

## ON A GENERALIZATION OF $W^*$ -MODULES

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**Abstract.** In a recent paper of the first author and Kashyap, a new class of Banach modules over dual operator algebras is introduced. These generalize the  $W^*$ -modules (that is, Hilbert  $C^*$ -modules over a von Neumann algebra which satisfy an analogue of the Riesz representation theorem for Hilbert spaces), which in turn generalize Hilbert spaces. In the present paper, we describe these modules, giving some motivation, and we prove several new results about them.

**1. Introduction and notation.** We begin with two definitions of great importance in  $C^*$ -algebra theory, which may be found in more detail in [14] for example. A (Hilbert)  $C^*$ -module is a right module over a  $C^*$ -algebra  $A$  with an  $A$ -valued inner product satisfying a variant of the usual axioms for a Hilbert space inner product. A  $W^*$ -module is a  $C^*$ -module over a von Neumann algebra which satisfies the analogue of the Riesz representation theorem for Hilbert spaces. Such spaces are far reaching and profound generalizations of Hilbert space, having orthonormal bases, etc. An earlier work [3] attempted to treat  $W^*$ -modules in the framework of dual Banach modules where this is possible, and where not possible then using the ‘operator space’ variant of dual Banach modules. Recently, the first author and Kashyap [6] generalized the notion, and a large part of the theory, of  $W^*$ -modules to the setting of ‘dual operator algebras’ more general than von Neumann algebras. The modules introduced there are called  $w^*$ -rigged modules. The present paper continues this work. In Section 2, which is pedagogical in nature in

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keeping with the nature of this volume, we motivate the definition of  $w^*$ -rigged modules by sketching a proof that  $w^*$ -rigged modules over von Neumann algebras are precisely the  $W^*$ -modules. Indeed we give a simplification of the main part of the proof from [6].

In Section 3, we prove that tensoring with a  $w^*$ -rigged module is bifunctorial in a certain sense. As an application, we complete the functorial characterization of Morita equivalence, sometimes called ‘Morita’s fundamental theorem’, in our setting of dual operator algebras. One direction of this characterization is the main theorem in [12], the other direction, from [5], was incomplete. In Section 4 we initiate the study of an interesting class of  $w^*$ -rigged modules, and prove an analogue of Paschke’s powerful characterization of  $W^*$ -modules as complemented submodules of ‘free’ modules [14]. We also give a short proof of a variant of the main theorem from [10], the stable isomorphism theorem. Finally, Section 5 characterizes the structure of surjective weak\* continuous linear complete isometries between bimodules, generalizing the well known ‘noncommutative Banach-Stone’ theorem for complete isometries between operator algebras. At first sight, it is not clear what a structure theorem for linear isomorphisms  $T$  between bimodules might look like. Some thought and experience reveals that the theorem one wants is precisely, or may be summarized succinctly by saying, that  $T$  is the ‘restriction to the 1-2 corner’ of a surjective (completely isometric in our case) homomorphism between the ‘linking algebras’, which maps each of the 4 corners to the matching corner. We recall that the linking algebra of a bimodule  $Y$  is an (operator) algebra which consists of  $2 \times 2$  matrices whose four entries (‘corners’) are  $Y$ , its ‘dual module’  $X = \check{Y}$ , and the two algebras acting on the left and the right of  $Y$  (see [5, Section 4] and 3.2 in [6] for more details in our setting). These linking algebras are a fundamental tool, being an operator algebra whose product encapsulates all the ‘data’ of the module. This ‘lifting of  $T$  to the linking algebra’ is in the spirit of Solel’s theorem from [17] concerning isometries between  $C^*$ -modules.

We will assume from Section 3 onwards that the reader has looked at the papers [5, 6], in which also further background and information may be found. We also assume that the reader is familiar with a few basic definitions from *operator space theory*, as may be found in e.g. [7, 15]. In particular, we assume that the reader knows what a dual operator space is, and is familiar with basic Banach space and operator space duality principles (as may be found for example within [7, Section 1.4, 1.6, Appendix A.2]). We will often abbreviate ‘weak\*’ to ‘ $w^*$ ’. Unless indicated otherwise, throughout the paper  $M$  denotes a *dual operator algebra*, by which we mean a weak\* closed subalgebra of  $B(H)$ , the bounded operators on a Hilbert space  $H$ . We take all dual operator algebras  $M$  to be *unital*, that is we assume they possess an identity of norm 1. Dual operator algebras may also be characterized abstractly (see e.g. [7, Section 2.7]). For the purposes of this paper, a (right) *dual operator module* over  $M$  is an (operator space and right  $M$ -module completely isometrically, weak\* homeomorphically, and  $M$ -module isomorphic to a) weak\* closed subspace  $Y \subset B(K, H)$ , for Hilbert spaces  $H, K$ , with  $Y\pi(M) \subset Y$ , where  $\pi : M \rightarrow B(H)$  is a weak\* continuous completely contractive unital homomorphism. Similarly for left modules. Dual operator modules may also be characterized abstractly (see e.g. [7, Section 3.8]). For right dual operator  $M$ -modules, the ‘space of morphisms’ for us will be  $w^*CB(Y, Z)_M$ , the weak\* continuous completely bounded right  $M$ -module

maps. We write  $\tilde{Y} = w^*CB(Y, M)_M$ , this is a left  $M$ -module, and plays the role in our theory of the ‘module dual’ of  $Y$ .

If  $n \in \mathbb{N}$  and  $M$  is a dual operator algebra, then we write  $C_n(M)$  for the first column of the space  $M_n(M)$  of  $n \times n$  matrices with entries in  $M$ . This is a right  $M$ -module, indeed is a dual operator  $M$ -module. As one expects,  $\tilde{Y}$  is the ‘row space’  $R_n(M)$  in this case. Similarly if  $n$  is replaced by an arbitrary cardinal  $I$ :  $C_I^w(M)$  may be viewed as one ‘column’ of the ‘infinite matrix algebra’  $\mathbb{M}_I(M) = M \bar{\otimes} B(\ell_1^2)$  (see [7, 2.7.5 (5)]). This is also a dual operator  $M$ -module (e.g. see 2.7.5 (5) and p. 140 in [7]). These modules are the ‘basic building blocks’ of  $w^*$ -rigged modules.

**2.  $W^*$ -rigged modules.** Although  $C^*$ -modules were generalized to the setting of non-selfadjoint operator algebras in the 1990s, for more than a decade after that it was not clear how generalize  $W^*$ -modules and their theory. In [6] we found the correct generalization, namely the  $w^*$ -rigged modules. There are now several equivalent definitions of these objects (see [6, Section 4]), of which the following is the most elementary.

**DEFINITION 2.1.** Suppose that  $Y$  is a dual operator space and a right module over a dual operator algebra  $M$ . Suppose that there exists a net of positive integers  $(n(\alpha))$ , and  $w^*$ -continuous completely contractive  $M$ -module maps  $\phi_\alpha : Y \rightarrow C_{n(\alpha)}(M)$  and  $\psi_\alpha : C_{n(\alpha)}(M) \rightarrow Y$ , with  $\psi_\alpha(\phi_\alpha(y)) \rightarrow y$  in the  $w^*$ -topology on  $Y$ , for all  $y \in Y$ . Then we say that  $Y$  is a *right  $w^*$ -rigged module* over  $M$ .

Our intention in this section is pedagogical. We will give some motivation for this definition, and use it to introduce some ideas in the theory. Suppose that  $M$  is a von Neumann algebra, acting on a Hilbert space  $H$ . Henceforth in this section, let  $Y$  be a right Banach  $M$ -module satisfying the Banach module variant of Definition 2.1, replacing ‘operator space’ by ‘Banach space’ and ‘completely contractive’ by ‘contractive’. That every right  $W^*$ -module over  $M$  is of this form follows exactly as in the Hilbert space case (where  $M = \mathbb{C}$ ), since there always exists an ‘orthonormal basis’  $(e_i)_{i \in I}$  (see [14]). In this case the net is indexed by the finite subsets of  $I$ , and it is an easy exercise to write down the maps  $\phi_\alpha, \psi_\alpha$  in terms of the  $e_i$ . We wish to show the nontrivial converse, that any such Banach  $M$ -module  $Y$  is a  $W^*$ -module. The bulk of this amounts to showing that the weak\* limit  $w^*\lim_\alpha \phi_\alpha(y)^* \phi_\alpha(z)$  exists in  $M$ , for all  $y, z \in Y$ : this expression then defines the  $W^*$ -module inner product. However the existence of this weak\* limit seems to be surprisingly difficult. We will sketch a proof, giving full details of a new proof of the main part of the argument.

It is a pleasant exercise for the reader (see the first lemma in [6] for a solution), to check the case that  $M = \mathbb{C}$ , that in this case  $Y$  is a Hilbert space with inner product  $\lim_\alpha \langle \phi_\alpha(z), \phi_\alpha(y) \rangle$ . In fact, this works with the same proof if the spaces  $C_{n(\alpha)}(\mathbb{C})$  are replaced by arbitrary Hilbert spaces  $H_\alpha$ . This case will be used later in the proof. The next thing to note is that  $\|y\| = \sup_\alpha \|\phi_\alpha(y)\|$ . Indeed, if  $\sup_\alpha \|\phi_\alpha(y)\| \leq t < \|y\|$ , then  $\|\psi_\alpha(\phi_\alpha(y))\| \leq t$  for all  $\alpha$ , and we obtain the contradiction  $\|y\| \leq t < \|y\|$  from Alaoglu’s theorem. We remark in passing that a similar argument shows that for any operator space

$Y$  satisfying Definition 2.1, we have

$$\| [y_{ij}] \|_{M_n(Y)} = \sup_{\alpha} \| [\phi_{\alpha}(y_{ij})] \|, \quad [y_{ij}] \in M_n(Y). \quad (1)$$

Note that equation (1) implies that such a  $Y$  is a dual operator module ( $Y$  is identified with a submodule of a direct sum of the operator modules  $C_{n(\alpha)}(M)$ ). In our (Banach module) case, we use (1) as a *definition* of matrix norms. This makes  $Y$  an operator space, a dual operator module, and it is easy to check that  $Y$  is now a  $w^*$ -rigged module.

We now mention that the kind of tensor product that appears in our theory is called the *module  $\sigma$ -Haagerup tensor product*  $Y \otimes_M^{\sigma_h} Z$  (see [10] and [5, Section 2]). We will not take time to properly introduce this here, suffice it to say that this tensor product is a dual operator space which is defined to have the universal property that it linearizes completely bounded separately weak\* continuous bilinear maps satisfying  $T(y_m, z) = T(y, mz)$  for  $y \in Y, z \in Z, m \in M$ . If  $Y$  is a  $w^*$ -rigged  $M$ -module, and if  $H$  is the Hilbert space that  $M$  acts on, define  $K = Y \otimes_M^{\sigma_h} H$ . By tensoring  $\phi_{\alpha}$  and  $\psi_{\alpha}$  on the right with the identity map on  $H$ , and using the simple fact that  $C_n(M) \otimes_M^{\sigma_h} H = H^{(n)}$  unitarily via the obvious map, one can show that  $K$  satisfies the conditions of the case discussed at the start of the last paragraph (beginning with ‘‘It is a pleasant exercise..’’). The hard part of this is that  $((\psi_{\alpha} \circ \phi_{\alpha}) \otimes I_H)(u) \rightarrow u$  weak\*, for any  $u \in K$ . We know no easy proof of this, and so will not discuss this detail here. Hence  $K$  is a Hilbert space with inner product

$$\langle y \otimes \zeta, z \otimes \eta \rangle = \lim_{\alpha} \langle (\phi_{\alpha} \otimes 1)(y \otimes \zeta), (\phi_{\alpha} \otimes 1)(z \otimes \eta) \rangle = \lim_{\alpha} \langle \phi_{\alpha}(z)^* \phi_{\alpha}(y) \zeta, \eta \rangle,$$

for  $y, z \in Y$  and  $\zeta, \eta \in H$ . This computation again uses the fact that  $C_n(M) \otimes_M^{\sigma_h} H = H^{(n)}$  unitarily. Thus  $\lim_{\alpha} \phi_{\alpha}(z)^* \phi_{\alpha}(y)$  exists in the weak\* topology of  $M$  as desired. Define  $\langle z, y \rangle$  to be this weak\* limit in  $M$ . It needs to be checked that this matches the original norm. This again uses the fact that this holds in the case  $M = \mathbb{C}$  (the exercise for the reader above), and the just mentioned ‘simple fact’, as follows: If  $\zeta \in \text{Ball}(H)$  then

$$\| \phi_{\alpha}(y) \zeta \|^2 = \| (\phi_{\alpha} \otimes 1)(y \otimes \zeta) \|^2 \leq \| y \otimes \zeta \|^2 = \langle \langle y, y \rangle \zeta, \zeta \rangle \leq \| y \|^2.$$

Taking a supremum over such  $\zeta$  and  $\alpha$ , and using (1) we obtain

$$\| \langle y, y \rangle \| = \sup_{\alpha} \| \phi_{\alpha}(y) \|^2 = \| y \|^2, \quad y \in Y.$$

Now it is clear that  $Y$  with its original norm is a  $C^*$ -module over  $M$ . That  $Y$  is a  $W^*$ -module will now be clear to experts; but in any case it is an easy fact that  $W^*$ -modules are the  $C^*$ -modules whose inner product is separately weak\* continuous (see e.g. Lemma 8.5.4 in [7]). The latter is clear in our case using a basic fact about  $Y \otimes_M^{\sigma_h} Z$ : the map  $\otimes$  is separately weak\* continuous (see [10] and [5, Section 2]). For example, if  $y_t \rightarrow y$  weak\* in  $Y$  then  $y_t \otimes \zeta \rightarrow y \otimes \zeta$  weakly in  $K$ , hence

$$\langle \langle z, y_t \rangle \zeta, \eta \rangle = \langle y_t \otimes \zeta, z \otimes \eta \rangle \rightarrow \langle y \otimes \zeta, z \otimes \eta \rangle = \langle \langle z, y \rangle \zeta, \eta \rangle.$$

REMARK. It is tempting to try to simplify the proof further by using ultrapowers. We thank Marius Junge for discussions on this; it seems that such a proof, while very interesting, may be more complicated.

**3. Bifactoriality of the tensor product, and an application.** The reader is directed to [6] for the basic theory of  $w^*$ -rigged modules. Turning to new results, we first prove the important but nontrivial fact that the module  $\sigma$ -Haagerup tensor product introduced in the last section is ‘bifunctorial’ in the following sense:

**THEOREM 3.1.** *Suppose that  $Y$  is a right  $w^*$ -rigged  $M$ -module, and that  $Z, W$  are left dual operator  $M$ -modules. If  $(S_t)$  is a net in  $w^*CB_M(Z, W)$  with weak\* limit  $S \in w^*CB_M(Z, W)$ , then  $I_Y \otimes S_t \rightarrow I_Y \otimes S$  weak\* in  $CB(Y \otimes_M^{\sigma h} Z, Y \otimes_M^{\sigma h} W)$ . Similarly, if  $T_t \rightarrow T$  weak\* in  $w^*CB(Y)_M$  then  $T_t \otimes I_Z \rightarrow T \otimes I_Z$  weak\*.*

*Proof.* The key new idea is to employ an isomorphism found in Theorem 3.5 of [6]: if  $\tilde{Y} = w^*CB(Y, M)_M$ , then the following map is an isometric weak\* homeomorphism:  $\theta_Z : Y \otimes_M^{\sigma h} Z \rightarrow w^*CB_M(\tilde{Y}, Z)$ , taking  $y \otimes z$ , for  $y \in Y, z \in Z$ , to the operator mapping an  $x \in \tilde{Y}$  to  $(x, y)z \in Z$ . If  $u = y \otimes z$ , for  $y, z$  as above, and  $x \in \tilde{Y}$ , then

$$(S \circ \theta_Z)(u)(x) = S((x, y)z) = (x, y)Sz = \theta_W((I \otimes S)(u)).$$

Thus  $S \circ \theta_Z(u) = \theta_W((I \otimes S)(u))$  for  $u = y \otimes z$ . By  $w^*$ -continuity and the density of elementary tensors, it follows that  $S \circ \theta_Z(u) = \theta_W((I \otimes S)(u))$ , for all  $u \in Y \otimes_M^{\sigma h} Z$  and  $S \in w^*CB_M(Z, W)$ . If  $S_t \rightarrow S$  weak\* in  $w^*CB_M(Z, W)$ , then  $S_t \circ \theta_Z(u) \rightarrow S \circ \theta_Z(u)$  weak\*, and hence  $\theta_W((I \otimes S_t)(u)) \rightarrow \theta_W((I \otimes S)(u))$  weak\*. Thus  $(I \otimes S_t)(u) \rightarrow (I \otimes S)(u)$  weak\*.

That  $T_t \otimes I_Z \rightarrow T \otimes I_Z$  weak\* is much shorter, following from facts about operator space multipliers, as in the proof of Theorem 2.4 in [6]. ■

**REMARK.** The variant of the last statement of the last theorem for a net  $(T_t)$  in  $w^*CB(Y, Y')$ , where  $Y'$  is a second  $w^*$ -rigged  $M$ -module, is also valid. This may be seen via the trick of viewing  $(T_t)$  in  $w^*CB_M(Y \oplus^c Y', Y \oplus^c Y')$ .

In [12], Kashyap proved the ‘difficult direction’ of the analogue of one of Morita’s famous theorems: dual operator algebras are weak\* Morita equivalent (which means that there exists a bimodule of the type mentioned just above Theorem 4.3 below) iff they are *left dual Morita equivalent* in the sense of [12, Definition 4.1] (that is, their categories of dual operator modules are equivalent via functors that are weak\* continuous on morphism spaces). Some of the ‘easy direction’ of the theorem was observed in [5]. However one aspect of this, namely the weak\* continuity of the functors implementing the categorial equivalence, stumped us until we were able in the present work to prove Theorem 3.1.

**COROLLARY 3.2.** *If  $M$  and  $N$  are weak\* Morita equivalent in the sense of [5], then their categories of dual operator modules are equivalent via functors that are weak\* continuous on morphism spaces. That is, they are left dual Morita equivalent in the sense of [12, Definition 4.1].*

*Proof.* Let  $Y$  be the equivalence  $N$ - $M$ -bimodule, with dual bimodule  $X$ . Then  $Y$  is a  $w^*$ -rigged  $M$ -module, and so the functor  $Y \otimes_M^{\sigma h} -$  is weak\* continuous on spaces of morphisms, by Theorem 3.1. Similarly,  $G = X \otimes_N^{\sigma h} -$  is weak\* continuous. That is, the functors implementing the categorial equivalence are weak\* continuous, which was the missing detail from our paper [5]. ■

**4. A variant of Paschke's 'free module' characterization.** In the last section of [6], we gave several examples of  $w^*$ -rigged modules, including the following two. First, if  $P$  is a weak\*-continuous completely contractive idempotent  $M$ -module map on  $C_I^w(M)$ , for a cardinal/set  $I$ , then  $\text{Ran}(P)$  is a  $w^*$ -rigged module. Second, suppose that  $Z$  is any WTRO (that is, a weak\* closed subspace of  $B(K, H)$  satisfying  $ZZ^*Z \subset Z$ ), and suppose that  $Z^*Z$  is contained in a dual operator algebra  $M$ . Then  $Y = \overline{ZM}^{w^*}$  is a  $w^*$ -rigged  $M$ -module, and such modules  $Y$  may be viewed as a one-sided generalization of the bimodules studied in [9, 10]. The following is the analogue of a famous theorem due to Paschke (see [14] or [7, Corollary 8.5.25]). We identify two  $w^*$ -rigged modules as *dual operator  $M$ -modules*, if there is a surjective weak\* homeomorphic completely isometric  $M$ -module map between them.

**THEOREM 4.1.** *As dual operator  $M$ -modules, the above two classes of  $w^*$ -rigged modules over  $M$  coincide, and coincide with the class of  $w^*$ -rigged module direct sums  $\bigoplus_{i \in I}^{wc} p_i M$ , for sets of projections  $\{p_i : i \in I\}$  in  $M$  (this sum coincides with the weak\* closure in  $C_I^w(M)$  of the algebraic module direct sum  $\bigoplus_{i \in I} p_i M$ ).*

*Proof.* Suppose that we have a projection  $P \in w^*CB(C_I^w(M))_M \cong \mathbb{M}_I(M)$  (see Corollary 3.6 in [6]). If  $P = [a_{ij}]$  then  $[a_{ij}] = [a_{ji}^*]$ , and so  $P \in \mathbb{M}_I(N)$  where  $N = M \cap M^*$ . Let  $Z = PC_I^w(N)$ , a WTRO. Then  $ZM \subset PC_I^w(M)$ . On the other hand, if  $\{e_i\}$  is the usual 'basis' for  $C_I^w(N)$  then  $Pe_i \in Z$ , and so  $Pe_i m \in ZM$  for all  $m \in M$ . Hence  $PC_I^w(M) \subset \overline{ZM}^{w^*}$ , and so  $PC_I^w(M) = \overline{ZM}^{w^*}$ .

Conversely, suppose that  $Y = \overline{ZM}^{w^*}$  as above. Set  $R = \overline{Z^*Z}^{w^*}$ . By the theorem of Paschke that we are modifying [14], there exist mutually orthogonal partial isometries  $(z_i)_{i \in I} \subset Z$  with  $\sum_i z_i z_i^* z = z$  for all  $z \in Z$ , and if  $P = [z_i^* z_j]$  then  $Z \cong PC_I^w(R)$ . The map  $\theta : Y \rightarrow C_I^w(M) : x \mapsto [z_i^* x]$  is a weak\* continuous complete isometry with left inverse  $[m_i] \mapsto \sum_i z_i m_i$ . It is easy to see that  $\theta(Y) \subset PC_I^w(M)$ . Also,  $\theta(z_j) = Pe_i$  for  $e_i$  as above, and so as at the end of the last paragraph we have  $PC_I^w(M) \subset \theta(Y)$ . Hence  $Y \cong PC_I^w(M)$  via  $\theta$ .

Finally, suppose that  $p_i = z_i^* z_i$ . We may view  $\bigoplus_i^{wc} p_i M$  as a submodule of  $C_I^w(M)$ . Clearly,  $\bigoplus_i^{wc} p_i M \subset PC_I^w(M)$ , and the reverse inclusion follows as in the first paragraph, since  $Pe_i \in \bigoplus_i^{wc} p_i M$ . ■

The modules considered in the last theorem form a very interesting subclass of the  $w^*$ -rigged modules, and we propose a study of this subclass. We shall call them *projectively  $w^*$ -rigged modules*.

**PROPOSITION 4.2.** *Let  $Y$  be a  $w^*$ -rigged right module over  $M$ , and let  $X = \tilde{Y}$ . Then  $Y$  is projectively  $w^*$ -rigged iff there exists a pair  $x = (x_i) \in \text{Ball}(C_I^w(X)), y = (y_i) \in \text{Ball}(R_I^w(Y))$ , for a cardinal  $I$ , such that  $x_i(y_j) = \delta_{ij} p_j$  for an orthogonal projection  $p_j \in M$ , for all  $i, j \in I$ , and  $\sum_{i \in I} y_i \otimes x_i = 1$  weak\* in  $w^*CB(Y)_M$ .*

*Proof.* If  $Y$  is projectively  $w^*$ -rigged, then we set  $y_i = z_i, x_i = z_i^*$  in the notation above. Conversely, if such  $x, y$  exist, then a slight variant of the second paragraph of the proof of the last theorem shows that  $Y$  may be identified with  $PC_I^w(M)$ , where  $P = [\delta_{ij} p_j]$ , as dual operator  $M$ -modules. ■

The following is a variant of the stable isomorphism theorem from [10]. The notation  $\mathbb{M}_I(M)$  is defined in the introduction. A *weak\* Morita equivalence bimodule* is a bimodule that is both a left and a right  $w^*$ -rigged module, and which satisfies a natural compatibility condition between the left and right actions (spelled out in [6, Section 5 (3)]).

**THEOREM 4.3.** *Suppose that  $Y$  is a weak\* Morita equivalence  $M$ - $N$ -bimodule, over dual operator algebras  $M$  and  $N$ , and suppose that  $Y$  is both left and right projectively  $w^*$ -rigged. Then  $M$  and  $N$  are stably isomorphic (that is,  $\mathbb{M}_I(M) \cong \mathbb{M}_I(N)$  completely isometrically and weak\* homeomorphically, for some cardinal  $I$ ).*

*Proof.* We will be brief, since this is well-trodden ground (see Theorem 8.5.28 and Theorem 8.5.31 in [7]). Since  $Y$  is projectively  $w^*$ -rigged over  $N$ , we have  $Y \cong PC_I^w(N)$  for a cardinal  $I$ . If  $Z = (I - P)C_I^w(N)$  then  $Y \oplus^c Z \cong C_I^w(N)$  as dual operator  $N$ -modules. Since  $Y$  is projectively  $w^*$ -rigged on the left over  $M$ , by the other-handed version of Proposition 4.2 there exists a pair  $x = (x_i) \in \text{Ball}(R_J^w(X)$ ,  $y = (y_i) \in \text{Ball}(C_J^w(Y))$ , for a cardinal  $J$ , such that  $y_j \otimes x_i = [y_j, x_i] = \delta_{ij}p_j$  for an orthogonal projection  $p_j \in M$ , for all  $i, j \in J$ , and  $\sum_{j \in J} x_j(y_j) = 1$  weak\* in  $M$ . Define maps  $\mu : N \rightarrow C_J^w(Y)$  and  $\rho : C_J^w(Y) \rightarrow N$  by  $\mu(n) = [y_j, n]$ , and  $\rho([z_j]) = \sum_{j \in J} x_j(z_j)$ , respectively. We obtain  $N \cong QC_J^w(Y)$ , where  $Q = \mu \circ \rho$ . We may identify  $Q$  with a diagonal matrix with projections  $q_j$  as the diagonal entries. If  $L = (I - Q)C_J^w(Y)$ , then  $N \oplus^c L \cong C_J^w(Y)$  as dual operator  $N$ -modules. The Eilenberg swindle, as used in the proof of [7, Theorem 8.5.28], then yields  $C_s^w(Y) \cong C_s^w(N)$ , and  $\mathbb{M}_s(Y) \cong \mathbb{M}_s(N)$ , as dual operator  $N$ -modules, for some cardinal  $s$ . By symmetry,  $\mathbb{M}_t(Y) \cong \mathbb{M}_t(M)$ , for some cardinal  $t$  which we can take to be equal to  $s$ . Thus  $M$  and  $N$  are stably isomorphic as in [7]. ■

**PROPOSITION 4.4.** *Not every  $w^*$ -rigged module is projectively  $w^*$ -rigged.*

*Proof.* Suppose by way of contradiction that every right  $w^*$ -rigged module  $Y$  is a projectively  $w^*$ -rigged module. Then every left  $w^*$ -rigged module is projectively  $w^*$ -rigged, by passing to the adjoint (or conjugate)  $\bar{Y}$ . Let  $Y$  be the  $M$ - $N$ -bimodule in Example (10) in [5, Section 3]. This is an example due to Eleftherakis, who showed that  $Y$  does not implement what he calls a  $\Delta$ -equivalence [9, 10]. We observed in [5] that  $Y$  is a weak\* Morita equivalence  $M$ - $N$ -bimodule. Hence it is both left and right  $w^*$ -rigged over  $M$  and  $N$  respectively. Thus  $Y$  is both left and right projectively  $w^*$ -rigged. By the last theorem,  $M$  and  $N$  are stably isomorphic, hence  $\Delta$ -equivalent in Eleftherakis' sense. This is a contradiction. ■

**5. Structure of isometries between bimodules.** A question of perennial interest is the structure of surjective linear isometries between various algebras. For  $C^*$ -algebras the appropriate theorem is Kadison's noncommutative Banach-Stone theorem. For non-selfadjoint algebras, the most general results on surjective isometries are due to Arazy and Solel [1], using deep techniques. One gets a much simpler structure, with a considerably easier proof, if one restricts attention to surjective *complete* isometries  $T : A \rightarrow B$  between unital operator algebras: the theorem here is that  $T$  is the product of a unitary in  $B \cap B^*$  and a surjective completely isometric homomorphism from  $A$  onto  $B$  (see e.g.

[7, Theorem 4.5.13]). In this section, we are interested in generalizing such results to maps between bimodules. Our bimodules will be  $w^*$ -rigged both as left and as right modules, that is they will be weak\* Morita equivalence bimodules, which were defined above Theorem 4.3. By considering 2 dimensional examples it seems that there is no characterization theorem in the case of general *one-sided*  $w^*$ -rigged modules. For  $C^*$ -modules (and hence  $W^*$ -modules), the structure of surjective complete isometries follows immediately from a theorem attributable to Hamana, Kirchberg, and Ruan, independently [7, Corollary 4.4.6]. The isometric case is due to Solel [17], who also shows that such isometries lift to the ‘linking  $C^*$ -algebras’. In [4], the first author characterized the structure of complete isometries between the strong Morita equivalence bimodules of [8]. Unfortunately the latter class of bimodules does not contain ours, and the proof technique used there fails in the ‘dual’ setting, for example it employs the noncommutative Shilov boundary, which has no weak\* topology variant to date. We show here how this can be circumvented. See the introduction for a discussion of the linking algebra (see [5, Section 4] for more details if desired).

**THEOREM 5.1.** *Let  $T : Y_1 \rightarrow Y_2$  be a surjective linear complete isometry between weak\* Morita equivalence bimodules (these were defined above Theorem 4.3). Suppose that  $Y_k$  is a weak\* equivalence  $M_k$ - $N_k$ -bimodule, for  $k = 1, 2$ . Then there exist unique surjective completely isometric homomorphisms  $\theta : M_1 \rightarrow M_2$  and  $\pi : N_1 \rightarrow N_2$  such that  $T(ayb) = \theta(a)T(y)\pi(b)$  for all  $a \in M_1, b \in N_1, y \in Y_1$ . If  $T$  is weak\* continuous then so are  $\theta$  and  $\pi$ , and moreover  $T$  is the 1-2-corner of a weak\* continuous surjective completely isometric homomorphism  $\rho : \mathcal{L}^w(Y_1) \rightarrow \mathcal{L}^w(Y_2)$  between the weak linking algebras which maps corners to the matching corner.*

*Proof.* We will use the machinery of ‘multipliers’ of operator spaces (see e.g. [4]), which hitherto has been the deepest tool in the theory of  $w^*$ -rigged modules [6]. From [6, Theorem 2.3], and [5, Theorem 3.6], we have that  $\mathcal{M}_\ell(Y_k) = w^*CB(Y_k)_{N_k} \cong M_k$ , and similarly  $\mathcal{M}_r(Y_k) \cong N_k$ , for  $k = 1, 2$ . By [7, Proposition 4.5.12], the map  $\mathcal{M}_\ell(Y_1) \rightarrow \mathcal{M}_\ell(Y_2) : u \mapsto TuT^{-1}$  is a completely isometric isomorphism. It is also weak\* continuous if  $T$  is, using [7, Theorem 4.7.4]. Putting these maps together, we obtain surjective completely isometric homomorphisms  $\theta : M_1 \rightarrow M_2$  and  $\pi : N_1 \rightarrow N_2$ , which are weak\* continuous if  $T$  is. We have  $\theta(a)T(y) = T(aT^{-1}(T(y))) = T(ay)$  for  $a \in M_1, y \in Y$  as desired, and a similar formula holds for  $\pi$ . The uniqueness of  $\theta$  and  $\pi$  are obvious.

Assuming  $T$  weak\* continuous, and with  $X_k = \tilde{Y}_k = w^*CB(Y_k, N_k)_{N_k}$ , define  $S : X_1 \rightarrow X_2$  by  $S(x)(z) = \pi(x(T^{-1}(z)))$ , for  $x \in X_1, z \in Y_2$ . It is routine to argue that  $S$  is a complete isometry, since  $T$  and  $\pi$  are. Clearly  $S(x)(T(y)) = \pi(x(y))$  for  $x \in X_1, y \in Y_1$ . It is simple algebra to check that  $S(bxa) = \pi(b)S(x)\theta(a)$ , and  $[T(y), S(x)] = \theta([y, x])$ . To see the latter, for example, note that for  $y' \in Y$ ,

$$[T(y), S(x)]T(y') = T(y)(S(x), T(y')) = T(y)\pi((x, y')) = T(y(x, y')) = T([y, x]y'),$$

which is just  $\theta([y, x])T(y')$ . Thus in a standard way, the maps under discussion become the four corners of a surjective homomorphism  $\rho$  between the weak linking algebras. It is easy to see that  $\rho$  is weak\* continuous, since each of its four corners is. That  $\rho$  is completely isometric follows by pulling back the operator space structure from  $\mathcal{L}^w(Y_2)$



via  $\rho$ , and using the fact that there is a unique operator space structure on the weak linking algebra making it a dual operator algebra, such that the matrix norms on each of the four corners are just the original norms of the four spaces appearing in those corners (see the third paragraph of [5, Section 4], which appeals to the idea in [7, p. 50–51]). ■

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