_M \subseteq i_N X + B;$$

(e) $\forall\, K\ \exists\, L\ \forall\, M\ \exists\, k\ \forall\, l\ \exists\, m, C > 0\ \forall\, y \in X'_K$

$$\|y\|^*_{L,l} \le C \max\left(\|y\|^*_{M,m}, \|y\|^*_{K,k}\right),$$

where $\|\cdot\|^*$ *denotes the dual norm.*

REMARK 3.3. The above result has many authors; see [101]. In particular, (a)⇒(b)⇒(c) ⇒(e) were proved by Vogt [96], [97]; (e)⇒(a) is due to Langenbruch [60], see also [101, Th. 3.2.18] (comp. earlier version due to Wengenroth [98]); (a)⇒(d) was obtained by Retakh [84]; (d)⇒(a) was obtained by Wengenroth [99]. For more refined criteria like (d) and (e) see [27], [41], [42] and [101].

EXAMPLE 3.4. The following PLS-spaces are ultrabornological:

(a) $\mathscr{E}_{\{\omega\}}(\Omega)$, $\mathscr{D}'_{(\omega)}(\Omega)$ and $\mathscr{D}'$ for non-quasianalytic weights ω (use the sequential representations as products of DFS-spaces, see Theorem 2.3);
(b) (Martineau [64]) $\mathscr{A}(\Omega)$ for arbitrary Ω and (Rösner [86]) $\mathscr{E}_{\{\omega\}}(\Omega)$ for Ω convex and ω a quasianalytic weight;

(c) (Vogt [96], [97]; see [101, p. 35]) $\Lambda(A)$ is ultrabornological if and only if

$$\forall\, K\ \exists\, L\ \forall\, M\ \exists\, k\ \forall\, l\ \exists\, m, C>0\ \forall\, j \qquad \frac{1}{a_{j,L,l}} \le C \max\left(\frac{1}{a_{j,M,m}}, \frac{1}{a_{j,K,k}}\right).$$

There are also negative examples.

EXAMPLE 3.5. The following spaces X are not ultrabornological.

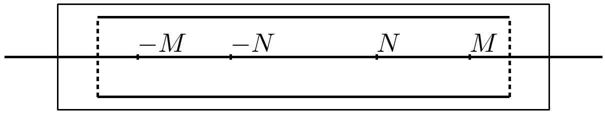
(a) The kernel of the differential operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ acting on the set $\Omega := \{(x,y,z) \in \mathbb{R}^3 : x^2 - y^2 - z^2 - 1 < 0\} \subseteq \mathbb{R}^3$ is a closed subspace with $\mathrm{Proj}^1 \neq 0$. Indeed, by [48, Example on p. 802] and [38, Proof of Cor. 3], $\mathscr{D}'(\Omega)/\ker P(D)$ is not complete. Then, by [32, Cor. 1.4], $\mathrm{Proj}^1\ \ker P(D) \neq 0$. Any other non-surjective linear partial differential operator with constant coefficients $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ which is surjective as an operator $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ will work as well (see [38] and [32]).

(b) There is a real analytic map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\{f \circ \varphi : f \in \mathscr{A}(\mathbb{R}^2)\}$ is a closed non-ultrabornological subspace of $\mathscr{A}(\mathbb{R}^2)$, see [30, Ex. 3.8, Remark 3.9].

(c) By [32, Cor. 1.4], if a quotient space Y/X is not complete for a closed subspace X of an ultrabornological PLS-space Y then X is a non-ultrabornological PLS-space. The first example of a non-complete quotient of $\mathscr{D}'(\Omega)$ was given by [44].

To show the strength of Theorem 3.2 we present a very elegant proof due to Vogt that $\mathrm{Proj}^1\ \mathscr{A}(\mathbb{R}) = 0$ (comp. also [35, 1.5]).

We will use condition (d) in Theorem 3.2. Observe that $\mathscr{A}(\mathbb{R}) = \mathrm{proj}_{N\in\mathbb{N}} H([-N,N])$. If $f \in H([-M,M])$ then $f \in H((-M-2\varepsilon, M+2\varepsilon) \times i(-2\varepsilon, 2\varepsilon))$ for some $\varepsilon > 0$. Let γ (denoted by ———) be horizontal and let η (denoted by - - - -) be vertical sides of the rectangle $(-M-\varepsilon, M+\varepsilon) \times (-\varepsilon, \varepsilon)$:



Then, by the Cauchy formula,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma\cup\eta} \frac{f(w)}{w-z}\, dw.$$

We define

$$f_1(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z}\, dw \in \mathscr{A}(\mathbb{R})$$

$$f_2(z) := \frac{1}{2\pi i} \int_{\eta} \frac{f(w)}{w-z}\, dw \in H^\infty(B(0, M+\varepsilon/2))$$

Clearly, there is a polynomial P such that $|f_2 - P| \le 1$ on $B(0,M) := \{z \in \mathbb{C} : |z| < M\}$. Finally,

$$f = (f_1 + P) + (f_2 - P),$$

where $f_1 - P \in \mathscr{A}(\mathbb{R})$ and $f_2 + P$ belongs to the 1-ball in $H^\infty(B(0,M))$ which is the bounded set B in $H([-N,N])$ we are looking for. This completes the proof.

The problem of surjectivity of classical operators was the main reason for introducing Proj^1 by Palamodov [79] into the theory of locally convex spaces. Let us recall that the following diagram of locally convex spaces and operators

$$0 \longrightarrow X \xrightarrow{\ j\ } Y \xrightarrow{\ q\ } Z \longrightarrow 0$$

is *exact* if the image of every map is equal to the kernel of the next map. The diagram is *topologically exact* if, additionally, all maps are open onto their images.

Let us consider an arbitrary operator (= linear continuous map) $T : Y \to Z$, Y, Z PLS-spaces. It follows from Theorem 3.2 and 3.1 that:

COROLLARY 3.6. *If* $\mathrm{Proj}^1\ Z = 0$*,* $T : Y \to Z$ *as above and surjective, then* T *is open.*

Let us assume additionally that T is open onto a dense subspace of Z. Then one can construct easily the following commutative diagram with all the rows except the first one topologically exact ($X = \ker T$; $X = \mathrm{proj}_{N\in\mathbb{N}} X_N$; $Y = \mathrm{proj}_{N\in\mathbb{N}} Y_N$; $Z = \mathrm{proj}_{N\in\mathbb{N}} Z_N$; X_N, Y_N, Z_N DFS-spaces), for a simple argument see [32, pp. 63–64]:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\ j\ } & Y & \xrightarrow{\ T\ } & Z & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_{N+1} & \xrightarrow{\ j_{N+1}\ } & Y_{N+1} & \xrightarrow{\ T_{N+1}\ } & Z_{N+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_N & \xrightarrow{\ j_N\ } & Y_N & \xrightarrow{\ T_N\ } & Z_N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

If the above diagram exists then we say that T is *locally surjective.*

The following result is one of the basic sources of applicability of Proj^1.

THEOREM 3.7 (Palamodov [79]). *Let* T *be locally surjective. If* $\mathrm{Proj}^1\ X = 0$ *then* T *is surjective. If* T *is surjective and* $\mathrm{Proj}^1\ Y = 0$ *then* $\mathrm{Proj}^1\ X = 0$.

COROLLARY 3.8 (Comp. [32, Cor. 1.4]). *Let* Y *be a PLS-space and* X *its closed subspace. The quotient* Y/X *is complete (or, equivalently, a PLS-space) whenever* $\mathrm{Proj}^1\ X = 0$*. If* $\mathrm{Proj}^1\ Y = 0$ *then the converse implication holds as well.*

4. Surjectivity of operators. Floret [38] observed in 1976 that a linear partial differential operator with constant coefficients $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is surjective if and only if $\mathscr{D}'(\Omega)/\ker P(D)$ is complete and $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective. Now, by Cor. 3.8, we know that the first condition is equivalent to $\mathrm{Proj}^1 \ker P(D) = 0$. Today the functor Proj^1 and criteria like (d) and (e) in Theorem 3.2 have become very powerful tools for proving surjectivity of linear partial differential operators, convolution and other operators (see, for instance, [101, Sec. 3.4], [13], [14], [24], [25], [26], [31, Sec. 4], [39], [60], [77], [79], [86], [97], [99]).

THEOREM 4.1 (Braun-Meise-Vogt [26], Braun [13], [14], Rösner [86]). *Let ω be an arbitrary weight, $\Omega \subseteq \mathbb{R}^d$ a convex open subset, $V := \{z \in \mathbb{C}^d : P(-z) = 0\}$. Then the following are equivalent:*
(a) $P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega)$ is surjective;
(b) $\operatorname{Proj}^1 \ker P(D) = 0$ (i.e., $\ker P(D)$ ultrabornological);
(c) The Phragmén-Lindelöf type condition:
$\forall\, K \Subset \Omega\ \ \exists\, \delta > 0, Q \Subset \Omega\ \ \forall\, \varepsilon, L > 0\ \ \exists\, C\ \ \forall\, \varphi$ *plurisubharmonic on V: (α)+$(\beta) \Rightarrow (\gamma)$*

$$(\alpha)\quad \forall\, z \in V \quad \varphi(z) \le \sup_{y\in K}\langle \operatorname{Im} z, y\rangle + \delta\omega(|z|)$$
$$(\beta)\quad \forall\, z \in V \quad \varphi(z) \le L|\operatorname{Im} z|$$
$$(\gamma)\quad \forall\, z \in V \quad \varphi(z) \le \sup_{y\in Q}\langle \operatorname{Im} z, y\rangle + \varepsilon\omega(|z|) + C.$$

REMARK 4.2. The case of ω non-quasianalytic was proved in [26] and [13] and the quasianalytic case in [86]. See also [14], [24] and [12]. In fact, (a)⇔(b) follows from the general theory of Proj^1 knowing that $P(D)$ is locally surjective and $\operatorname{Proj}^1 \mathscr{E}_{\{\omega\}}(\Omega) = 0$ (see Theorem 3.7). Especially, the second part is far from being trivial. In order to get (c), one has to translate via the Fourier-Laplace transform the condition (e) in Theorem 3.2 to some condition on holomorphic functions f on V and then to some condition on plurisubharmonic functions $\log|f|$ on V [71]. It is again not obvious at all.

It seems that the paper of Hörmander [45] on the characterization of surjectivity in case of $P(D) : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$, Ω convex, is the first one in which conditions of type (c) appeared in the context of surjectivity of partial differential operators. For $\Omega = \mathbb{R}^2$ it is known that $P(D) : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is always surjective [28]. Evaluation of the obtained conditions for $\Omega = \mathbb{R}^d$ was obtained for $d = 3$ by Zampieri [103] and Braun [11]; for $d = 4$ by Braun, Meise and Taylor in [20] and [23]. Quite recently Langenbruch [60] (comp. earlier paper [59]) has characterized surjectivity of $P(D) : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$, Ω an arbitrary domain in $\mathbb{R}^d$, using also conditions similar in spirit to the above ones. Other characterizations of surjectivity of $P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega)$ in the non-quasianalytic case but for arbitrary open Ω were given by Langenbruch in [57], [58]. Various Phragmén-Lindelöf type conditions were analyzed in a sequence of papers (see, for instance, [11], [12], [13], [14], [16], [17], [18], [22], [23], [24], [26], [45], [68], [70], [71], [72], [75], [82], [86], [103]).

Similar methods work also for convolution operators.

THEOREM 4.3 (Napalkov-Rudakov [78]). *Let $F(D)$ be like in Theorem 2.8. The infinite order differential operator $F(D) : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is surjective if and only if there is $r : \mathbb{R}_+ \to \mathbb{R}_+$, $r(x) = o(x)$, $\delta > 0$ such that if $F(z) = 0$ then either $|\operatorname{Im} z| > \delta|\operatorname{Re} z|$ or $|\operatorname{Im} z| < r(|z|)$.*

REMARK 4.4. The theorem above was generalized for arbitrary convolution operators on $\mathscr{A}(\mathbb{R})$ by Langenbruch [56] and for quasianalytic classes $\mathscr{E}_{\{\omega\}}(\mathbb{R})$ by Meyer [77]. The result for non-quasianalytic classes $\mathscr{E}_{\{\omega\}}(\mathbb{R})$ and convolution operators is contained in [25]. A similar result for convolution operators on $\mathscr{D}'_{(\omega)}(\mathbb{R})$ is proved in [39].

The proof of Theorem 4.3 uses the following three main ingredients:

- We prove local surjectivity of $F(D)$.
- We apply Theorem 2.8 to get an explicit sequential representation of $\ker F(D)$ as $\Lambda(A)$.
- We apply Proj^1 machinery via Example 3.4 (c) and Theorem 3.7.

5. Dependence of the solution on parameters. The last section of the survey will be devoted to the study of dependence on parameters of solutions of differential equations. It turns out that the developed theory is also useful in this problem. Let us consider a linear partial differential equation

$$P(D)u = f \tag{1}$$

One of the typical problems was posed by L. Schwartz, who asked when there exists a linear continuous right inverse for $P(D)$ (or, when we can make the solution u depend linearly and continuously on the right hand side of (1)). This problem was solved by Meise, Taylor and Vogt [70] for $P(D) : \mathscr{D}' \to \mathscr{D}'$ and in [72] for $P(D) : \mathscr{D}'_{(\omega)} \to \mathscr{D}'_{(\omega)}$ or $P(D) : \mathscr{E}_{\{\omega\}}(\Omega) \to \mathscr{E}_{\{\omega\}}(\Omega)$ for non-quasianalytic ω. By now there are plenty of papers evaluating the conditions obtained for given polynomials P (see, for instance, [18], [19], [21], [40], [69], [73], [74]). The characterizations are given in terms of Phragmén-Lindelöf conditions similar to those characterizing surjectivity of partial differential operators on $\mathscr{E}_{\{\omega\}}$. This is not just a coincidence. It is shown in [13] that for a non-quasianalytic weight ω and a homogeneous polynomial P the differential operator $P(D) : \mathscr{E}_{\{\omega\}}(\mathbb{R}^N) \to \mathscr{E}_{\{\omega\}}(\mathbb{R}^N)$ has a linear continuous right inverse if and only if $P(D) : \mathscr{A}(\mathbb{R}^M) \to \mathscr{A}(\mathbb{R}^M)$ (or, $P(D) : \mathscr{E}_{\{\omega\}}(\mathbb{R}^M) \to \mathscr{E}_{\{\omega\}}(\mathbb{R}^M)$) is surjective for $M > N$. A similar problem was also considered for convolution operators on $\mathscr{A}(\mathbb{R})$ [55]. In case when (1) is an overdetermined system of equations we obtain the following complex, $P(D) = P_0(D)$:

$$\begin{array}{l} 0 \longrightarrow \ker P_0(D) \longrightarrow \mathscr{D}'(\Omega) \xrightarrow{P_0(D)} \mathscr{D}'(\Omega) \xrightarrow{P_1(D)} \mathscr{D}'(\Omega) \xrightarrow{P_2(D)} \ldots \\ \qquad\qquad\qquad\qquad\qquad\qquad \ldots \xrightarrow{P_n(D)} \mathscr{D}'(\Omega) \longrightarrow 0. \end{array} \tag{2}$$

The book of Tarkhanov [89] explains in detail the theory of such complexes providing plenty of examples.

Surprisingly we have the following very strong result.

THEOREM 5.1 (Domański-Vogt [33]). *If (2) is an exact complex then the operators*

$$P_j(D) : \mathscr{D}'(\Omega) \to \operatorname{im} P_j(D)$$

have linear continuous right inverses for every $j \geq 1$.

REMARK 5.2. The result holds also for complexes over $\mathscr{D}'_{(\omega)}(\Omega)$. The most curious fact is that $P_j(D)$ being differential operators plays no role! A condition characterizing the existence of a linear continuous right inverse for $P_0(D)$ is also obtained in [33]. Earlier, Palamodov [82] proved this result for Ω convex and differential complexes using a lot of "hard analysis" (Fourier-Laplace transform) but he obtained a better condition for right inverse of $P_0(D)$ in this special case in terms of Phragmén-Lindelöf type conditions.

The proof of Theorem 5.1 is purely functional analytic. First, one proves inductively that the exact complex (2) must be topologically exact. Indeed, since $\mathrm{Proj}^1\ \mathscr{D}'(\Omega) = 0$ (see Example 3.4 (a)), $P_n(D)$ is open by Corollary 3.6. By Theorem 3.7, $\mathrm{Proj}^1\ (\ker P_n(D))$ $= 0$, but $\ker P_n(D) = \operatorname{im} P_{n-1}(D)$ and again $P_{n-1}(D)$ is open. Step by step the whole complex turns out to be topologically exact. In the next part of the proof one uses the sequential representation of $\mathscr{D}(\Omega) \simeq (s')^{\mathbb{N}}$ (Theorem 2.3). This representation allows us to prove, by a repeated application of the Vogt-Wagner (Ω)-(DN) splitting theorem [76, 30.1], the following result which seems to be the most difficult part of the story:

THEOREM 5.3 (Wengenroth [100], Domański-Vogt [32]). *If X, Y, Z are PLS-spaces, X is a quotient of $\mathscr{D}'(\Omega)$ and Z is a subspace of $\mathscr{D}'(\Omega)$, then every topologically exact sequence of the form*

$$0 \to X \to Y \to Z \to 0$$

splits.

Finally, Theorem 5.1 follows from Theorem 5.3 applied to the sequence

$$0 \longrightarrow \ker P_j(D) \longrightarrow \mathscr{D}'(\Omega) \xrightarrow{P_j(D)} \operatorname{im} P_j(D) \longrightarrow 0.$$

REMARK 5.4. Let us mention here that Proj^1 stands behind the whole splitting business (see [95], comp. [42] and [101]) for Fréchet spaces, i.e., also behind the Vogt-Wagner Theorem.

Another problem of a similar form was posed by Hörmander [46, vol. II, p. 59]. Let

$$P(z, D) = \sum_{|\alpha| \le m} a_\alpha(z)(-i)^{|\alpha|}\partial^\alpha,$$

$a_\alpha(z)$ holomorphic, be a linear partial differential operator depending analytically on a parameter $z \in U$. Let δ be the Dirac distribution. Hörmander asked if it is possible to find a holomorphic map $g : U \to \mathscr{D}'(\mathbb{R}^d)$ such that

$$P(z, D)g(z) = \delta.$$

Treves (see [90] and [91]) proved that $P(z, D)$ must be "equally strong" for every z and that this necessary condition is sufficient for local dependence.

THEOREM 5.5 (Mantlik [62], [63]). *The answer to Hörmander's question is yes for every Stein manifold U whenever $P(z, D)$ is of "equal strength" for every $z \in U$.*

We finish with a result on analytic dependence of solutions of $P(D) : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ using the theory of vector valued real analytic functions:

THEOREM 5.6 (Bonet-Domański [6], [7]). *Let U be a Stein manifold with the strong Liouville property. Let $T : \mathscr{A}(\Omega_1) \to \mathscr{A}(\Omega_2)$ be a surjective operator. For every holomorphic function $f \in H(U, \mathscr{A}(\Omega_2))$ there is $g \in H(U, \mathscr{A}(\Omega_1))$ such that*

$$T(g(z)) = f(z) \qquad \text{for every } z \in U.$$

REMARK 5.7. A Stein manifold U has the strong Liouville property if every plurisubharmonic function on U bounded from above is constant. As I learned from J. Siciak (personal communication) there are examples of domains of holomorphy in $\mathbb{C}^2$ which have the standard Liouville property (= every bounded holomorphic function is constant) but without the strong Liouville property.

As we have seen, behind Theorem 5.1 stands a splitting Theorem 5.3. Similarly, one can get vector valued surjectivity results (like Theorem 5.6) using other splitting results. The splitting theory for Fréchet spaces is rather complete, see [95], [94], [42]. We have even presentations in a book form in [76] and [101].

PROBLEM 5.8. *Construct a splitting theory for PLS-spaces, i.e., find pairs of PLS-spaces E, F such that every short exact sequence*

$$0 \longrightarrow F \longrightarrow X \longrightarrow E \longrightarrow 0$$

splits.

The problem is of special interest if one of the spaces E or F is a classical PLS-space like $\mathscr{D}'$, $\mathscr{D}'_{(\omega)}$, $\mathscr{E}_{\{\omega\}}$, $\mathscr{A}(\Omega)$. So far we have only some scattered results which do not give any complete theory. In [32] and [100] (comp. [101, Sec. 5.3]) some splitting results in the PLS-category are given in case F or E is isomorphic to $\mathscr{D}'(\Omega)$ or $E \simeq \omega$ (= the space of all sequences). This theory also works for $\mathscr{D}'_{(\omega)}$. The dissertation of Kunkle [53] gives a surprisingly complete solution for the case where E *and* F are the so-called power series Köthe PLS-sequence spaces. It is based on an analogue of Retakh's characterization of the vanishing of Proj^1 (Theorem 3.2 (d)) for webbed spaces, see [41]. Other specific splitting results in the category of PLS-spaces are contained in [29] and in some unpublished notes.

The proof of Theorem 5.6 is based on the observation done in [6] that, in fact, we are looking for surjectivity of $T \otimes \mathrm{id} : \mathscr{A}(\Omega_1, X) \to \mathscr{A}(\Omega_2, X)$ for $X = H(U)$, where

$$\mathscr{A}(\Omega, X) := \mathscr{A}(\Omega)\hat{\otimes}_\pi X = \{f : \Omega \to X : \ \forall\, u \in X', u \circ f \in \mathscr{A}(\Omega)\}.$$

On the other hand, using heavily the Proj^1 machinery, one can prove [6, Th. 36] that $T \otimes \mathrm{id} : \mathscr{B}\mathscr{A}(\Omega, X) \to \mathscr{B}\mathscr{A}(\Omega, X)$ is surjective where

$$\mathscr{B}\mathscr{A}(\Omega, X) := \{f : \Omega \to X : f \ \text{analytic}\}.$$

These observations reduce our problem to the question when $\mathscr{A}(\Omega, X) = \mathscr{B}\mathscr{A}(\Omega, X)$.

THEOREM 5.9 (Bonet-Domański [6], [7]). *For a Fréchet space $(X, (\|\cdot\|_n)_{n\in\mathbb{N}})$ we have*

$$\mathscr{A}(\Omega, X) = \mathscr{B}\mathscr{A}(\Omega, X)$$

if and only if $X \in (DN)$, i.e.,

$$\exists\, k\ \forall\, l\ \exists\, m, C : \|\cdot\|_l^2 \le C\|\cdot\|_k\|\cdot\|_m.$$

REMARK 5.10. It is known that $H(U)$ has (DN) if and only if U has the strong Liouville property [102].

A completely different field of applications of vector valued real analytic functions one can find in [51] and [52]. For more information on vector valued real analytic functions, see [5], [8] and [37].

Theorem 5.9 has an unexpected link to the mathematical work of Władysław Orlicz. Soon after I had just started my study of the spaces of real analytic functions I found the following admirable result:

THEOREM 5.11 (Alexiewicz-Orlicz [1]). *For every Banach space X we have*

$$\mathscr{B}\mathscr{A}(\Omega, X) = \mathscr{A}(\Omega, X).$$

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References

[1] A. Alexiewicz and W. Orlicz, *On analytic vector-valued functions of a real variable*, Studia Math. 12 (1951), 108–111.

[2] A. Andreotti and M. Nacinovich, *Complexes of partial differential operators*, Ann. Scuola Norm. Sup. Pisa Sec. IV (4) 3 (1976), 553–622.

[3] C. A. Berenstein and B. A. Taylor, *A new look at interpolation theory for entire functions in one variable*, Adv. in Math. 33 (1979), 109–143.

[4] K. D. Bierstedt, *An introduction to locally convex inductive limits*, in: Functional Analysis and its Applications, H. Hogbe-Nlend (ed.), World Sci., Singapore, 1988, 35–133.

[5] J. Bochnak and J. Siciak, *Analytic functions in topological vector spaces*, Studia Math. 39 (1971), 77–111.

[6] J. Bonet and P. Domański, *Real analytic curves in Fréchet spaces and their duals*, Mh. Math. 126 (1998), 13–36.

[7] J. Bonet and P. Domański, *Parameter dependence of solutions of partial differential equations in spaces of real analytic functions*, Proc. Amer. Math. Soc. 129 (2000), 495–503.

[8] J. Bonet, P. Domański and D. Vogt, *Interpolation of vector valued real analytic functions*, J. London Math. Soc. 66 (2002), 407–420.

[9] J. Bonet, A. Galbis and R. Meise, *On the range of convolution operators on nonquasianalytic ultradifferentiable functions*, Studia Math. 126 (1997), 171–198.

[10] J. Bonet and R. Meise, *Ultradistributions of Roumieu type and projective descriptions*, J. Math. Anal. Appl. 255 (2001), 122–136.

[11] R. W. Braun, *Hörmander's Phragmén-Lindelöf principle and irreducible singularities of codimension 1*, Boll. Un. Mat. Ital. (7) 6-A (1992), 339–348.

[12] R. W. Braun, *A partial differential operator which is surjective on Gevrey classes* $\Gamma^d(\mathbb{R}^3)$ *with* $1 \leq d < 2$ *and* $d \geq 6$ *but not for* $2 \leq d < 6$, Studia Math. 107 (1993), 159–169.

[13] R. W. Braun, *Surjektivität partieller Differentialoperatoren auf Roumieu-Klassen*, Habilitationsschrift, Düsseldorf 1993.

[14] R. W. Braun, *Surjectivity of partial differential operators on Gevrey classes*, in: Functional Analysis, Proc. of the First International Workshop held at Trier University, S. Dierolf, P. Domański, S. Dineen (eds.), Walter de Gruyter, Berlin 1996, 69–80.

[15] R. W. Braun, R. Meise and B. A. Taylor, *Ultradifferentiable functions and Fourier analysis*, Results Math. 17 (1990), 207–237.

[16] R. W. Braun, R. Meise and B. A. Taylor, *Algebraic varieties on which the classical Phragmén-Lindelöf estimates hold for plurisubharmonic functions*, Math. Z. 232 (1999), 103–135.

[17] R. W. Braun, R. Meise and B. A. Taylor, *A radial Phragmén-Lindelöf estimate for plurisubharmonic functions on algebraic varieties*, Ann. Polon. Math. 72 (1999), 159–179.

[18] R. W. Braun, R. Meise and B. A. Taylor, *Characterization of the homogeneous polynomials P for which $(P+Q)(D)$ admits a continuous linear right inverse for all lower order perturbations Q*, Pac. J. Math. 192 (2000), 201–218.

[19] R. W. Braun, R. Meise and B. A. Taylor, *A perturbation result for linear differential operators admitting a global right inverse on $\mathscr{D}'$*, in: Analysis, Geometry, Number Theory: the Mathematics of Leon Ehrenpreis (Philadelphia, PA, 1998), Contemp. Math. 251, Amer. Math. Soc., Providence, 2000, 93–106.

[20] R. W. Braun, R. Meise and B. A. Taylor, *Surjectivity of constant coefficient partial differential operators on $\mathscr{A}(\mathbb{R}^4)$ and Whitney's C_4-cone*, Bull. Soc. Roy. Sci. Liège 70 (2001), 195–206.

[21] R. W. Braun, R. Meise and B. A. Taylor, *Perturbation of differential operators admitting a continuous linear right inverse on ultradistributions*, Pacific J. Math. 212 (2003), 25–48.

[22] R. W. Braun, R. Meise and B. A. Taylor, *Perturbation results for the local Phragmén-Lindelöf condition and stable homogeneous polynomials*, Rev. R. Acad. Cien. Serie A Mat. 97 (2003), 189–208.

[23] R. W. Braun, R. Meise and B. A. Taylor, *The geometry of analytic varieties satisfying the local Phragmén-Lindelöf condition and a geometric characterization of the partial differential operators that are surjective on $\mathscr{A}(\mathbb{R}^4)$*, Trans. Amer. Math. Soc. 356 (2004), 1315–1383.

[24] R. W. Braun, R. Meise and D. Vogt, *Applications of the projective limit functor to convolution and partial differential equations*, in: Advances in the Theory of Fréchet Spaces, T. Terzioğlu (ed.), Kluwer, Dordrecht, 1989, 29–46.

[25] R. W. Braun, R. Meise and D. Vogt, *Existence of fundamental solutions and surjectivity of convolution operators on classes of ultradifferentiable functions*, Proc. London Math. Soc. 61 (1990), 344–370.

[26] R. W. Braun, R. Meise and D. Vogt, *Characterization of the linear partial differential operators with constant coefficients which are surjective on non-quasianalytic classes of Roumieu type on $\mathbb{R}^N$*, Math. Nach. 168 (1994), 19–54.

[27] R. W. Braun and D. Vogt, *A sufficient condition for $Proj^1 X = 0$*, Mich. Math. J. 44 (1997), 149–156.

[28] E. De Giorgi and L. Cattabriga, *Una dimostrazione diretta dell'esistenza di soluzioni analitiche nel piano reale di equazioni a derivate parziali a coefficienti costanti*, Boll. Un. Mat. Ital. (4) 4 (1971), 1015–1027.

[29] P. Domański, L. Frerick and D. Vogt, *Fréchet quotients of spaces of real analytic functions*, Studia Math. 159 (2003), 229–245.

[30] P. Domański and M. Langenbruch, *Composition operators on spaces of real analytic functions*, Math. Nach. 254-255 (2003), 68–86.

[31] P. Domański and M. Langenbruch, *Coherent analytic sets and composition of real analytic functions*, J. Reine Angew. Math., to appear.

[32] P. Domański and D. Vogt, *A splitting theory for the space of distributions*, Studia Math. 140 (2000), 57–77.

[33] P. Domański and D. Vogt, *Distributional complexes split for positive dimensions*, J. Reine Angew. Math. 522 (2000), 63–79.

[34] P. Domański and D. Vogt, *The space of real analytic functions has no basis*, Studia Math. 142 (2000), 187–200.

[35] P. Domański and D. Vogt, *Linear topological properties of the space of real analytic functions on the real line*, in: Recent Progress in Functional Analysis, K. D. Bierstedt, J. Bonet, M. Maestre, J. Schmets (eds.), Proc. Valencia 2000, North-Holland Math. Studies, Amsterdam, 2001, 113–132.

[36] P. Domański and D. Vogt, *A non-trivial Fréchet quotient of the space of real analytic functions*, Arch. Math. 81 (2003), 208–214.

[37] P. Domański and D. Vogt, *Infinite systems of linear equations for real analytic functions*, Proc. Amer. Math. Soc., to appear.

[38] K. Floret, *On well-located subspaces of distribution spaces*, Math. Ann. 221 (1976), 147–151.

[39] U. Franken and R. Meise, *Generalized Fourier expansions for zero-solutions of surjective convolution operators on* $\mathscr{D}'(\mathbb{R})$ *and* $\mathscr{D}'_{(\omega)}(\mathbb{R})$, Note Mat. (Lecce) 10 Suppl. 1 (1990), 251–272.

[40] U. Franken and R. Meise, *Continuous linear right inverses for homogeneous linear partial differential operators on bounded convex open sets and extension of zero-solutions*, in: Functional Analysis, Proc. of the First International Workshop held at Trier University, S. Dierolf, P. Domański, S. Dineen (eds.), Walter de Gruyter, Berlin, 1996, 153–161.

[41] L. Frerick, D. Kunkle and J. Wengenroth, *The projective limit functor for spectra of webbed spaces*, Studia Math. 158 (2003), 117–129.

[42] L. Frerick and J. Wengenroth, *A sufficient condition for the vanishing of the derived projective limit functor*, Arch. Math. 67 (1996), 296–301.

[43] *Handbook of the Geometry of Banach Spaces*, Vol. I, W. B. Johnson and J. Lindenstrauss (eds.), North-Holland, Amsterdam, 2001.

[44] Ch. Harvey and R. F. Harvey, *Open mappings and the lack of full completeness of* $\mathcal{D}'(\Omega)$, Proc. Amer. Math. Soc. 25 (1970), 786–790.

[45] L. Hörmander, *On the existence of real analytic solutions of partial differential equations with constant coefficients*, Inventiones Math. 21 (1973), 151–182.

[46] L. Hörmander, *The Analysis of Linear Partial Differential Operators II*, Springer, Berlin, 1983.

[47] H. Jarchow, *Locally Convex Spaces*, B. G. Teubner, Stuttgart, 1981.

[48] M. J. Kascic, jr. and B. Roth, *A closed subspace of* $\mathscr{D}(\Omega)$ *which is not an LF-space*, Proc. Amer. Math. Soc. 24 (1970), 801–802.

[49] H. Komatsu, *Ultradistributions I. Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo, Sec. 1 A. Math. 20 (1973), 25–105.

[50] S. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, Birkhäuser, Basel, 1992.

[51] A. Kriegl and P. W. Michor, *The convenient setting for real analytic mappings*, Acta Math. 165 (1990), 105–159.

[52] A. Kriegl and P. W. Michor, *The Convenient Setting of Global Analysis*, American Mathematical Society, Providence, 1997.

[53] D. Kunkle, *Splitting of power series spaces of (PLS)-type*, Dissertation, Wuppertal, 2001.

[54] M. Langenbruch, *Bases in solution sheaves of systems of partial differential equations*, J. Reine Angew. Math. 373 (1987), 1–36.

[55] M. Langenbruch, *Continuous linear right inverses for convolution operators in spaces of real analytic functions*, Studia Math. 110 (1994), 65–82.

[56] M. Langenbruch, *Hyperfunction fundamental solutions of surjective convolution operators on real analytic functions*, J. Funct. Anal. 131 (1995), 78–93.

[57] M. Langenbruch, *Surjective partial differential operators on spaces of ultradifferentiable functions of Roumieu type*, Results Math. 29 (1996), 254–275.

[58] M. Langenbruch, *Surjective partial differential operators on Gevrey classes and their localizations at infinity*, Linear Topological Spaces and Complex Analysis 3 (1997), 95–111.

[59] M. Langenbruch, *Localizations of partial differential operators and surjectivity on real analytic functions*, Studia Math. 140 (2000), 15–40.

[60] M. Langenbruch, *Characterization of surjective partial differential operators on spaces of real analytic functions*, Studia Math. 162 (2004), 53–96.

[61] M. Langenbruch and S. Momm, *Complemented submodules in weighted spaces of analytic functions*, Math. Nach. 157 (1992), 263–276.

[62] F. Mantlik, *Partial differential operators depending analytically on a parameter*, Ann. Inst. Fourier (Grenoble) 41 (1991), 577–599.

[63] F. Mantlik, *Fundamental solutions for hypoelliptic differential operators depending analytically on a parameter*, Trans. Amer. Math. Soc. 334 (1992), 245–257.

[64] A. Martineau, *Sur la topologie des espaces de fonctions holomorphes*, Math. Ann. 163 (1966), 62–88.

[65] R. Meise, *Sequence space representations for (DFN)-algebras of entire functions modulo closed ideal*, J. Reine Angew. Math. 363 (1985), 59–95.

[66] R. Meise, *Sequence space representations for zero-solutions of convolution equations on ultradifferentiable functions of Roumieu type*, Studia Math. 92 (1989), 211–230.

[67] R. Meise and B. A. Taylor, *Sequence space representations for (FN)-algebras of entire functions modulo closed ideals*, Studia Math. 85 (1987), 203–227.

[68] R. Meise and B. A. Taylor, *Phragmén-Lindelöf conditions for graph varieties*, Results Math. 36 (1999), 121–148.

[69] R. Meise, B. A. Taylor and D. Vogt, *Partial differential operators with continuous linear right inverses*, in: Advances in the Theory of Fréchet Spaces, T. Terzioğlu (ed.), Kluwer, Dordrecht, 1989, 47–62.

[70] R. Meise, B. A. Taylor and D. Vogt, *Characterisation of the linear partial differential operators with constant coefficients that admit a continuous linear right inverse*, Ann. Inst. Fourier, Grenoble 40 (1990), 619–655.

[71] R. Meise, B. A. Taylor and D. Vogt, *Equivalence of analytic and plurisubharmonic Phragmén-Lindelöf conditions*, Proc. Symposia Pure Math. 52 (1991), part 3, 287–308.

[72] R. Meise, B. A. Taylor and D. Vogt, *Continuous linear right inverses for partial differential operators on non-quasianalytic classes and on ultradistributions*, Math. Nachr. 180 (1996), 213–242.

[73] R. Meise, B. A. Taylor and D. Vogt, *Continuous linear right inverses for partial differential operators of order 2 and fundamental solutions in half spaces*, Manuscr. Math. 90 (1996), 449–464.

[74] R. Meise, B. A. Taylor and D. Vogt, *Right inverses for linear, constant coefficient partial differential operators on distributions over open half spaces*, Arch. Math. (Basel) 68 (1997), 311–319.

[75] R. Meise, B. A. Taylor and D. Vogt, *Phragmén-Lindelöf principles on algebraic varieties*, J. Amer. Math. Soc. 11 (1998), 1–39.

[76] R. Meise and D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford, 1997.

[77] T. Meyer, *Surjectivity of convolution operators on spaces of ultradifferentiable functions of Roumieu type*, Studia Math. 125 (1997), 101–129.

[78] V. V. Napalkov and I. A. Rudakov, *A convolution operator in the space of real analytic functions*, Mat. Zametki 49 (1991), no. 3, 57–65, 159 (in Russian); English transl., Math. Notes 49 (1991), no. 3-4, 266–271.

[79] V. P. Palamodov, *Functor of projective limit in the category of topological vector spaces*, Mat. Sb. 75 (1968), 567–603 (in Russian); English transl., Math. USSR Sbornik 17 (1972), 289–315.

[80] V. P. Palamodov, *Homological methods in the theory of locally convex spaces*, Uspekhi Mat. Nauk 26 (1) (1971), 3–66 (in Russian); English transl., Russian Math. Surveys 26 (1) (1971), 1–64.

[81] V. P. Palamodov, *On a Stein manifold the Dolbeault complex splits in positive dimensions*, Mat. Sb. 88 (1972), 287–315 (in Russian); English transl., Math. USSR Sbornik 17 (1972), 289–315.

[82] V. P. Palamodov, *A criterion for splitness of differential complexes with constant coefficients*, in: Geometrical and Algebraical Aspects in Several Variables, C. A. Berenstein, D. Struppa (eds.), EditEl, 1991, 265–291.

[83] A. Pietsch, *Nuclear Locally Convex Spaces*, Akademie Verlag, Berlin, 1972.

[84] V. S. Retakh, *Subspaces of a countable inductive limit*, Doklady AN SSSR 194 (1970) (in Russian); English transl., Soviet Math. Dokl. 11 (1970), 1384–1386.

[85] S. Rolewicz, *Metric Linear Spaces*, PWN, Warszawa, 1984.

[86] T. Rösner, *Surjektivität partieller Differentialoperatoren auf quasianalytischen Roumieu-Klassen*, Dissertation, Düsseldorf, 1997.

[87] C. Roumieu, *Sur quelques extensions de la notion de distributions*, Ann. Sci. Ecole Norm. Sup. Paris, 3 Sér., 77 (1960), 41–121.

[88] H. H. Schaefer, *Topological Vector Spaces*, Springer, New York, 1971.

[89] N. N. Tarkhanov, *Complexes of Differential Operators*, Kluwer, Dordrecht, 1995.

[90] F. Trèves, *Un théorème sur les équations aux dérivées partielles à coefficients constants dépendant de paramètres*, Bull. Soc. Math. France 90 (1962), 471–486.

[91] F. Trèves, *Fundamental solutions of linear partial differential equations with constant coefficients depending on parameters*, Amer. J. Math. 84 (1962), 561–577.

[92] M. Valdivia, *Representaciones de los espacios $\mathscr{D}(\Omega)$ y $\mathscr{D}'(\Omega)$*, Revista Real Acad. Sci. Exactas, Fis. y Nat. (Madrid) 72 (1978), 385–414.

[93] D. Vogt, *Sequence space representations of spaces of test functions and distributions*, in: Functional Analysis, Holomorphy and Approximation Theory, G. L. Zapata (ed.), Lecture Notes Pure Appl. Math. 83, Marcel Dekker, New York, 1983, 405–443.

[94] D. Vogt, *Some results on continuous linear maps between Fréchet spaces*, in: Functional Analysis: Surveys and Recent Results III, K. D. Bierstedt, B. Fuchssteiner (eds.), North-Holland, Amsterdam, 1984, 349–381.

[95] D. Vogt, *On the functors $Ext^1(E, F)$ for Fréchet spaces*, Studia Math. 85 (1987), 163–197.

[96] D. Vogt, *Lectures on projective spectra of DF-spaces*, Seminar lectures, AG Funktionalanalysis, Düsseldorf/Wuppertal, 1987.

[97] D. Vogt, *Topics on projective spectra of LB-spaces*, in: Advances in the Theory of Fréchet Spaces, T. Terzioğlu (ed.), Kluwer, Dordrecht, 1989, 11–27.

[98] J. Wengenroth, *Acyclic inductive spectra of Fréchet spaces*, Studia Math. 120 (1996), 247–258.

[99] J. Wengenroth, *A new characterization of* $\mathrm{Proj}^1\ \mathcal{X} = 0$ *for countable spectra of (LB)-spaces*, Proc. Amer. Math. Soc. 127 (1999), 737–744.

[100] J. Wengenroth, *A splitting theorem for subspaces and quotients of* $\mathscr{D}'$, Bull. Pol. Acad. Sci. Math. 49 (2001), 349–354.

[101] J. Wengenroth, *Derived Functors in Functional Analysis*, Lecture Notes Math. 1810, Springer, Berlin, 2003.

[102] V. P. Zaharjuta, *Spaces of analytic functions and complex potential theory*, in: Linear Topological Spaces and Complex Analysis 1, A. Aytuna (ed.), METU-TÜBİTAK, Ankara, 1994, 74–146.

[103] G. Zampieri, *Propagation of singularity and existence of real analytic solutions of locally hyperbolic equations*, J. Fac. Sci. Univ. Tokyo, Sec. I. A, Math. 31 (1984), 373–390.