

THE REVIVAL OF THE RIEMANNIAN APPROACH TO INTEGRATION

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Riemannian sums were used in a definition of a nonabsolutely convergent integration on \mathbb{R} almost fifty years ago and it was proved that the new integration is equivalent to Perron integration. Therefore it looked like there appeared just a new approach to integration. But the Riemannian approach proved to be very flexible: (i) it is the basis of numerous integrations of real functions of a real variable, (ii) it opens new ways in integration of functions of several variables, (iii) it is well applicable to integration of vector valued functions, (iv) it admits a great degree of abstraction. This paper is concentrated on (i) with a special accent on structures (convergence, topology) on the space of integrable functions. Section 5 contains a brief information concerning (ii).

1. Preliminaries. Let $I = [a, b] \subset \mathbb{R}$ and let $|E|$ denote the Lebesgue measure of $E \subset I$. A set $A \subset I$ is a *figure* if it is a finite union of compact intervals.

A *partition in I* is a set $\Delta = \{(t_1, A_1), \dots, (t_k, A_k)\}$ where $t_i \in I, A_i$ is a figure and $|A_i \cap A_j| = 0$ for $i \neq j$. If in addition $\bigcup A_i = I$ then Δ is a *partition of I*. Let $\xi : I \rightarrow (0, \infty)$. Then Δ is called *ξ -fine* if $A_i \subset (t_i - \xi(t_i), t_i + \xi(t_i))$ for $i = 1, 2, \dots, k$.

If $H : I \rightarrow \mathbb{R}, A = [c_1, d_1] \cup \dots \cup [c_k, d_k]$ where $a \leq c_1 < d_1 \leq \dots \leq c_k < d_k \leq b$, we put

$$H(A) = \sum_{j=1}^k (H(d_j) - H(c_j)).$$

2. \mathcal{Y} -integration. Let \mathcal{L} be the set of partitions in I , and let \mathcal{HK} be the set of partitions $\theta = \{(s_1, J_1), \dots, (s_l, J_l)\}$ in I such that $s_m \in J_m$ and J_m is an interval for $m = 1, 2, \dots, l$. A set \mathcal{Y} of partitions Δ in I is called an *integration base* if $\mathcal{HK} \subset \mathcal{Y}$, if $\Delta \in \mathcal{Y}$ and $\Psi \subset \Delta$ implies $\Psi \in \mathcal{Y}$ and if $\Delta \cup \theta \in \mathcal{Y}$ for $\Delta \in \mathcal{Y}, \theta \in \mathcal{HK}$ where $|A_i \cap J_m| = 0$ for all i, m .

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2.1. DEFINITION. Let $f : I \rightarrow \mathbb{R}$, $w \in \mathbb{R}$. Then f is called \mathcal{Y} -integrable on I and w is called the \mathcal{Y} -integral of f over I ($w = (\mathcal{Y}) \int_I f dt$) if there exists $\delta : \mathbb{N} \times I \rightarrow (0, \infty)$ such that

$$\left| w - \sum_i f(t_i) |A_i| \right| \leq 2^{-j} \quad \text{for } j \in \mathbb{N}, \Delta \in \mathcal{Y},$$

Δ being a $\delta(j, \cdot)$ -fine partition of I .

2.2. NOTE. Definition 2.1 with $\mathcal{Y} = \mathcal{HK}$ was introduced in [Kur 57] and it was proved that \mathcal{HK} -integration is Perron integration (see also [He 61], [He 63]). The motivation came from a convergence problem in the theory of ordinary differential equations (see [Kur 57] or [Kur]). Riemannian approach to Lebesgue integration appeared in [Mc Sh 69], other integrations were introduced in [Bo-Pf 92] and [Bo 96]. For a survey of results on the Riemannian approach to integration see also [Bo 03].

2.3. Elementary properties

(2.1) w is unique since for every $\xi : I \rightarrow (0, \infty)$ there exists a ξ -fine partition Δ of I , $\Delta \in \mathcal{HK}$.

(2.2) As a direct consequence of $\mathcal{HK} \subset \mathcal{Y}$ we get: if f is \mathcal{Y} -integrable over I , then f is \mathcal{HK} -integrable over I and $(\mathcal{Y}) \int_I f dt = (\mathcal{HK}) \int_I f dt$.

(2.3) If f is \mathcal{Y} -integrable over I then it is \mathcal{Y} -integrable over every subinterval of I .

(2.4) Let f be \mathcal{Y} -integrable over I . Set $F(a) = 0, F(s) = (\mathcal{Y}) \int_{[a,s]} f dt$ for $a < s \leq b$; F is called the primitive of f .

F is continuous and differentiable a.e., $\dot{F} = f$ a.e.

$P_{\mathcal{Y}}$ is the set of primitives of functions which are \mathcal{Y} -integrable over I .

(2.5) $P_{\mathcal{L}} \subset P_{\mathcal{Y}} \subset P_{\mathcal{HK}}$ (since $\mathcal{HK} \subset \mathcal{Y} \subset \mathcal{L}$).

(2.6) $P_{\mathcal{L}}$ is the set of absolutely continuous functions $F : I \rightarrow \mathbb{R}, F(a) = 0$ (cf. [Mc Sh 69]).

$P_{\mathcal{HK}}$ is the set of ACG*-functions (since Perron integration is equivalent to Denjoy integration in the restricted sense, cf. [Sa 37]).

The emphasis will be put on the concept of the primitive (in order to avoid classes of equivalent functions).

2.4. REMARK. A fundamentally different characterization of $P_{\mathcal{HK}}$ was given in [B-P-S 95]:

Let $F : I \rightarrow \mathbb{R}, F(a) = 0$. Then $F \in P_{\mathcal{HK}}$ if and only if F generates an absolutely continuous variational measure. See also [Pf 99].

3. Convergence in $P_{\mathcal{Y}}$. Let $\delta : \mathbb{N} \times I \rightarrow (0, \infty), \xi : I \rightarrow (0, \infty)$. By $Q^*(\mathcal{Y}, \delta, \xi)$ denote the set of $F : I \rightarrow \mathbb{R}$ such that there exist $f : I \rightarrow \mathbb{R}$ and $N \subset I, |N| = 0$ fulfilling

$$(3.1) \quad \sum_i |F(A_i)| \leq 2^{-j} + \sum_i \xi(t_i) |A_i|$$

for $j \in \mathbb{N}$ and every $\delta(j, \cdot)$ -fine $\Delta = \{(t_i, A_i)\} \in \mathcal{Y}$,

$$(3.2) \quad \sum_i |F(A_i) - f(t_i) |A_i|| \leq 2^{-j}$$

for $j \in \mathbb{N}$ and every $\delta(j, \cdot)$ -fine $\Delta = \{(t_i, A_i)\} \in \mathcal{Y}$ where $t_i \in I \setminus N$ for all i . Then

$$(3.3) \quad P_{\mathcal{Y}} = \bigcup_{\delta, \xi} Q^*(\mathcal{Y}, \delta, \xi),$$

(3.4) if $F_p \in Q^*(\mathcal{Y}, \delta, \xi)$ for $p \in \mathbb{N}$ and if $F : I \rightarrow \mathbb{R}$ is bounded, $\|F_p - F\|_{\text{sup}} \rightarrow 0$ for $p \rightarrow \infty$ then $F \in P_{\mathcal{Y}}$ and $f_p \rightarrow f$ in measure for $p \rightarrow \infty$, where F_p and F are the primitives of f_p, f .

The definition of $Q^*(\mathcal{Y}, \delta, \xi)$ differs only slightly from the definition of $Q_{\mathcal{Y}}(\delta)$ in [Kur 02], Definition 2.6. The proofs of (3.3) and (3.4) are analogous to the proofs in [Kur 02], Theorems 2.9 and 2.10. (3.3) may be interpreted as an equivalent definition of $P_{\mathcal{Y}}$ and (3.4) implies that $Q^*(\mathcal{Y}, \delta, \xi)$ is closed in the space of continuous functions; in fact, it is compact.

3.1. DEFINITION. Let $F_p \in P_{\mathcal{Y}}$ for $p \in \mathbb{N}$, $F \in P_{\mathcal{Y}}$. The sequence F_p is called Q_y^* -convergent to F , $F_p \xrightarrow{Q_y^*} F$, if there exist δ and ξ such that

$$(3.5) \quad F_p \in Q^*(\mathcal{Y}, \delta, \xi) \quad \text{for } p \in \mathbb{N}$$

and

$$(3.6) \quad \|F_p - F\|_{\text{sup}} \rightarrow 0 \quad \text{for } p \rightarrow \infty.$$

Q_y^* -convergence resembles the two-norm convergence (cf. [Al 50], [W 61]).

3.2. REMARK. Convergence on $P_{\mathcal{Y}}$ can be introduced in another way. Call the sequence F_p \mathbb{E}_y -convergent to F , $F_p \xrightarrow{\mathbb{E}_y} F$, if there exist $f_p, f : I \rightarrow \mathbb{R}$ such that

(3.7) there exists $\delta : \mathbb{N} \times I \rightarrow (0, \infty)$ such that

$$\left| F_p(I) - \sum_i f_p(t_i) |A_i| \right| \leq 2^{-j}$$

for $p, j \in \mathbb{N}$, $\Delta = \{(t_i, A_i)\} \in \mathcal{Y}$, Δ being a $\delta(j, \cdot)$ -fine partition of I ;

(3.8) $f_p(t) \rightarrow f(t)$ for $p \rightarrow \infty$, $t \in I$.

3.3. REMARK. $F_p \xrightarrow{\mathbb{E}_y} F$ implies that $F_p \xrightarrow{Q_y^*} F$; on the other hand if $F_p \xrightarrow{Q_y^*} F$ then there exists a subsequence $q(p)$ such that $F_{q(p)} \xrightarrow{\mathbb{E}_y} F$.

3.4. REMARK. $F : I \rightarrow \mathbb{R}$ is an ACG_* -function if there exists a sequence of sets $E_i \subset I$ such that F in relation to each E_i has a property which resembles absolute continuity. In this way a descriptive characterization of functions from $P_{\mathcal{HK}}$ is obtained. The concept of controlled convergence has the same roots (cf. [D 51], [Ch-D 78], [L 89]). In general it can be said that controlled convergence is equivalent to $\mathbb{E}_{\mathcal{HK}}$ -convergence. For a general approach see [Bo-Pi 91]. There is treated a more general case of integration of f with respect to a function $G : I \rightarrow \mathbb{R}$, the most important results being valid if $G \in P_{\mathcal{HK}}$. See also [G 91] and [G 92].

4. Topologization of $P_{\mathcal{Y}}$. Let $\mathcal{T}(\mathcal{Y})$ be the set of locally convex topologies τ on $P_{\mathcal{Y}}$ such that

$$F_p \xrightarrow{Q_y^*} F \text{ implies } F_p \rightarrow F \text{ in } (P_{\mathcal{Y}}, \tau).$$

Since $\mathcal{T}(\mathcal{Y}) \neq \emptyset$ there exists a unique locally convex topology $\mathcal{U}_{LC}(\mathbb{Q}_y^*) \in \mathcal{T}(\mathcal{Y})$ which is finer than any topology $\tau \in \mathcal{T}(\mathcal{Y})$. If $\mathcal{U}_{LC}(\mathbb{E}_y)$ has an analogous meaning then $\mathcal{U}_{LC}(\mathbb{E}_y) = \mathcal{U}_{LC}(\mathbb{Q}_y^*)$, cf. Remark 3.3.

4.1. THEOREM. $\mathcal{U}_{LC}(\mathbb{E}_{\mathcal{L}})$ is the topology induced on $P_{\mathcal{L}}$ by the norm $\|\cdot\|_{\text{var}}$ where $\|F\|_{\text{var}} = \text{var } F$ (cf. [Kur 02], Theorems 5.2 and 5.3).

4.2. THEOREM. $\mathcal{U}_{LC}(\mathbb{E}_{\mathcal{HK}})$ is the topology induced on $P_{\mathcal{HK}}$ by the norm $\|\cdot\|_{\text{sup}}$ (cf. [Kur 02], Theorem 8.2 and Note 8.1 (ii)).

Observe that $(P_{\mathcal{L}}, \mathcal{U}_{LC}(\mathbb{E}_{\mathcal{L}}))$ is complete and $(P_{\mathcal{HK}}, \mathcal{U}_{LC}(\mathbb{E}_{\mathcal{HK}}))$ is not complete. It is desirable to get sufficient conditions for the completeness or noncompleteness of $(P_{\mathcal{Y}}, \mathcal{U}_{LC}(\mathbb{E}_y))$. A theoretical approach to this problem was obtained in [Kur 02], Chapters 4 and 7 for a rather wide class of integration bases \mathcal{Y} but only some concrete results will be described here.

4.3. NOTATION. Let Λ be the set of $\lambda : [0, \infty) \rightarrow [0, \infty)$ nondecreasing, $\lambda(\sigma) > 0$ for $\sigma > 0$. Denote by $\text{dist}(t, E)$ the distance of t from E where $t \in I$, $E \subset I$.

For $\lambda \in \Lambda$ let $\mathcal{S}(\lambda)$ be the set of partitions $\Delta = \{(t_i, A_i)\} \in \mathcal{L}$ such that

$$(4.1) \quad \text{if } (t_i, A_i) \in \Delta \text{ then } A_i \text{ is an interval,}$$

$$(4.2) \quad \sum_i \lambda(\text{dist}(t_i, A_i)) \leq 1.$$

4.4. THEOREM. Let $\lambda \in \Lambda$. Then $(P_{\mathcal{S}(\lambda)}, \mathcal{U}_{LC}(\mathbb{E}_{\mathcal{S}(\lambda)}))$ is not complete (cf. Remark 3.2 and [Kur 02], Theorem 8.11).

4.5. REMARK. Put $\lambda(i, \sigma) = 2^{-i}\sigma$ for $\sigma \geq 0$, $i \in \mathbb{N}$. In [Bo 96], $*$ -integration was defined. It is a direct consequence of the definition that F is the primitive of a $*$ -integrable f if and only if

$$(4.3) \quad F \in \bigcap_i P_{\mathcal{S}(\lambda(i, \cdot))}.$$

Moreover, F is the primitive of a $*$ -integrable f if and only if there exist an everywhere differentiable G and an absolutely continuous H such that

$$(4.4) \quad F = G + H$$

(cf. [B-P-P 00]).

Let $\lambda, \eta \in \Lambda$, $\lambda(\sigma)/\eta(\sigma) \rightarrow 0$ for $\sigma \rightarrow 0$. Then

$$(4.5) \quad P_{\mathcal{S}(\eta)} \subset P_{\mathcal{S}(\lambda)}, \quad P_{\mathcal{S}(\lambda)} \setminus P_{\mathcal{S}(\eta)} \neq \emptyset \text{ and } \mathcal{U}_{LC}(\mathbb{E}_{\mathcal{S}(\eta)}) \text{ is finer than the restriction } \mathcal{U}_{LC}(\mathbb{E}_{\mathcal{S}(\lambda)})|_{P_{\mathcal{S}(\eta)}}$$

(cf. [Kur 02], Theorem 8.11). It can also be proved that

$$(4.6) \quad \bigcap_{\lambda \in \Lambda} P_{\mathcal{S}(\lambda)} = P_{\mathcal{L}}.$$

Therefore $P_{\mathcal{S}(\lambda)}$ seems to be rather close to $P_{\mathcal{L}}$ if $\lambda \rightarrow 0$ rapidly but the results on completeness are different.

4.6. NOTATION. The *regularity* of a couple (t, A) where $t \in I$ and A is a figure is defined by

$$(4.7) \quad \text{reg}(t, A) = \frac{|A|}{\#\partial A \cdot \text{diam}(\{t\} \cup A)}$$

where $\#\partial A$ is the number of boundary points of A and $\text{diam } E$ is the diameter of E . (We have $0 < \text{reg}(t, A) \leq \frac{1}{2}$ for every couple (t, A) .)

Let Ω be the set of nondecreasing functions $\omega : (0, \infty) \rightarrow (0, \frac{1}{2}]$. For $\omega \in \Omega$ let $\mathcal{R}(\omega)$ be the set of partitions $\Delta \in \mathcal{L}$ such that

$$(4.8) \quad \text{reg}(t, A) \geq \omega(\text{diam}(\{t\} \cup A)) \quad \text{for } (t, A) \in \Delta$$

and let $\mathcal{R}(\omega, 1)$ be the set of partitions $\Delta \in \mathcal{R}(\omega)$ such that

$$(4.9) \quad t \in A \quad \text{for } (t, A) \in \Delta.$$

4.7. THEOREM. *Neither $(P_{\mathcal{R}(\omega)}, \mathcal{U}_{LC}(\mathbb{E}_{\mathcal{R}(\omega)}))$ nor $(P_{\mathcal{R}(\omega, 1)}, \mathcal{U}_{LC}(\mathbb{E}_{\mathcal{R}(\omega, 1)}))$ is complete (cf. [Kur 02], Theorem 9.2, Lemma 9.4, Note 9.5).*

4.8. REMARK. $(P_{\mathcal{HK}}, \mathcal{U}_{LC}(\mathbb{E}_{\mathcal{HK}}))$ is not complete. On the other hand there exists a topology τ on $P_{\mathcal{HK}}$ such that

$$(4.10) \quad (P_{\mathcal{HK}}, \tau) \text{ is a complete topological vector space,}$$

$$(4.11) \quad \text{if } F_p \xrightarrow{\mathbb{E}_{HK}} F \text{ then } F_p \rightarrow F \text{ in } (P_{\mathcal{HK}}, \tau)$$

(cf. (2.6) and [Kur 00], Theorem 8.1).

4.9. REMARK. Let P^Δ be the set of primitives with respect to the integration on one-dimensional intervals which was introduced in [Bo-Pf 92]. We have

$$(4.12) \quad P^\Delta = \bigcap_j P_{\mathcal{R}(\omega_j, 1)}$$

where $\omega_j(\sigma) = 2^{-j}$ for $j \in \mathbb{N}$, $\sigma > 0$.

Let $F_p, F \in P^\Delta$ for $p \in \mathbb{N}$. The sequence F_p is called \mathbb{E}^Δ -convergent to F , $F_p \xrightarrow{\mathbb{E}^\Delta} F$, if $F_p \xrightarrow{\mathbb{E}_{\mathcal{R}(\omega_j, 1)}} F$ for every $j \in \mathbb{N}$. The $\mathcal{U}_{LC}(\mathbb{E}^\Delta)$ is defined in the same way as $\mathcal{U}_{LC}(\mathbb{E}_y)$ in Theorem 3.1. It can be proved by methods from [Kur 02], Ch. 10 that $(P^\Delta, \mathcal{U}_{LC}(\mathbb{E}^\Delta))$ is not complete (this result may differ from Theorem 10.6, Remark 1 [Kur 02] since the topologies $\mathcal{U}_{LC}(\mathbb{E}^\Delta)$ and $\mathcal{U}_\chi(\mathcal{R}^*(1))$ from the theorem mentioned above need not coincide).

Theorems 4.4 and 4.7 evoke the following question: Does there exist an integration base \mathcal{Y} such that $P_{\mathcal{Y}} \neq P_{\mathcal{L}}$ and $(P_{\mathcal{Y}}, \mathcal{U}_{LC}(\mathbb{Q}_y^*))$ is complete?

The answer is affirmative.

Let $\mathcal{M}(\{a\})$ be the set of partitions $\Delta = \{(t_i, A_i)\} \in \mathcal{L}$ such that

$$(4.13) \quad \text{if } t_i = a \text{ then } A_i = [a, s], a < s \leq b.$$

4.10. THEOREM. *$P_{\mathcal{M}(\{a\})}$ is the set of $F : I \rightarrow \mathbb{R}$, $F(a) = 0$ such that F is continuous and the restriction $F|_{[s, b]}$ is absolutely continuous for $s \in (a, b)$. The topology $\mathcal{U}_{LC}(\mathbb{E}_{\mathcal{M}(\{a\})})$ is induced by the set of seminorms $\{\|\cdot\|_{\text{sup}}, \|\cdot\|_s^*; s \in (a, b)\}$ where $\|F\|_s^*$ is the variation of F on $[s, b]$. Moreover, $(P_{\mathcal{M}(\{a\})}, \mathcal{U}_{LC}(\mathbb{E}_{\mathcal{M}(\{a\})}))$ is complete.*

4.11. REMARK. Let $S \subset I$ be countable and closed. $\mathcal{M}(S)$ can be defined analogously to $\mathcal{M}(\{a\})$ and the above result can be extended to the integration base $\mathcal{M}(S)$ (cf. [Kur 00], Chapter 6).

4.12. REMARK. Nonabsolute integration was approached by the method of truncation (cf. [Ti 29], [O 54], [U 56], [A-A 65]). For the concept of a ranked space and its application to nonabsolute integration see [Kun 54], [Kun 56], [N 78,79], [N 90] and for numerous results which include the completion of st with respect to a quasinorm and the ranked completion of st see [N 01]. Here st is the set of step functions.

5. A short excursion to integration in \mathbb{R}^k . \mathcal{HK} -integration in \mathbb{R}^1 has the following two properties:

(5.1) If F is differentiable at every $t \in I$ then

$$(\mathcal{HK}) \int_I \dot{F} dt = F(b) - F(a).$$

(5.2) The classical transformation formula is valid for diffeomorphisms.

Thus \mathcal{HK} -integration on intervals of dimension 1 can be transferred to integration on curves.

Let now $k > 1$,

$$(5.3) \quad I = [a_1, b_1] \times \cdots \times [a_k, b_k], \quad J = [c_1, d_1] \times \cdots \times [c_k, d_k]$$

where $a_i \leq c_i < d_i \leq b_i$, $i = 1, 2, \dots, k$,

$$(5.4) \quad \text{reg } J = \frac{\min\{d_i - c_i; i = 1, 2, \dots, k\}}{\max\{d_i - c_i; i = 1, 2, \dots, k\}}.$$

(A) If in the definition of the \mathcal{HK} -integral I and J are interpreted as in (5.3) then the resulting integration is equivalent to Perron integration (cf. e.g. [Kur 80]). An analogue of the Fubini theorem is valid but neither (5.1) or (5.2) extend to this integration.

(B) If $0 < \alpha < 1$ and if the assumption

$$(5.5) \quad \text{reg } J \geq \alpha \quad (\text{for all } J \text{ which are used in the partitions of } I)$$

is added then the analogue of the Fubini theorem gets lost (cf. [J-K-S 83]) but on the other hand we have

(5.6) If $F : I \rightarrow \mathbb{R}^k$ is a vector field which is differentiable at every $t \in I$ then

$$\begin{aligned} \int_I \text{div } F d\lambda_k \text{ exists and} \\ \int_I \text{div } F d\lambda_k = \int_{\partial I} F \cdot n d\lambda_{k-1} \end{aligned}$$

where λ_k is the Lebesgue measure on \mathbb{R}^k and λ_{k-1} is the Lebesgue measure on $(k-1)$ dimensional hyperplanes, n is the exterior normal to ∂I and $F \cdot n$ denotes the scalar product (cf. [Ma 81]). (5.6) is an obvious extension of (5.1) and the formula in (5.6) is a strange version of the Stokes formula since $\text{div } F$ is general but the domain of integration is very special.

(C) A satisfactory solution in this situation is due to B. Bongiorno and W. Pfeffer (cf. [Bo-Pf 92]). Without going into details one can say that the role of the intervals is played by *BV*-sets (which are closed sets A in \mathbb{R}^k such that the $(m - 1)$ -dimensional Hausdorff outer measure of the essential boundary of A is finite). In this interpretation of I the formula from (5.6) is valid and (5.2) also extends (cf. also [Pf 93], Ch. 13).

6. S-integration; local systems. **S**-integration is defined for some maps **S** from \mathbb{R} to the set of subsets of \mathbb{R} . A very brief information on **S**-integration is given in this section.

Assume that for every $x \in \mathbb{R}$

$$(6.1) \quad \mathbf{S}(x) \text{ is a filter on } \mathbb{R},$$

$$(6.2) \quad \{x\} \notin \mathbf{S}(x),$$

$$(6.3) \quad \text{if } s \in \mathbf{S}(x) \text{ then } x \in s,$$

$$(6.4) \quad \text{if } s \in \mathbf{S}(x) \text{ and } \sigma > 0 \text{ then } s \cap (x - \sigma, x + \sigma) \in \mathbf{S}(x).$$

In the terminology of [Bo-Pi-Sk 2002] **S** is called a *filtering local system*.

Let $I = [a, b] \subset \mathbb{R}$. A function $\gamma : I \rightarrow 2^{\mathbb{R}}$ with $\gamma(x) \in \mathbf{S}(x)$ is called an **S-choice**.

With each choice γ we associate the family

$$(6.5) \quad \beta_\gamma = \{(x, [u, v]); x = u, v \in \gamma(x) \text{ or } x = v, u \in \gamma(x) \text{ where } x \in I\}.$$

A partition $\Delta = \{(x, A)\}$ of I is called a β_γ -partition of I if $\Delta \subset \beta_\gamma$.

Assume in addition that

$$(6.6) \quad \mathbf{S} \text{ is } \textit{bilateral} \text{ (i.e. every } s \in \mathbf{S}(x) \text{ contains points on either side of } x),$$

$$(6.7) \quad \mathbf{S} \text{ satisfies } \textit{the intersection condition} \text{ (i.e. for every choice } \gamma \text{ there exists a positive function } \delta \text{ such that } 0 < y - x < \min\{\delta(x), \delta(y)\} \text{ implies } \gamma(x) \cap \gamma(y) \cap [x, y] \neq \emptyset).$$

Then

$$(6.8) \quad \text{for any choice } \gamma \text{ there exists a } \beta_\gamma\text{-partition of } I.$$

6.1. DEFINITION. Let $f : I \rightarrow \mathbb{R}$, $w \in \mathbb{R}$. If for every $\varepsilon > 0$ there exists an **S-choice** γ such that

$$\left| w - \sum_{\Delta} f(x)|J| \right| \leq \varepsilon$$

for every β_γ -partition $\Delta = \{(x, J)\}$ of I then f is **S-integrable with the integral** w , $w = (\mathbf{S}) \int_I f dt$.

The **S**-integral has some of the usual elementary properties: (i) If f is **S-integrable** on I then f is **S-integrable** on any subinterval of I ; F defined by $F(s) = (\mathbf{S}) \int_{[a,s]} f dt$ is called the **S-primitive** of f . (ii) F is **S-continuous** and the **S-derivative** of F exists and is equal to f a.e.

S-primitives were examined in connection with a new concept of *ACG*-functions (these functions need not be continuous), **S**-derivatives, approximate derivatives, **S ACG** functions, variational measures (see [B-P-S 02], [E 95]). Further results were obtained in the case that **S** is generated by a path system or if **S** is a weak path system or if **S** is a p -adic

path system or a dyadic path system (see [Th 90], [Th 82/83a, Th 82/83b], [B-P-S 02], [B-P-S 03]).

7. N-integration. \mathcal{HK} -integration can be applied to $\int_I f dh$ where $f, h : I \rightarrow \mathbb{R}$ in an obvious way. If h is continuous then **N**-integration and \mathcal{HK} -integration are equivalent. For some discontinuous h the **N**-integration is a true extension of the \mathcal{HK} -integration.

Discontinuous processes are caused by friction, hysteresis, plasticity, action of regulators etc. In terms of mathematics discontinuities appear if we have to do with variational inequalities, the play operator and limit cases of differential equations with a small parameter at derivatives. There is an increasing interest of mathematicians in these problems (cf. [Kr-L 02], [Br-Kr 03]). The concept of regulated functions is a useful tool in this direction.

$g : [a, b] \rightarrow \mathbb{R}$ is called a *regulated function on $[a, b]$* if the following limits exist:

$$(7.1) \quad g(t+) = \lim_{s \rightarrow t+} g(s) \quad \text{for } a \leq t < b,$$

$$(7.2) \quad g(t-) = \lim_{s \rightarrow t-} g(s) \quad \text{for } a < t \leq b;$$

$G(a, b)$ is the space of regulated functions on $[a, b]$.

$G(a, b)$ is usually endowed with the supremum norm. For results on linear functionals, operators and integral equations see e.g. [Ho 75], [Sch 92a, Sch 92b], [Tv 96], [Tv 98].

Let $f, g : I \rightarrow \mathbb{R}$,

$$g(t) = 0 \quad \text{for } t \in I \setminus M, \quad M \text{ countable.}$$

It is well known that the Young integral $(Y) \int_I f dg$ exists for any $f : I \rightarrow \mathbb{R}$ and, moreover,

$$(7.3) \quad (Y) \int_{[a,b]} f dg = f(b)g(b) - f(a)g(a).$$

However, $(\mathcal{HK}) \int_I f dg$ need not exist even if f is continuous. An example of such f and g was based on the bad properties of integral sums

$$(7.4) \quad \sum_{i=1}^k f(t_i)[g(x_i) - g(x_{i-1})]$$

where $a = x_0 < x_1 < \dots < x_k = b$, $t_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \dots, k$ and $x_i \in M$ for a large number of i 's (cf. [Kr-Kur 02]).

This situation gave an impulse to an extension of the concept of $(\mathcal{HK}) \int_I f dg$ (by avoiding the possibility that $x_i \in M$ for any $i = 1, 2, \dots, k - 1$).

Let **N** be a system of subsets of I fulfilling

$$(7.5) \quad \text{cl}(I \setminus E) = I \quad \text{for } E \in \mathbf{N},$$

$$(7.6) \quad \text{if } E, C \in \mathbf{N} \text{ then } E \cup C \in \mathbf{N}.$$

7.1. DEFINITION. Let **N** fulfil (7.6) and (7.7), $\xi : I \rightarrow (0, \infty)$, $E \in \mathbf{N}$. Let $\Delta = \{(t_i, [x_{i-1}, x_i]); i = 1, 2, \dots, k\} \in \mathcal{HK}$, $a = x_0 < x_1 < \dots < x_k = b$.

Δ is called (ξ, E) -fine if

$$(7.7) \quad \Delta \text{ is } \xi\text{-fine,}$$

$$(7.8) \quad x_i \in I \setminus E \quad \text{for } i = 1, 2, \dots, k - 1,$$

$$(7.9) \quad t_i \in (x_{i-1}, x_i) \quad \text{for } i = 2, 3, \dots, k - 1.$$

7.2. DEFINITION. Let \mathbf{N} fulfil (7.5) and (7.6), $f, g : I \rightarrow \mathbb{R}$, $w \in \mathbb{R}$. w is called the \mathbf{N} -integral over I of f with respect to g , $w = (\mathbf{N}) \int_I f dg$, if for every $\varepsilon > 0$ there exist $\xi : I \rightarrow (0, \infty)$ and $E \in \mathbf{N}$ such that

$$(7.10) \quad \left| w - \sum_i f(t_i)(g(x_i) - g(x_{i-1})) \right| \leq \varepsilon$$

for every (ξ, E) -fine partition

$$\Delta = \{(t_i, [x_{i-1}, x_i]); i = 1, 2, \dots, k\}$$

of I . (w is unique since for every (ξ, E) there exists a (ξ, E) -fine partition Δ of I .)

7.3. THEOREM. Let \mathbf{N} fulfil (7.5) and (7.6). Let $f, g : I \rightarrow \mathbb{R}$, g continuous. If $(\mathbf{N}) \int_I f dg$ exists, then $(\mathcal{HK}) \int_I f dg$ exists and both integrals are equal (cf. [Kr 03]; Krejčí writes KN -integration and $(KN) \int_I f dg$ instead of \mathbf{N} -integration and $(\mathbf{N}) \int_I f dg$).

7.4. THEOREM. Let \mathbf{N} be the system of countable subsets of I , $f : I \rightarrow \mathbb{R}$, $g \in G(a, b)$. If $(Y) \int_I f dg$ exists then $(\mathbf{N}) \int_I f dg$ exists and both integrals are equal.

7.5. REMARK. Let Iv be the set of nonempty compact subintervals of $I = [a, b]$. A general scheme for Riemannian integration scheme was formulated in [Th 82/83b].

Let $B = \{\beta\}$ be a filter base on $I \times Iv$ which has the partitioning property (i.e. for every $\beta \in B$ and every $K \in Iv$ there exists a partition $\Delta = \{(t, J)\}$ of K such that $\Delta \subset \beta$.)

Let $f : I \rightarrow \mathbb{R}$, $G : I \rightarrow \mathbb{R}$, $w \in \mathbb{R}$. f is called B -integrable on I with respect to G and w is the B -integral, $w = (B) \int_I f dg$, if for every $\varepsilon > 0$ there exists $\beta \in B$ such that

$$\left| w - \sum_{\Delta} f(t)G(J) \right| \leq \varepsilon$$

for every partition Δ of I , $\Delta \subset \beta$.

In order to interpret the \mathbf{N} -integral as a B -integral put

$$(7.11) \quad \begin{aligned} \beta_{\xi, E} = & \{(t, [c, d]); t - \xi(t) < c < t < d < t + \xi(t) \text{ with } c, d \in I \setminus E\} \\ & \cup \{(t, [a, d]); t - \xi(t) < a \leq t < d < t + \xi(t) \text{ with } d \in I \setminus E\} \\ & \cup \{(t, [c, b]); t - \xi(t) < c < t \leq d < t + \xi(t) \text{ with } c \in I \setminus E\}, \end{aligned}$$

$$B = \{\beta_{\xi, E}; \xi : I \rightarrow (0, \infty), E \in \mathbf{N}\}.$$

Then the set of partitions $\Delta = \{(t, [x_{i-1}, x_i])\}$ of I which are (ξ, E) -fine is just the set of partitions Δ of I which fulfil $\Delta \subset \beta$ and B -integration is \mathbf{N} -integration. (If \mathbf{S} fulfils (6.1)-(6.4), (6.6), (6.7) and if $B = \{\beta_{\gamma}; \gamma \text{ is an } \mathbf{S}\text{-choice}\}$ then \mathbf{S} -integration is B -integration.)

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