APPLICATIONS OF SPHERICAL DESIGNS
TO BANACH SPACE THEORY

HERMANN KÖNIG

Mathematisches Seminar, Universität Kiel, 24 098 Kiel, Germany
E-mail: hkoenig@math.uni-kiel.de

Abstract. Spherical designs constitute sets of points distributed on spheres in a regular way. They can be used to construct finite-dimensional normed spaces which are extreme in some sense: having large projection constants, big or small Banach-Mazur distance to Hilbert spaces or ℓ_p-spaces. These examples provide concrete illustrations of results obtained by more powerful probabilistic techniques which, however, do not exhibit explicit examples. We give a survey of such constructions where the geometric invariants can be estimated quite precisely.

1. Introduction. Dvoretzky's theorem and the study of Hilbertian subspaces of high-dimensional Banach spaces has been the subject of intensive investigation which typically involve probabilistic techniques and the concentration of measure phenomenon, [D], [MS]. Below a certain dimension, "typical" subspaces are Hilbertian whereas for higher dimensions, the maximal Banach-Mazur distance to Hilbert spaces can now be estimated quite precisely, at least for classical spaces like the ℓ_p-spaces. Very often such spaces are very badly complemented and thus have very large projection constants, e.g. for Kashin's subspaces of ℓ_1, [Ka], or Gluskin's spaces yielding the correct diameter estimate for the Banach-Mazur compactum, [G]. "Typical" subspaces here is meant in a probabilistic way, i.e. with respect to the Haar measure on the Graßmann manifold of n-dimensional subspaces of N-dimensional spaces where $n < N$.

In this note, we present some more explicit constructions of Hilbertian subspaces of classical Banach spaces and of spaces with large projection constants which rely on the existence of regular point distributions on spheres—spherical designs—instead of random point set distributions.

As for notation, ℓ_p^N denotes $\mathbb{K}^N$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, equipped with the norm

$$\|x\|_p = \Big(\sum_{j=1}^{N} |x_j|^p \Big)^{1/p}, \quad x = (x_j)_{j=1}^N \in \mathbb{K}^N.$$

The Banach-Mazur distance between two isomorphic Banach spaces X and Y is defined to be

$$d(X, Y) = \inf\{\|T\| \, \|T^{-1}\| \mid T : X \to Y \text{ linear isomorphism}\}.$$

Let $n, k, N \in \mathbb{N}$. A *spherical design* of order $2k$ in n variables with N nodes consists of N points on the euclidean unit sphere $x_1, \ldots, x_N \in S^{n-1}(\mathbb{K})$ and a discrete probability measure on them, $\mu_1, \ldots, \mu_N > 0$, $\sum_{s=1}^{N} \mu_s = 1$ such that all even polynomials of degree $2k$ in n variables $\mathcal{P}_{2k,n}^{even}$ are integrated exactly over the normalized Haar measure σ on the sphere S^{n-1} by the quadrature formula defined by (x_s, μ_s), i.e.

$$\sum_{s=1}^{N} \mu_s p(x_s) = \int_{S^{n-1}} p(y) d\sigma(y), \quad p \in \mathcal{P}_{2k,n}^{even}. \tag{1.1}$$

For $\mathbb{K} = \mathbb{C}$, $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ are identified for the purpose of integration; "even polynomials" then means polynomials in $(z, \overline{z})$ which are homogeneous of degree k each in z and in $\overline{z}$. It suffices to have (1.1) for polynomials which are homogeneous of the highest degree $2k$, since lower order (2ℓ)—homogeneous parts can be multiplied by $1 = (y, y)^{k-\ell}$ to get the same degree $(2k)$ of homogenity when restricted to S^{n-1}. Spherical designs in the classical sense of Delsarte, Goethals and Seidel [DGS] are more restricted having $\mu_s = 1/N$ but (1.1) is more suitable for our purpose. In the case of $k = 1$, polynomials of degree 2, take $p(y) = (y \otimes y)_{ij} = y_i y_j$ for $y = (y_j)_{j=1}^{n} \in \mathbb{K}^n$. Since $\int_{S^{n-1}} p(y) d\sigma(y) = 1/n \, \delta_{ij}$, a spherical design of degree 2 just means

$$n \sum_{s=1}^{N} \mu_s x_s \otimes x_s = \mathrm{Id}_{\mathbb{K}^n}. \tag{1.2}$$

John's theorem typically yields such points which as *contact* points of a norm additionally satisfy $\|x_s\|_X = \|x_s\|_{X^*} = 1$. The points (x_s) in (1.2) are orthogonal projections of an orthonormal basis in ℓ_2^N onto a suitable n-dimensional subspace:

$$P := (\mu_s \langle x_s, x_t \rangle)_{s,t=1}^{N} : \ \mathbb{K}^N \to \mathbb{K}^N.$$

Here $\mathbb{K}^n$ should be thought of being imbedded into $\mathbb{K}^N$ by $x \mapsto (\mu_s \langle x, x_s \rangle)_{s=1}^{N}$.

2. Hilbertian subspaces of ℓ_p-spaces.

Let $\varepsilon > 0$ and $1 \leq p < \infty$. By Dvoretzky's theorem for ℓ_p-spaces, given $n \in \mathbb{N}$ there exists $c_p > 0$ such that if $N \in \mathbb{N}$ satisfies

$$N \geq \left\{ \begin{array}{cc} c_p n & 1 \leq p \leq 2 \\ c_p n^{p/2} & 2 \leq p < \infty \end{array} \right\},$$

the N-dimensional space ℓ_p^N contains almost Hilbertian subspaces $Y_n \subset \ell_p^N$ of $\dim Y_n = n$, $d(Y_n, \ell_2^n) \leq 1 + \varepsilon$, [FLM]. We say that ℓ_2^n embeds $(1 + \varepsilon)$-isomorphically into ℓ_p^N.

We consider the case of isometric imbeddings, i.e. $\varepsilon = 0$ and $n \geq 2$. Do isometric imbeddings of ℓ_2^n into ℓ_p^N exsist? The answer is yes if and only if p is an even integer, $p = 2k \in 2\mathbb{N}$. That p needs to be even can be found in Lyubich's paper [L], that isometric imbeddings of ℓ_2^n into ℓ_p^N exist for $p = 2k$ was shown by Gromov, Milman [M]. In fact, even in the infinite-dimensional case, isometric imbeddings of ℓ_2^2 into ℓ_p only exist if $p = 2k \in 2\mathbb{N}$ as was proved by Delbaen, Jarchow and Pełczyński [DJP]. They also prove

that general n-dimensional subspaces of L_p are imbedded isometrically into ℓ_{2k}^N if N is sufficiently large.

The following proposition states the equivalence of imbedding ℓ_2^n isometrically into ℓ_{2k}^N with the existence of appropriate spherical designs. It is a generalization of results of Delsarte, Goethals and Seidel [DGS] and Reznick [R] and was stated in [K]. The equivalence $(1) \Leftrightarrow (2)$ was also obtained by Lyubich and Vaserstein [LV].

PROPOSITION 1. *Let $n, k, N \in \mathbb{N}$. Then the following are equivalent:*

(1) *There exists an isometric imbedding of ℓ_2^n into ℓ_{2k}^N.*
(2) *There exists a spherical design $(x_s, \mu_s)_{s=1}^N$ of order $2k$ in n variables with N nodes.*
(3) *There exist points $x_1, \ldots, x_N \in S^{n-1}(\mathbb{K})$ and $\mu_s > 0$, $\sum_{s=1}^N \mu_s = 1$ such that*

$$\sum_{s,t=1}^N \mu_s \mu_t |\langle x_s, x_t \rangle|^{2k} = \int_{S^{n-1}} \int_{S^{n-1}} |\langle x, y \rangle|^{2k} d\sigma(x) d\sigma(y) =: c_{nk}. \qquad (2.1)$$

Condition (2.1) gives a criterion to check whether (x_s, μ_s) is a spherical design which defines an imbedding. We sketch the important directions $(3) \Rightarrow (2)$ and $(2) \Rightarrow (1)$ in the real case; $(1) \Rightarrow (3)$ is not difficult either.

Proof. $(3) \Rightarrow (2)$. For $x \in \mathbb{R}^n$ and $\ell \in N$, denote $x^{\otimes \ell} = x \otimes \cdots \otimes x \in \mathbb{R}^{n^\ell}$ the ℓ-fold tensor product. Then $\langle x^{\otimes \ell}, y^{\otimes \ell} \rangle = \langle x, y \rangle^\ell$ which for even ℓ coincides with $|\langle x, y \rangle|^\ell$. Given points $x_s \in S^{n-1}$ and $\mu_s > 0$, $\sum_{s=1}^N \mu_s = 1$, form the tensor

$$\xi := \sum_{s=1}^N \mu_s x_s^{\otimes 2k} - \int_{S^{n-1}} y^{\otimes 2k} d\sigma(y) \in \mathbb{R}^{n^{2k}}.$$

Evaluating the scalar product $0 \leq \langle \xi, \xi \rangle$ in $\mathbb{R}^{n^{2k}}$, we find

$$0 \leq \sum_{s,t=1}^N \mu_s \mu_t |\langle x_s, x_t \rangle|^{2k} - 2 \sum_{s=1}^N \mu_s \int_{S^{n-1}} |\langle x_s, y \rangle|^{2k} d\sigma(y) + \int_{S^{n-1}} \int_{S^{n-1}} |\langle x, y \rangle|^{2k} d\sigma(y).$$

The integral in the second sum does not depend on s since $|x_s| = 1$ and σ is rotation invariant; it equals c_{nk} as defined in (2.1). Hence

$$0 \leq \sum_{s,t=1}^N \mu_s \mu_t |\langle x_s, x_t \rangle|^{2k} - c_{nk} \qquad \text{(Sidelnikov's inequality)}.$$

Thus (2.1) means that $\xi = 0$. In coordinates this means that all homogeneous polynomials of degree $2k$ are integrated exactly by the quadrature formula defined by (x_s, μ_s) and (2) follows from this.

$(2) \Rightarrow (1)$. Given the design (x_s, μ_s), consider the imbedding

$$i : \ell_2^n \to \ell_{2k}^N, \qquad x \mapsto ((\mu_s/c_{nk})^{1/2k} \langle x, x_s \rangle)_{s=1}^N,$$

Denoting $p_x(y) := \langle x, y \rangle^{2k}$ for any fixed $x \in \mathbb{R}^n$, we have

$$||ix||_{2k}^{2k} = \sum_{j=1}^N (\mu_s/c_{nk}) p_x(x_s) = (1/c_{nk}) \int_{S^{n-1}} p_x(y) d\sigma(y) = |x|^{2k},$$

i.e. i is isometric. ∎

REMARK. If (x_s, μ_s) are given explicitly, the imbedding i is defined explicitly as the proof shows. The usefulness of (2.1) is best illustrated by a few examples.

EXAMPLES.

(1) $\mathbb{K} = \mathbb{R}$, $n = 2$, $N = k+1$. Take $x_s = \exp\left(\frac{2\pi i}{N}s\right)_{s=0}^{k} \in \mathbb{C} \simeq \mathbb{R}^2$, $\mu_s = 1/N$. It is easy to check that (2.1) holds, so $\ell_2^2 \hookrightarrow \ell_{2k}^{k+1}$ imbeds isometrically. In particular $i : \ell_2^2 \hookrightarrow \ell_4^3$ with $i(\ell_2^2) = \{y \in \ell_4^3 \mid \sum_{s=1}^3 y_s = 0, \ y = (y_s)_{s=1}^3\}$.

(2) $\mathbb{K} = \mathbb{C}$, $n = 2$, $k = 2$, $N = 4$: $\ell_2^2(\mathbb{C}) \hookrightarrow \ell_4^4(\mathbb{C})$ by using $\mu_s = 1/4$ and the vectors $x_1 = (1, 0)$, $x_s = \frac{1}{\sqrt{3}}(1, \sqrt{2}\,\omega^{s-2})$, $s = 2, 3, 4$ with $\omega = \exp(2\pi i/3)$.

(3) $\mathbb{K} = \mathbb{R}$, $n = 24$, $k = 5$, $N = 98280$: this uses vectors of the Leech lattice, see [R] or [DGS].

It is of course of interest to determine the minimal dimension N of the ℓ_p^N-space, $p = 2k$, where ℓ_2^n imbeds isometrically into; for quadrature formulas this means the minimal number of nodes.

DEFINITION. $N(n, k) := \min\{N \in \mathbb{N} \mid \ell_2^n \hookrightarrow \ell_{2k}^N \text{ imbeds isometrically}\}$.

PROPOSITION 2. Let $2 \le n, k \in \mathbb{N}$. Then

$$\binom{n+k-1}{k} \le N(n, k) \le \binom{n+2k-1}{2k}, \quad \mathbb{K} = \mathbb{R},$$

$$\binom{n + [\frac{k+1}{2}] - 1}{[\frac{k+1}{2}]}\binom{n + [\frac{k}{2}] - 1}{[\frac{k}{2}]} \le N(n, k) \le \binom{n+k-1}{k}^2, \quad \mathbb{K} = \mathbb{C}. \tag{2.2}$$

For the upper bound, see [M], for the lower e.g. in the real case [R]. Note that $\binom{n+k-1}{k}$ is the dimension of the (real) polynomials in n variables which are homogeneous of degree k. For fixed k and $n \to \infty$, the lower and upper bounds differ in an essential way. The lower bound is $O(n^k)$ whereas the upper bound is $O(n^{2k})$. Integrating polynomials of degree $2k$ with dimension $O(n^{2k})$, one might expect the upper bound to be tight; looking at Dvoretzky's theorem, just putting $\varepsilon = 0$, one might suspect the lower bound $O(n^k) = O(n^{p/2})$ to be tight. The right order is unknown so far, except for $k = 2$. The lower bound is an equality for tight designs; they have been characterized by Bannai [B] and Hogger [H]:

PROPOSITION 3. Let $n, k \in \mathbb{N}$ and assume that the lower bound in (2.2) is an equality. Then the nodes $(x_s)_{s=1}^N \subseteq S^{n-1}$ and the measure $(\mu_s)_{s=1}^N$ of any design with (2.1) satisfy $\mu_s = 1/N$ and for any $1 \le s \ne t \le N$ the value $|\langle x_s, x_t\rangle|$ is a zero of a certain explicit polynomial of degree k (in terms of Jacobi polynomials even/odd if k is even/odd). For $k = 2$, $|\langle x_s, x_t\rangle| = 1/\sqrt{k+2}(\mathbb{R})$ or $= 1/\sqrt{k+1}(\mathbb{C})$ for $s \ne t$ and $N = n(n+1)/2(\mathbb{R})$ or $n^2(\mathbb{C})$. This is equivalent to the existence of the maximal possible number of "equiangular lines" $\mathbb{K}x_s$ in $\mathbb{K}^n$.

Designs attaining the lower bound are called *tight*. They do not exist for $k > 5$, but there are a few cases if $k \le 5$ ([B], [H]): they exist for

$\mathbb{K} = \mathbb{R}$: $\quad k = 2$, $n = 2, 3, 7, 23$ when $N = 3, 6, 28, 276$ (equiangular lines)

$\qquad\quad k = 3$, $n = 8, 23$ when $n = 120, 2300$

$\qquad\quad k = 5$, $n = 24$ when $N = 98280$ (points in the Leech lattice)

$\qquad\quad n = 2$, $N = k + 1$ $\quad$ (n-the root of unity in $\mathbb{C} \simeq \mathbb{R}^2$)

$\mathbb{K} = \mathbb{C}$: $\quad k = 2$, $n = 2, 3, 8$ when $N = 4, 9, 64$

$\qquad\quad k = 3$, $n = 4, 6$ when $N = 40, 126$

$\qquad\quad n = 2$, $k = 2, 3, 5$ when $N = 4, 6, 12$.

In these cases, $\ell_2^n \to \ell_{2k}^N$ imbeds isometrically and N is the smallest dimension for this to hold. The examples following Proposition 1 were of this form.

To find a sequence of almost tight 4-designs in $\mathbb{K}^n$ with (2.1) we use sets of equiangular vectors in $\mathbb{C}^n$ which attain almost the maximal possible number n^2 of such vectors. We use the number-theoretic B_2-sequences employed by Rudin in his constructions of $\Lambda(4)$-sets.

PROPOSITION 4 ([K]). *Let $n = p^m + 1$, p prime. Then there exists an isometric imbedding $\ell_2^n \hookrightarrow \ell_4^{n^2+1}$ over $\mathbb{C}$.*

Proof. Let $N = n^2 - n + 1$. There exist integers $0 \le d_1 < \cdots d_n < N$ such that all numbers from 1 to $N - 1 = n(n-1)$ occur exactly once as residue mod N of a difference $d_j - d_k$ $(j \ne k)$, of which there are $N - 1 = n(n-1)$, cf [HR]. Define

$$x_s := \frac{1}{\sqrt{n}} \left(\exp\left(\frac{2\pi i}{N} d_j s\right) \right)_{j=1}^n \in S^{n-1}(\mathbb{C}) \subseteq \mathbb{C}^n; \quad s = 1, \ldots, N. \qquad (2.3)$$

These vectors are equiangular with $|\langle x_s, x_t \rangle| = \sqrt{n-1}/n$ for all $1 \le s \ne t \le N$ as seen by calculating $|\langle x_s, x_t \rangle|^2$ by using that $\sum_{j \ne k} \exp(\frac{2\pi i}{N}(d_j - d_k)\Theta) = -1$ for all $1 \le \Theta \le N - 1$. These vectors do not quite suffice to have equality in (2.1) with $\mu_s = 1/N$. However, adding the n unit vectors $e_j = (0 \ldots 0, 1, \ldots 0)$, $j = 1, \ldots, n$; i.e. letting $x_{N+j} := e_j$, we have $n^2 + 1$ vectors which satisfy (2.1) if one puts in addition $\mu_s = \frac{n}{n+1} \frac{1}{N}$ for $1 \le s \le N$ and $\mu_{N+j} = \frac{n}{n+1} \frac{1}{n^2}$ for $1 \le j \le n$. There $c_{n,2} = 2/[n(n+1)]$ for $\mathbb{K} = \mathbb{C}$. Hence $\ell_2^n \hookrightarrow \ell_4^{n^2+1}$ over $\mathbb{C}$. ∎

These complex cubature formulas give rise to real cubature formulas of degree 4 in $m = 2n$ real variables with $M = 3(n^2 + 1) = \frac{3}{4}(m^2 + 4)$ nodes as follows from the chain of imbeddings

$$\ell_2^m(\mathbb{R}) = \ell_2^n(\mathbb{C}) \hookrightarrow \ell_4^{n^2+1}(\mathbb{C}) = \ell_4^{n^2+1}(\ell_2^2(\mathbb{R})) \hookrightarrow \ell_4^{n^2+1}(\ell_4^3(\mathbb{R})) = \ell_4^{3(n^2+1)}(\mathbb{R}).$$

Also these have only $O(m^2)$ nodes and thus asymptotically are of the lower asymptotic order in Proposition 2 as $m \to \infty$.

Similar constructions for B_k-sequences for $k \ge 3$ do not seem to work; one would have to add more vectors than just the unit vectors e_j. For $k = 3$ one can add $O(n^2)$ vectors but to satisfy (2.1) some of the weights μ_s would have to be negative which is not allowed since μ would not define a (positive) probability measure. It is an important open problem whether the lower bound in Proposition 3 gives the right asymptotic order as $n \to \infty$ also if $k \ge 3$ is fixed. A positive answer would mean that one could put $\varepsilon = 0$ in Dvoretzky's theorem for ℓ_p^N if $0 = 2k$.

3. Projection constants. Spherical designs and equiangular lines also show up in estimating projection constants and in the construction of spaces with maximal projection constants. Any separable, in particular any finite-dimensional Banach space X may be imbedded into ℓ_∞. Take any fixed linear isometric imbedding $X \subseteq \ell_\infty$ and define the *projection constant* of X by

$$\lambda(X) := \inf\{||P|| \mid P^2 = P : \ell_\infty \to X \subseteq \ell_\infty \text{ is a linear projection onto } X\}.$$

This value is independent of the particular imbedding and ℓ_∞ is the "worst" superspace: if $\dim X < \infty$ and X is isometrically imbedded into some Banach space Y, there is a projection $Q^2 = Q : Y \to X \subseteq Y$ onto X with $||Q|| \leq \lambda(X)$. For n-dimensional Hilbert spaces, $\lambda(\ell_2^n) \sim \sqrt{\frac{2}{\pi}}\sqrt{n}$ ($\mathbb{K} = \mathbb{R}$) and $\sim \frac{\sqrt{\pi}}{2}\sqrt{n}$ ($\mathbb{K} = \mathbb{C}$). Kadec and Snobar [KS] showed that $\lambda(X) \leq \sqrt{n}$ holds for any n-dimensional space X. This estimate was improved in [KT1] where the following dual characterization of extremal projection constants was proved; the supremum on the right side is attained in some cases by spherical designs:

PROPOSITION 5 ([KT1]). *Let $n \in \mathbb{N}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then*

$$\sup\{\lambda(X_n) \mid \dim X_n \leq n\} = n \sup\Big\{ \sum_{s,t=1}^{N} \mu_s\mu_t|\langle x_s, x_t\rangle| \,\Big|\, N \in \mathbb{N}\cup\{\infty\},\ \mu_t > 0, \sum_{t=1}^{N}\mu_t = 1,$$

$$\mathrm{Id}_n = n\sum_{t=1}^{N}\mu_t x_t \otimes x_t \Big\}. \quad (3.1)$$

Both suprema are in fact attained as maxima.

The proof uses the trace-dual characterization of projection constants in terms of 1-summing norms; $\mu = (\mu_t)$ is the corresponding Pietsch measure. Since

$$n \sum_{s,t=1}^{N} \mu_s\mu_t|\langle x_s, x_t\rangle| \leq n\Big(\sum_{s,t=1}^{N} \mu_s\mu_t|\langle x_s, x_t\rangle|^2 \Big)^{1/2} = \sqrt{n}$$

under the constraints on (x_t, μ_t), the Kadec–Snobar result is an immediate consequence of (3.1). An improvement can be derived by estimating $|u| \leq a + bu^2 - cu^4$ for appropriate $a, b, c > 0$ with $u = \langle x_s, x_t\rangle$ and by using the Sidelnikov inequality mentioned in the proof of Proposition 1. This way one proves

THEOREM 6 ([KT1]). *Let $n = \mathbb{N}$. Then for any n-dimensional Banach space X_n*

$$\lambda(X_n) \leq G(n) := \left\{ \begin{array}{ll} (2 + (n-1)\sqrt{n+2})/(n+1), & \mathbb{K} = \mathbb{R} \\ (1 + (n-1)\sqrt{n+1})/n, & \mathbb{K} = \mathbb{C} \end{array} \right\} < \sqrt{n}.$$

There is $X_n, \dim X_n = n$, with $\lambda(X_n) = G(n)$ if and only if there exist $N(n)$ equiangular lines in $\mathbb{K}^n$ where

$$N(n) = \left\{ \begin{array}{ll} n(n+1)/2, & \mathbb{K} = \mathbb{R} \\ n^2, & \mathbb{K} = \mathbb{C} \end{array} \right\},$$

i.e. if and only if there is a tight 4-design in $S^{n-1}(\mathbb{K}) \subseteq \mathbb{K}^n$.

If $(x_s, \mu_s = \frac{1}{N(n)})_{s=1}^{N(n)}$ is such a tight design, the norm of $X_n = (\mathbb{K}^n, ||\cdot||)$ with $\lambda(X_n) = G(n)$ may be defined as $||x|| = \max_{1 \leq s \leq N(n)} |\langle x, x_s\rangle|$. Thus $X_n \subset \ell_\infty^{N(n)}$ is very

badly complemented. Taking the ℓ_4-norm, instead of the ℓ_∞-norm, however, one gets isometrically a Hilbertian subspace of $\ell_4^{N(n)}$ onto which in addition uniformly bounded projections exist. For $\mathbb{K} = \mathbb{R}$ and $n = 2, 3$ spaces with maximal projection constants derived from this are those whose unit balls are the regular hexagon and the regular dodecahedron. However, in general spaces with extremal projection constants are not unique since a corresponding imbedding into ℓ_1^N instead of ℓ_∞^N may yield a different space with the same projection constant:

PROPOSITION 7 ([KT2]). *Let* $X_\infty \subseteq \ell_\infty^N$, $\dim X_\infty = n$ *be such that* $\lambda(X_\infty) = \sup\{\lambda(Y) \mid \dim Y = n\}$. *Then there exists a probability measure* $\mu = (\mu_s)_{s=1}^N$ *on* $\{1, \ldots, N\}$ *such that when the same space* X_∞ *is considered as a subspace of* $\ell_1^N(\mu)$, *then written as* $X_1 \subseteq \ell_1^N(\mu)$, *one has* $\lambda(X_\infty) = \lambda(X_1)$.

For $\mathbb{K} = \mathbb{R}$ and $n = 2$ one gets the hexagonal space in both cases. In general, however, the spaces X_∞ and X_1 are very different. For $\mathbb{K} = \mathbb{R}$ and $n = 3$, $x_s =$ diagonals of the icosahedron, X_∞ is the space with dodecahedron unit ball and X_1 the space whose unit ball is the dodecahedron intersected with the ϕ-multiple of the face-centrally inscribed icosahedron, $\phi = (1 + \sqrt{5})/2$. The second unit ball has as its faces 12 pentagons—as the dodecahedron—plus 20 triangles, cf. [KT2].

Taking the almost maximal number of equiangular vectors x_s given by (2.3) in $\mathbb{C}^n$ where $n = p^m + 1$ and $s = 1, \ldots, N := n^2 - n + 1$, we may define a space $X_\infty = (\mathbb{C}^n, ||\cdot||) \subseteq \ell_\infty^N$ by letting

$$||x|| := \sup_{1 \leq s \leq N} |\langle x, x_s \rangle|.$$

In this case, one can check that X_∞ has almost extremal projection constant. More precisely, cf. [KT3],

$$\lambda(X_\infty) \geq G(n) - 1/n^{3/2} \geq \sqrt{n} - 1/\sqrt{n}.$$

In particular, $\lim_{n \to \infty} \lambda(X_\infty)/\sqrt{n} = 1$ when n is a sequence of integers of the form $n = p^m + 1$ tending to infinity. The spaces X_∞ also have extremal distance to ℓ_2^n and extremal unconditional basis constant ubc, i.e.

$$d(X_\infty, \ell_2^n) = \sqrt{n}, \qquad ubc(X_\infty) = \sqrt{n}.$$

As subspaces of ℓ_p^N, denoted X_p they have uniformly bounded Banach-Mazur distance to ℓ_2^n provided that $1 \leq p \leq 4$; onto these spaces, as subspaces of ℓ_p^N, there are also uniformly bounded projections. For these facts see [KT2], [KT3].

Further beautiful applications of spherical designs to the Borsuk and Knaster conjectures are given in papers of Hinrichs and Richter [HR1], [HR2].

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