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A STUDY OF SOME CONSTANTS CHARACTERIZING THE WEIGHTED HARDY INEQUALITY

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Abstract. The modern form of Hardy's inequality means that we have a necessary and sufficient condition on the weights u and v on $[0,b]$ so that the mapping

$$H : L^p(0,b;v) \to L^q(0,b;u)$$

is continuous, where $Hf(x) = \int_0^x f(t)dt$ is the Hardy operator. We consider the case $1 < p \leq q < \infty$ and then this condition is usually written in the Muckenhoupt form

$$(*) \qquad A_1 := \sup_{0<x<b} A_M(x) < \infty.$$

In this paper we discuss and compare some old and new other constants A_i of the form $(*)$, which also characterize Hardy's inequality. We also point out some dual forms of these characterizations, prove some new compactness results and state some open problems.

1. Introduction. We consider the general one-dimensional Hardy inequality

$$(1.1) \qquad \left(\int_0^b |f(t)|^q\, u(t)dt\right)^{1/q} \leq C\left(\int_0^b |f'(t)|^p\, v(t)dt\right)^{1/p}$$

with a fixed b, $0 < b \leq \infty$, for measurable functions f satisfying $f(0) = 0$, weights u and v and for parameters p, q satisfying $1 < p \leq q < \infty$. (1.1) is usually characterized by the

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(Muckenhoupt) condition

$$A_1 := \sup_{0 \le x \le b} A_M(x) < \infty, \tag{1.2}$$

where

$$A_M(t) = \left(\int_x^b u(t)dt\right)^{1/q} \left(\int_0^x v^{1-p'}(t)dt\right)^{1/p'}.$$

It is nowadays also well-known that A_1 can be defined in terms of other functions of t than $A_M(t)$ (see e.g. the book [10] and the PhD thesis [17]). In this paper we prove a new scale of conditions $A_4(s) < \infty$ $(1 < s < p)$, which can replace (1.2) [and so that $A_4(p) = A_1$], see Theorem 1, Remark 2 and cf. also [17]. Moreover, by using standard duality arguments we can even point out some more conditions of the type (1.2). We sum up our investigations by formulating a more general theorem, where the Hardy inequality in (1.1) is characterized not only by (1.2) but by seven different (mutually equivalent) conditions and a corresponding better estimate of the best constant in (1.1); see the "seven conditions" Theorem 2.

It is also well-known that some of the constants A_i (or more precisely, the corresponding functions $A_i(x)$) mentioned below are useful also in other circumstances (e.g. when studying limiting procedures and compactness properties of the Hardy operators in weighted Lebesgue spaces). Therefore it is of interest to further discuss and compare the constants above and in particular the equipped functions corresponding to $A_M(t)$. In Section 3 we present and discuss some results in this direction and, in particular, our main compactness result corresponding to Theorem 1 is proved there (see Theorem 3) and also its dual form is pointed out (see Theorem 4). Finally, in Section 4 we make some concluding remarks and raise some open questions.

2. Some weight characterizations of Hardy's inequality. First we present a new weight characterization of Hardy's inequality which also can be found in [17] for the case $b = \infty$.

THEOREM 1. *Let* $1 < p \le q < \infty$, $s \in (1, p)$, *and* $0 < b \le \infty$. *Then the inequality*

$$\left(\int_0^b \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{1/q} \le C\left(\int_0^b f^p(x)v(x)dx\right)^{1/p} \tag{2.1}$$

holds for all $f \ge 0$ *iff*

$$A_W(s,q,p) := \sup_{0<t<b} V(t)^{s-1/p}\left(\int_t^b u(x)V(x)^{q(\frac{p-s}{p})}dx\right)^{1/q} < \infty, \tag{2.2}$$

where $V(t) = \int_0^t v(x)^{1-p'}dx$. *Moreover, if* C *is the best possible constant in (2.1), then*

$$\sup_{1<s<p}\left(\frac{(\frac{p}{p-s})^p}{(\frac{p}{p-s})^p + \frac{1}{s-1}}\right)^{1/p} A_W(s,q,p) \le C \le \inf_{1<s<p}\left(\frac{p-1}{p-s}\right)^{1/p'} A_W(s,q,p). \tag{2.3}$$

We note that the inequalities (1.1) (with $f(0) = 0$) and (2.1) are equivalent.

Proof. Let $f^p(x)v(x) = g(x)$ in (2.1). Then (2.1) is equivalent to

$$\left(\int_0^b \left(\int_0^x g(t)^{\frac{1}{p}} v(t)^{-\frac{1}{p}} dt\right)^q u(x)dx\right)^{1/q} \leq C\left(\int_0^b g(x)dx\right)^{1/p}. \tag{2.4}$$

Assume that (2.2) holds. By applying Hölder's inequality, the fact that $V'(t) = v(t)^{1-p'} = v(t)^{-p'/p}$ and Minkowski's inequality, we have

$$\left(\int_0^b \left(\int_0^x g(t)^{\frac{1}{p}} v(t)^{-\frac{1}{p}} dt\right)^q u(x)dx\right)^{1/q} =$$

$$= \left(\int_0^b \left(\int_0^x g(t)^{\frac{1}{p}} V(t)^{\frac{s-1}{p}} V(t)^{-\frac{s-1}{p}} v(t)^{-\frac{1}{p}} dt\right)^q u(x)dx\right)^{1/q}$$

$$\leq \left(\int_0^b \left(\int_0^x g(t)V(t)^{s-1}dt\right)^{q/p} \left(\int_0^x V(t)^{-\frac{(s-1)p'}{p}} v(t)^{-\frac{p'}{p}} dt\right)^{q/p'} u(x)dx\right)^{1/q}$$

$$= \left(\frac{p}{p-(s-1)p'}\right)^{1/p'} \left(\int_0^b \left(\int_0^x g(t)V(t)^{s-1}dt\right)^{q/p} V(x)^{\frac{p-(s-1)p'}{p}\frac{q}{p'}} u(x)dx\right)^{1/q}$$

$$\leq \left(\frac{p-1}{p-s}\right)^{1/p'} \left(\int_0^b g(t)V(t)^{s-1}\left(\int_t^b V(x)^{q(\frac{p-s}{p})} u(x)dx\right)^{p/q} dt\right)^{1/p}$$

$$\leq \left(\frac{p-1}{p-s}\right)^{1/p'} A_W(s,q,p)\left(\int_0^b g(t)dt\right)^{1/p}.$$

Hence (2.4) and, thus, (2.1) holds with a constant satisfying the right hand side inequality in (2.3).

Now we assume that (2.1) and thus (2.4) holds and choose the test function

$$g(x) = \left(\frac{p}{p-s}\right)^p V(t)^{-s} v(x)^{1-p'} \chi_{(0,t)}(x) + V(x)^{-s} v(x)^{1-p'} \chi_{(t,b)}(x),$$

where t is a fixed number > 0. By inserting this function into (2.4) we find that

$$\left(\frac{p}{p-s}\right)\left(\int_t^b V(x)^{(1-\frac{s}{p})q} u(x)dx\right)^{1/q} \leq C\left(\left(\frac{p}{p-s}\right)^p + \frac{1}{s-1}\right)^{1/p} V(t)^{1-s/p},$$

i.e. that

$$\left(\frac{p}{p-s}\right)\left(\left(\frac{p}{p-s}\right)^p + \frac{1}{s-1}\right)^{-1/p} V(t)^{s-1/p}\left(\int_t^b V(x)^{(1-\frac{s}{p})q} u(x)dx\right)^{1/q} \leq C$$

or, equivalently, that

$$\left(\frac{(\frac{p}{p-s})^p}{(\frac{p}{p-s})^p + \frac{1}{s-1}}\right)^{1/p} V(t)^{s-1/p}\left(\int_t^b V(x)^{(1-\frac{s}{p})q} u(x)dx\right)^{1/q} \leq C.$$

We conclude that (2.2) and the left hand side of the estimate of (2.3) hold. The proof is complete.

Next we note that the function $A_M(x)$ from the introduction can be written as

$$A_M(x) = \left(\int_x^b u(t)dt\right)^{1/q} V^{1/p'}(x),$$

where

$$V(x) := \int_0^x v^{1-p'}(t)dt < \infty \quad \text{for every } x \in (0,b).$$

Here, and in the sequel, $p' = p/(p-1)$. We also need to introduce the following additional auxiliary functions:

$$A_T(x) := \left(\int_0^x u(t)V^q(t)dt \right)^{1/q} V^{-1/p}(x),$$
$$A_B(x;g) := \left(\frac{1}{g(x)} \int_0^x u(t)\left[g(t)+V(t)\right]^{q/p'+1} dt \right)^{1/q'}, \quad g(x) > 0;$$

and

$$A_W(x,s) := \left(\int_x^b u(t)V^{q(p-s)/p}(t)dt \right)^{1/q} V^{(s-1)/p}(x), \quad 1 < s < p.$$

Further we denote

$$\text{(2.5)} \qquad \begin{cases} A_1 := \sup\limits_{x\in(0,b)} A_M(x), \\ A_2 := \sup\limits_{x\in(0,b)} A_T(x), \\ A_3 := \inf\limits_{g(x)>0} \sup\limits_{x\in(0,b)} A_B(x;g), \\ A_4(s) := \sup\limits_{x\in(0,b)} A_W(x,s), 1 < s < p. \end{cases}$$

We have noted in our introduction and Theorem 1 that each of the conditions $A_1 < \infty$ and $A_4(s) < \infty$, respectively are necessary and sufficient for the Hardy inequality (1.1) to hold. It is also known that the conditions $A_2 < \infty$ and $A_3 < \infty$ characterize Hardy's inequality (see [7] and [14]).

REMARK 1. Here the subscripts M, T, B and W are equipped with the names B. Muckenhoupt, G. Tomaselli, P. R. Beesack and A. Wedestig. The oldest result is due to Beesack [2] (1961) with $A_B(x)$ for $p = q$, while the form given here for $p \le q$ is due to P. Gurka [7] (1984). G. Tomaselli [16] characterized Hardy's inequality via the function $A_T(x)$ for $p = q$ in 1969 and L.-E. Persson and V. D. Stepanov [14] for the case $p \le q$ in 2002. B. Muckenhoupt published his result connected to the function $A_M(x)$ for $p = q$ in 1972 [13] and in 1978 J. S. Bradley [4] generalized this result for the case $p \le q$. For the case $b = \infty$ the newest condition $A_4(s) < \infty$ was proved in the PhD thesis of A. Wedestig [17] and our proof here in Theorem 1 is similar. More details concerning the history of Hardy's inequality can be found in [9].

REMARK 2. Note that $A_W(x,p) = A_M(x)$, so that $A_1 < \infty$ may be regarded as a limiting case of the Wedestig scale of conditions $A_4(s) < \infty$ $(1 < s \le p)$. Moreover, we have

$$\text{(2.6)} \qquad A_W(x,s) \ge A_M(x), \quad 1 < s \le p.$$

In fact, since $V(t)$ is increasing and $p \ge s$, we have

$$\text{(2.7)} \qquad A_W(x,s) = \left(\int_x^b u(t)V^{q(p-s)/p}(t)dt \right)^{1/q} V^{(s-1)/p}(x)$$

$$\geq \left(\int_x^b u(t)dt \right)^{1/q} V^{(p-s)/p}(x) V^{(s-1)/p}(x)$$
$$= \left(\int_x^b u(t)dt \right)^{1/q} V^{(p-1)/p}(x) = A_M(x).$$

In fact, by modifying this argument we see that $A_W(x,s)$ is nonincreasing in s on $(1,p]$.

Next, we note that the constants A_1, A_2, A_3 and $A_4(s)$ can all also be used to estimate the best constant C in (1.1). For example the following is known (see e.g. [10] and [17] and the references given there):

$$l_1 \leq \frac{C}{A_1} \leq u_1,$$

with

$$l_1 = l_1(p,q) = 1 \text{ and } u_1 = u_1(p,q) = \left(\frac{\Gamma(\frac{pq}{(q-p)}}{\Gamma(\frac{q}{q-p})\Gamma(\frac{p(q-1)}{q-p})} \right)^{q-p/qp}, \tag{2.8}$$

and

$$l_2 \leq \frac{C}{A_2} \leq u_2,$$

with

$$l_2 = l_2(p,q) = 1, \qquad u_2 = u_2(p,q) = p'; \tag{2.9}$$

furthermore

$$l_3 \leq \frac{C}{A_3} \leq u_3,$$

with

$$l_3 = l_3(p,q) = \frac{(p'/q)^{1/q}}{(1+\frac{q}{p'})^{1/q}(1+\frac{p'}{q})^{1/p'}}, \qquad u_3 = u_3(p,q) = \left(\frac{p'}{q} \right)^{1/q}, \tag{2.10}$$

and, for $1 < s < p$,

$$l_4(s) \leq \frac{C}{A_4(s)} \leq u_4(s),$$

with

$$\begin{aligned} l_4(s) = l_4(s,p,q) &= \left(\frac{p}{p-s} \right) \left(\left(\frac{p}{p-s} \right)^p + \frac{1}{s-1} \right)^{-1/p}, \\ u_4(s) = u_4(s,p,q) &= \left(\frac{p-1}{p-s} \right)^{1/p'}. \end{aligned} \tag{2.11}$$

Next we will point out the fact that there are also other criteria for the validity of (1.1). These are based on the following *duality principle*:

Let $p, q > 1$ and consider the Hardy operator

$$(Hf)(x) := \int_0^x f(t)dt.$$

If the mapping

$$H : L^p(0,b;v) \to L^q(0,b,u) \tag{2.12}$$

is continuous, then the conjugate Hardy operator $(\widetilde{H}f)(x) = \int_x^b f(t)dt$ *maps the space* $L^{q'}(0,b,u^{1-q'})$ *continuously into* $L^{p'}(0,b;v^{1-p'})$ *with the same operator norm (=the least constant such that the corresponding inequality holds), see e.g. [9, p. 13].*

We consider the following (dual) analogue of $A_T(x)$:

$$A_T^*(x) = \left(\int_x^b v^{1-p'}(t)\left(\int_t^b u(s)ds\right)^{p'} dt\right)^{1/p'} \left(\int_x^b u(t)dt\right)^{-1/q'}.$$

Moreover, the dual analogues to $A_B(x;g)$ and $A_W(x,s)$ read:

$$A_B^*(x;g) = \left(\frac{1}{g(x)}\int_x^b v^{1-p'}(t)\left[g(t) + \int_t^b u(s)ds\right]^{p'/q+1} dt\right)^{1/p'}, \quad g(x) > 0,$$

and, for $1 < s < q'$,

$$A_W^*(x,s) = \left(\int_0^x v^{1-p'}(t)\left(\int_t^b u(y)dy\right)^{p'(q'-s)/q'} dt\right)^{1/p'} \left(\int_x^b u(t)dt\right)^{(s-1)/q'},$$

respectively. We also put

$$\text{(2.13)} \qquad \begin{cases} A_5 := \sup\limits_{0<x\le b} A_T^*(x), \\ A_6 := \inf\limits_{g(x)>0} \sup\limits_{0<x\le b} A_B^*(x,g), \\ A_7(s) := \sup\limits_{0<x\le b} A_W^*(x,s), 1 < s < q'. \end{cases}$$

With these notations and in view of the duality principle we conclude that the Hardy inequality (2.1) holds if and only if any of the conditions $A_5 < \infty$, $A_6 < \infty$ or $A_7(s) < \infty$, $1 < s < q'$, is satisfied. Moreover, we have the following estimates:

$$l_5 \le \frac{C}{A_5} \le u_5,$$

with

$$\text{(2.14)} \qquad l_5 = l_5(p,q) = 1, \qquad u_5 = u_5(p,q) = q;$$

furthermore

$$l_6 \le \frac{C}{A_6} \le u_6,$$

with

$$\text{(2.15)} \qquad l_6 = l_6(p,q) = \frac{(q/p')^{1/p'}}{(1+\frac{q}{p'})^{1/q}(1+\frac{p'}{q})^{1/p'}}, \qquad u_6 = u_6(p,q) = \left(\frac{q}{p'}\right)^{1/p'},$$

and, for $1 < s < q'$,

$$l_7(s) \le \frac{C}{A_7(s)} \le u_7(s),$$

with

$$\text{(2.16)} \qquad \begin{aligned} l_7(s) = l_7(s,p,q) &= \left(\frac{q'}{q'-s}\right)\left(\left(\frac{q'}{q'-s}\right)^{q'} + \frac{1}{s-1}\right)^{-1/q'}, \\ u_7(s) = u_7(s,p,q) &= \left(\frac{q'-1}{q'-s}\right)^{1/q}. \end{aligned}$$

Summing up the investigations we are now ready to formulate the following improvement of the usual characterization of Hardy's inequality mentioned in the introduction as the following seven conditions Theorem:

THEOREM 2. *Let* $1 < p \le q < \infty$. *Then*

$$\left(\int_0^b \left(\int_0^x f(t)dt \right)^q u(x)dx \right)^{1/q} \le C \left(\int_0^b f^p(x)v(x)dx \right)^{1/p} \tag{2.17}$$

holds for all measurable functions $f \ge 0$ *if and only if any of the (mutually equivalent) numbers* $A_1, A_2, A_3, A_4(s)$, $1 < s < p, A_5, A_6$ *or* $A_7(r)$, $1 < r < q'$, *is finite. Moreover, the best constant* C *in (2.17) can be estimated as follows:*

$$\sup_i A_i l_i \le C \le \inf_i A_i u_i,$$

where A_i *are defined by (2.5) and (2.13)* $[A_4 = A_4(s),\ A_7 = A_7(r)]$, l_i *and* u_i *are defined by (2.8)-(2.11) and (2.14)-(2.16)* $[l_4 = l_4(s),\ l_7 = l_7(r),\ u_4 = u_4(s), u_7 = u_7(r)]$ *and the infimum is taken over* $i = 1, 2, ..., 7$, $1 < s < p$ *and* $1 < r < q'$.

REMARK 3. Analogously to Remark 2 we note that

$$A_W^*(x, q') = A_M(x).$$

Moreover, since the function $U(t) = \int_t^b u(s)ds$ is decreasing, we find as in Remark 2 that $A_W^*(x, s)$ is nonincreasing in s on $(1, q']$ so that in particular

$$A_W^*(x, s) \ge A_M(x), \qquad 1 < s \le q'.$$

3. Some comparisons of the conditions and compactness results. First we note that according to our Theorem 2 the (best) constant C in (1.1) satisfies $C \approx A_i$ for $i = 1, 2, ..., 7$, so that we can easily estimate any of the constants A_i with help of any other A_j, $j \ne i$. However, it would be useful to estimate mutually not only the suprema of the functions involved, but also be able to compare the functions themselves. One reason for this claim is based on the fact that the mapping (2.12) is continuous due to the Hardy inequality

$$\left(\int_0^b |(Hf)(x)|^q u(x)dx \right)^{1/q} \le C \left(\int_0^b |f(x)|^p v(x)dx \right)^{1/p}$$

if and only if any of the numbers A_i is finite and is, moreover, compact if and only if in addition

$$\lim_{x \to 0+} A_M(x) = 0, \qquad \lim_{x \to b-} A_M(x) = 0, \tag{3.1}$$

see e.g. [11, Theorem 7.3]. This fact reflects the question whether or not also the other "defining" functions corresponding to some of the constants A_i characterizing the Hardy inequality (1.1) have some additional properties of the type (3.1) which guaranties also the compactness of the Hardy operator between weighted Lebesgue spaces. One situation which directly gives a positive result is when the defining functions are equivalent but this is not always true. For example we know that $A_1 < \infty$ is equivalent to that $A_2 < \infty$

but it does not in general hold that

$$A_M(x) \approx A_T(x),$$

i.e., that there exist positive constants C_1, C_2 such that

$$C_1 A_M(x) \le A_T(x) \le C_2 A_M(x). \tag{3.2}$$

EXAMPLE 1. Let us take $u(t) = t^\alpha$, $v(t) = t^\beta$, $\alpha, \beta \in \mathbb{R}$. The condition $V(x) = \int_0^x v^{1-p'}(t)dt < \infty$ leads to the restriction

$$\beta < p - 1.$$

(i) Let us first consider the case $b = \infty$, i.e. investigate the validity of (1.1) on the interval $(0, \infty)$. Then all functions $A_M(x)$, $A_T(x)$, $A_W(x, s)$ and also $A_B(x; g)$ for $g = V$ are of the form

$$A_i(x) = C_i x^{\alpha + 1/q - \frac{\beta+1}{p} + 1};$$

for simplicity, we omit the s and put $V = g$ in $A_W(x, s)$, $A_B(x, g)$ writing simply $A_i(x)$, with some positive constants C_i. Consequently, the numbers $A_i := \sup x > 0 A_i(x)$ are finite if and only if the pair $\{\alpha, \beta\}$ satisfies

$$\frac{\alpha+1}{q} - \frac{\beta+1}{p} + 1 = 0.$$

For this pair, inequality (1.1) holds and the mapping (2.12) is continuous, but cannot be compact since the conditions (3.1) cannot be satisfied.

(ii) Let us now consider the interval $(0, b)$ with $b < \infty$, say $b = 1$. Then we have again

$$A_T(x) = C_T x^\lambda, \quad \lambda = \frac{\alpha+1}{q} - \frac{\beta+1}{p} + 1$$

and the mapping H will be continuous provided

$$\frac{\alpha+1}{q} - \frac{\beta+1}{p} + 1 \ge 0.$$

On the other hand,

$$A_M(x) = C_M \left(\frac{1 - x^{\alpha+1}}{\alpha+1} \right)^{1/q} x^{-(\beta+1)/p+1}$$

with $\alpha > -1$ and $\beta < p - 1$. Consequently, the mapping H will be continuous if (3.2) holds, and moreover, it will be compact since $A_M(0) = A_M(1) = 0$. But a comparison of the formulas $A_M(x)$ and $A_T(x)$ shows that we never can expect that it could be $A_T(x) \approx A_M(x)$ for $x \in (0, 1)$ due to the behavior in the neighborhood of $x = 1$ where $A_M(x)$ vanishes but $A_T(1) = C_T > 0$.

Next we note that according to Remarks 2 and 3 we have only the estimates

$$A_W(x, s) \ge A_M(x) \text{ and } A_W^*(x, s) \ge A_M(x) \tag{3.3}$$

so we cannot use (3.1) to directly obtain a similar compactness result by using the functions $A_W(x, s)$ or $A_W^*(x, s)$. However, the following compactness result hold:

THEOREM 3. *Let $1 < p \le q < \infty$, $1 < s < p$ and let u and v be weight functions on $(0, b)$. Moreover, let*

$$A_W(x,s) := \left(\int_x^b u(t)V^{q(p-s)/p}(t)dt\right)^{1/q} V^{(s-1)/p}(x),$$

and denote

$$A_4(s) := \sup_{x\in(0,b)} A_W(x,s).$$

Then the Hardy operator (2.12) is compact if and only if

$$\lim_{x\to 0+} A_W(x,s) = 0, \quad \lim_{x\to b-} A_W(x,s) = 0.$$

Before we give the proof we formulate the following assertions whose proofs can be found e.g. in N. Dunford and J.T. Schwartz [5]:

(A) Let $1 \le p < \infty$. The sequence $\{u_n\} \subset L^p(a,b)$ converges weakly to $u \in L^p(a,b)$ if and only if the following two conditions are fulfilled:

(α) $\sup_n \|u_n\|_{L^p(a,b)} < \infty$;

(β) $\int_M u_n(t)dt \to \int_M u(t)dt$ for every measurable subset $M \subset (a,b)$.

(B) Let T: $X \to Y$. $\{u_n\} \subset X$, u_n converges weakly to u. Then Tu_n converges strongly to Tu.

(C) Let T: $X \to Y$. Then T is compact if and only if $\widetilde{T} : \widetilde{Y} \to \widetilde{X}$, with $\widetilde{T}$ the conjugate operator of T and $\widetilde{X}, \widetilde{Y}$ duals of X, Y, is compact.

Proof. I) Assume that H is compact. Then (3.1) is true and it follows that $A_1 < \infty$.

Now,

$$\begin{aligned}
A_W(x,s) &= \left(\int_x^b u(t)V^{q(p-s)/p}(t)dt\right)^{1/q} V^{(s-1)/p}(x) \\
&= \left(\int_x^b u(t)A_M^{q\left(\frac{p-s}{p-1}\right)}(t)\left(\int_t^b u(y)dy\right)^{-\left(\frac{p-s}{p-1}\right)} dt\right)^{1/q} V^{(s-1)/p}(x) \\
&\le A_1^{\frac{p-s}{p-1}}\left(\int_x^b u(t)\left(\int_t^b u(y)dy\right)^{-\left(\frac{p-s}{p-1}\right)} dt\right)^{1/q} V^{(s-1)/p}(x) \\
&= \left(\frac{p-1}{s-1}\right)^{1/q} A_1^{\frac{p-s}{p-1}}\left(\int_x^b u(t)dt\right)^{(s-1)/q(p-1)} V^{(s-1)/p}(x) \\
&= \left(\frac{p-1}{s-1}\right)^{1/q} A_1^{\frac{p-s}{p-1}}\left(\left(\int_x^b u(t)dt\right)^{1/q} V^{1/p'}(x)\right)^{(s-1)/(p-1)} \\
&= \left(\frac{p-1}{s-1}\right)^{1/q} A_1^{\frac{p-s}{p-1}} A_M^{(s-1)/(p-1)}(x).
\end{aligned}$$

So in view of (3.1) it follows that $\lim_{x\to 0+} A_W(x,s) = 0$ and $\lim_{x\to b-} A_W(x,s) = 0$.

II) Now assume that $A_4(s) < \infty$ and

$$\lim_{x\to 0+} A_W(x,s) = 0, \quad \lim_{x\to b-} A_W(x,s) = 0.$$

According to Remark 2 it yields that $A_W(x,s) \ge A_M(x)$ for $0 \le x \le b$. This implies that (3.1) is satisfied and we conclude that the operator $H : L^p(0,b;v) \to L^q(0,b;u)$ is compact. The proof is complete.

Finally, we state that a similar result holds also with $A_4(s)$ replaced by $A_7(s)$ and correspondingly $A_W(x,s)$ replaced by $A^*_W(x,s)$.

THEOREM 4. *Let* $1 < p \le q < \infty$, $1 < s < q'$ *and let* u *and* v *be weight functions on* $(0,b)$. *Moreover, let*

$$A^*_W(x,s) = \left(\int_0^x v^{1-p'}(t)\left(\int_t^b u(s)ds\right)^{p'(q'-s)/q'} dt\right)^{1/p'} \left(\int_x^b u(t)dt\right)^{(s-1)/q'},$$

where

$$U(t) = \int_t^b u(s)dx,$$

and denote

$$A_7(s) := \sup_{0<x\le b} A^*_W(x,s).$$

Then the Hardy operator (2.12) is compact if and only if

$$\lim_{x\to 0+} A^*_W(x,s) = 0, \quad \lim_{x\to b-} A^*_W(x,s) = 0.$$

Proof. The proof is completely similar to that of Theorem 3; this time we just use the relation $A^*_W(x,s) \ge A_M(x)$ pointed out in Remark 3 and the following observation:

$$\begin{aligned}
A^*_W(x,s) &= \left(\int_0^x v^{1-p'}(t)\left(\int_t^b u(y)dy\right)^{p'(q'-s)/q'} dt\right)^{1/p'} \left(\int_x^b u(t)dt\right)^{(s-1)/q'} \\
&= \left(\int_0^x v^{1-p'}(t) A_M^{p'(q'-s)(q-1)}(t) V^{-q(q'-s)/q'}(t)dt\right)^{1/p'} \left(\int_x^b u(t)dt\right)^{(s-1)/q'} \\
&\le A_1^{(q'-s)(q-1)}\left(\int_0^x v^{1-p'}(t) V^{-q(q'-s)/q'}(t)dt\right)^{1/p'} \left(\int_x^b u(t)dt\right)^{(s-1)/q'} \\
&= \left(\frac{1}{(s-1)(q-1)}\right)^{1/p'} A_1^{(q'-s)(q-1)} V(x)^{(s-1)(q-1)/p'}(t)\left(\int_x^b u(t)dt\right)^{(s-1)/q'} \\
&= \left(\frac{1}{(s-1)(q-1)}\right)^{1/p'} A_1^{(q'-s)(q-1)} \left(V(x)^{1/p'}(t)\left(\int_x^b u(t)dt\right)^{1/q}\right)^{(s-1)(q-1)} \\
&= \left(\frac{1}{(s-1)(q-1)}\right)^{1/p'} A_1^{(q'-s)(q-1)} A_M^{(s-1)(q-1)}(x).
\end{aligned}$$

4. Concluding remarks and open questions

REMARK 4. The reason to introduce the constant A_2 in [14] for characterizing (1.1) was the idea to find, via a limiting procedure, a weight characterization for the weighted Pólya-Knopp inequality

$$\left(\int_0^b \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)^q u(t)dt\right)^{1/q} \le C\left(\int_0^b f(x)^p v(x)dx\right) szczyl^{1/p}. \tag{4.1}$$

The problem which appears when we try to solve the same problem by using the usual Muckenhoupt condition (1.2) is described in detail in [17] and necessary and sufficient conditions could only be proved in the power weight case (cf. also [8]).

REMARK 5. The new scale of conditions $A_4(s) < \infty$ introduced in [17] for $b = \infty$ for characterizing (1.1) has the same property as A_2 concerning the limiting procedure described in Remark 4 and we end up with another weight characterization of (4.1). Actually, $A_4(s)$ are constructed in such a way that the same characterization as that by B. Opic and P. Gurka [11] is obtained (but with a somewhat better lower bound for C). For details see [17].

REMARK 6. Let us note that some analogues to the "dual" criteria pointed out in Section 2 appear also in slightly modified forms in papers by other authors, e.g. by G. Bennett [1] for the discrete analogue of Hardy's inequality, by E. Sawyer [15] for the two-dimensional Hardy inequality (see (4.2) and Question 3 below). Also in S. Bloom and R. Kerman [3] we can find a criterion corresponding to $A_T^*(x)$.

In the previous sections we have given some relations between the constants A_i (and the corresponding defining functions) which all characterizes the Hardy inequality (1.1) but obviously many open questions remains to be solved. Here we just point out the following:

QUESTION 1. Is it possible to find a fixed constant C so that

$$A_W(x,s) \le CA_M(x) \quad \text{or} \quad A_W^*(x,s) \le CA_M(x),$$

i.e. $A_W(x,s) \approx A_M(x)$, $1 < s < p$, or $A_W^*(x,s) \approx A_M(x)$, $1 < s < q'$ (see (3.3))? More generally, is it possible to estimate some of the defining functions $A_i(x)$ from above and from below by some other function $A_j(x)$, $j \ne i$?

QUESTION 2. Our Example 2.1 shows that the function $A_T(x)$ cannot satisfy a condition like (3.1) so the question is whether it is possible to characterize compactness of the Hardy operator in terms of A_2 and its corresponding defining function $A_T(x)$.

In view of the new explanations and complements of the remarkable E. Sawyer result [15] for the two-dimensional Hardy operator H_2 :

$$H_2 f(x_1,x_2) := \int_0^{x_1}\int_0^{x_2} f(y_1,y_2)dy_1dy_2, \tag{4.2}$$

which was presented in [17] we suggest that some similar results as those presented here can also hold in some two-dimensional cases. We only pose the following:

QUESTION 3. Is it possible to characterize the compactness of the mapping

$$H_2 : L^p([0,b_1]\times[0,b_2];v) \to L^q([0,b_1]\times[0,b_2];u),$$

where H_2 is defined by (4.2), in a similar way as in the one-dimensional case (see Theorem 7.3 in [12] and (3.1))? According to the result in [17] it seems to be easier to solve this question when v is of product type i.e. $v(x_1,x_2) = v_1(x_1)v_2(x_2)$.

FINAL REMARK. Some complementary result concerning necessary and sufficient conditions for the weighted Hardy inequality for the case $1 < p \le q < \infty$ can also be found in the paper [6].

Finally we remark that this paper coincides with a talk given by the second named author at the Władysław Orlicz Centenary Conference and Function Spaces VII in Poznań,

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